Math 302: Abstract Algebra
Problems 37-42 Solutions

37. By the FTCG, there is exactly one subgroup of $\mathbb{Z}_{20}$ for each divisor of 20. There are six divisors of 20, so there are six subgroups. Each subgroup can be generated by a divisor of 20 (recalling that we are working modulo 20, so the trivial subgroup would be generated by 0 rather than 20).

38.

$$\langle 1 \rangle$$

$$\langle 2 \rangle \qquad\qquad \langle 3 \rangle$$

$$\langle 6 \rangle$$

$$\langle 4 \rangle \qquad\qquad .$$

$$\langle 0 \rangle$$

(Here, there should be lines connecting

- $\langle 1 \rangle - \langle 2 \rangle$, $\langle 1 \rangle - \langle 3 \rangle$,
- $\langle 2 \rangle - \langle 6 \rangle$, $\langle 3 \rangle - \langle 6 \rangle$,
- $\langle 2 \rangle - \langle 4 \rangle$, and
- $\langle 4 \rangle - \langle 0 \rangle$, $\langle 6 \rangle - \langle 0 \rangle$.)

39. The subgroup of order 8 in $\mathbb{Z}_{24}$ is generated by $\frac{24}{8} = 3$. An element $3k \in \langle 3 \rangle$ is a generator of $\langle 3 \rangle$ if and only if $\gcd(8, k) = 1$. Thus, k can be 1, 3, 5, or 7, so the generators are 3, 9, 15, and 21.

By the exact same reasoning, if $|a| = 24$, the subgroup of order 8 in $\langle a \rangle$ is generated by $a^{\frac{24}{8}} = a^3$. An element $(a^3)^k$ is a generator of $\langle a^3 \rangle$ if and only if $\gcd(8, k) = 1$. Thus k can be 1, 3, 5, or 7, so the generators are a^3, a^9, a^{15}, and a^{21}.

40. If G is cyclic and 6 divides the order of G then the number of elements of order 6 is equal to the number of integers less than 6 that are relatively prime to 6. In this case, the number is 2 (1 and 5 are both relatively prime to 6). This is because by the FTCG, there is exactly one subgroup of order 6, so the elements of order 6 in G will correspond exactly to the generators of that subgroup. If a is a generator of the unique subgroup of order 6, then the only other generator will be a^5. By the exact same reasoning, if 8 divides $|G|$ then the nubmer of elements of order 8 in G is equal to the number of integers less than 8 that are relatively prime to 8. In this case, the number is 4 (1, 3, 5, and 7 are all relatively prime to 8). If a is one element of order 8 the other elements of order 8 are a^3, a^5 and a^7 because 3, 5, and 7 are all relatively prime to 8.

41. The subgroup of order 8 in $\mathbb{Z}_{8000000}$ is generated by 1000000. Since this is the *only* subgroup of order 8, any other element of order 8 must generate this subgroup. An element $1000000k$ in this subgroup is a generator of the subgroup if and only if $\gcd(8, k) = 1$. Thus, the generators are 1000000, 3000000, 5000000, and 7000000.

    By the same reasoning, if $|a| = 8000000$, then $|a^{\frac{8000000}{8}}| = 8$. Any other element of order 8 in $\langle a \rangle$ must generate $\langle a^{1000000} \rangle$. An element $(a^{10000000})^k$ is a generator of $\langle a^{1000000} \rangle$ if an only if $\gcd(8, k) = 1$. Thus the elements of order 8 in $\langle a \rangle$ are $a^{1000000}$, $a^{3000000}$, $a^{5000000}$, and $a^{7000000}$.

42. Let $s = \text{lcm}(m, n)$. The subgroup $\langle m \rangle \cap \langle n \rangle$ is generated by s. Indeed, suppose that some integer t is an element of $\langle m \rangle \cap \langle n \rangle$. Then $t = pm$ and $t = qn$ for some integers p and q. But from our first homework set, we know that t must be a multiple of s. Thus $t \in \langle s \rangle$ and $\langle m \rangle \cap \langle n \rangle \subset \langle s \rangle$.

    On other hand suppose that ks is a generic element of $\langle s \rangle$. Then since s is a multiple of m and n, $ks = kam$ and $ks = kbn$ for some integers a, b. But that means that ks is an element of both $\langle m \rangle$ and $\langle n \rangle$ so $ks \in \langle m \rangle \cap \langle n \rangle$, so $\langle s \rangle \subset \langle m \rangle \cap \langle n \rangle$.