MATH 122: Calculus II

Some Notes on Assignment 29

I: Section 8.8: 30, 48

Exercise 30: Using the power series representation of the exponential function, we have $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + x^2/2! - x^3/3! + x^4/4! - ....$ Then $1/e = e^{-1} = 1 - 1 + 1/2! - 1/3! + 1/4! - ...$ First two nonzero terms are 1 - 1 = 0. Error is no larger than third term,1/2.

Exercise 48: Power series for $\arctan x$, given by 8.48(e) is $x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - ....$ Using these 5 nonzero terms, we have $\arctan \frac{1}{2} \approx \frac{74783}{161280} \approx .463684$, $\arctan \frac{1}{3} \approx \frac{1994903}{6200145} \approx .321751$ and $\pi = 4(\arctan \frac{1}{2} + \arctan \frac{1}{3} \approx 3.14174$.

II: Section 8.9; 20, 26, 28

Exercise 20: Using earlier analysis of the sine function, we have $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{\sin z}{8!}x^8$

Exercise 26: The first several derivatives of $f(x) = \sqrt{4-x}$ are
$f'(x) = \frac{-1}{2\sqrt{4-x}}, f''(x) = \frac{-1}{4(\sqrt{4-x})^3}, f^{(3)}(x) = \frac{-3}{8(\sqrt{4-x})^5}, f^{(4)}(x) = \frac{-15}{16(\sqrt{4-x})^7}$
Thus $f(0) = \sqrt{4} = 2$, $f'(0) = -1/4$, $f''(0) = \frac{-1}{4 \cdot 4^{3/2}} = -\frac{1}{32}$, $f^{(3)}(0) = \frac{-3}{8 \cdot 4^{5/2}} = -\frac{3}{256}$
The Taylor Series ($n = 3$) with remainder $R_3(x)$

$$2 - \frac{1}{4}x - \frac{1/32}{2!}x^2 - \frac{3/256}{3!}x^3 + \left(\frac{-15}{16(4-z)^{7/2}}\right)\frac{1}{4!}x^4 = 2 - \frac{1}{4}x - \frac{1}{64}x^2 - \frac{1}{512}x^3 + \frac{-15}{128(4-z)^{7/2}}x^4$$

Exercise 28: The first several derivatives of $f(x) = e^{-x^2}$ are
$f'(x) = -2x\, e^{-x^2}$, $f''(x) = e^{-x^2}(-2 + 4x^2)$, $f^{(3)}(x) = e^{-x^2}(12x - 8x^3)$, $f^{(4)}(x) = 4e^{-x^2}(3 - 12x + 4x^2)$.
Thus $f(0) = 1, f'(0) = 0, f''(0) = -2, f^{(3)}(0) = 0$. The Taylor Series ($n = 3$) with remainder $R_3(x)$:

$$1 + \frac{0}{1!}x - \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{f^{(4)}(z)}{4!}x^4 = 1 - x^2 + \frac{4e^{-z^2}(3 - 12z + 4z^2)}{4 \cdot 3!}x^4 = 1 - x^2 + \frac{e^{-z^2}(3 - 12z + 4z^2)}{6}x^4$$

III: Section 9.2: 1,7, 10

Exercise 1: From $x = t^2 + 1$ and $y = t^2 - 1$, we have $dx/dt = 2t, dy/dt = 2t$ so slope of tangent line is $dy/dt = \frac{dx/dt}{dy/dt} = \frac{2t}{2t} = 1$. Slope of normal line is then (-1/1) = -1.

Exercise 7: From $x = 2\sin t$ and $y = 3\cos t$, we have $dx/dt = 2\cos t$, $dy/dt = -3\sin t$ so slope of tangent line is $\frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t$ At $t = 1$, slope of tangent line is $-\frac{3}{2}\tan 1$; slope of normal line is thus $\frac{2}{3}\cot 1$.

Exercise 10: Here $x = t^2 + t$ gives $dx/dt = 2t + 1$ and $y = 5t^2 - 3$ has $dy/dt = 10t$ so slope of tangent line is $\frac{10t}{2t+1}$ which equals 4 when $10t = 4(2t + 1) = 8t + 4$ and hence $2t = 4$ so $t = 2$. When $t = 2$, $x = 2^2 + 2 = 6$, $y = 5(2^2) - 3 = 17$. The point is (6,17).