MATH 122: Calculus II

Some Notes on Assignment 2

I: Measuring Inequality: 5, 6.

5: One type of function often used to model Lorenz curves is $f(x) = ax + (1-a)x^p$. Suppose that $a = 1/4$ and that the Gini index for the distribution of wealth in a country is known to be 9/16. (a) Find the value of p that fits this situation. (b) According to this model, how much of the wealth is owned by the wealthiest 5% of the population?

Solution: (a) We have $\frac{9}{16} = 2\int_0^1 x - \left(\frac{1}{4}x + \frac{3}{4}x^p\right) dx = 2\int_0^1 \frac{3}{4}x - \frac{3}{4}x^p dx = 2\left(\frac{3}{4}\right)\int_0^1 x - x^p dx = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right)$. Thus $\frac{9}{16} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right)$. Solving arithmetically for p, we have $p = 7$.

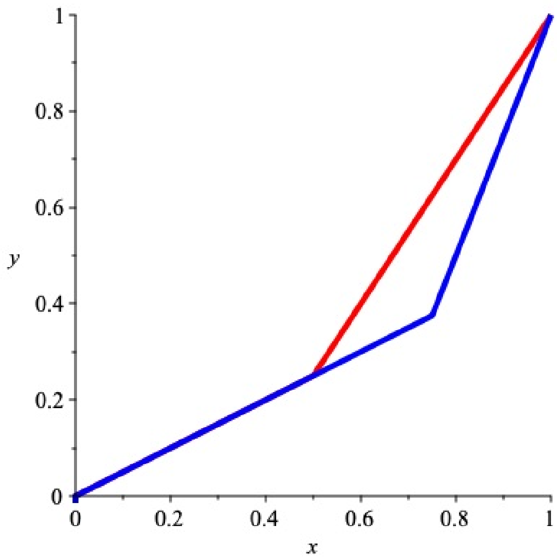
(b) The wealthiest 5% own 1 - f(.95) of the total wealth, which is $1 - f\left(\frac{19}{20}\right) = 1 - \left(\frac{1}{4}\right)\frac{19}{20} - \left(\frac{3}{4}\right)\left(\frac{19}{20}\right)^7 = \frac{1222384783}{5120000000} \approx 0.2387$ or roughly 24%.

6: Two-class societies. In theory, it could happen that one portion of the total resources is distributed equally among one class, with the rest being shared equally by another class. Here are functions that represent two different two-class societies:

$$f_1(x) = \begin{cases} \frac{x}{2} & 0 \le x \le \frac{1}{2} \\ \frac{3x-1}{2} & \frac{1}{2} \le x \le 1 \end{cases} \text{ and } f_2(x) = \begin{cases} \frac{x}{2} & 0 \le x \le \frac{3}{4} \\ \frac{5x-3}{2} & \frac{3}{4} \le x \le 1 \end{cases}$$

Compute the Gini index for each and decide which is the more equitable society, In each case, how much of the total resources are owned by the richest half of the population? ?

Solution: Note that $f_2(x) \le f_1(x)$ for all x in interval [0,1] so the f_2 society is less equitable than the f_1 one.



Graph of f1 and **Graph of f2**

In computing the Gini indices, note that $\int_0^1 f_1(x)dx = \int_0^{1/2} f_1(x)dx + \int_{1/2}^1 f_1(x)dx$ and $\int_0^1 f_2(x)dx = \int_0^{3/4} f_2(x)dx + \int_{3/4}^1 f_2(x)dx$; Recall Theorem 4.24 (page 391 of text).
The Gini index for f_1 is $\frac{1}{4}$ and the Gini index for f_2 is $\frac{3}{8}$ Since $f_1(1/2) = f_2(1/2) = 1/4$, the richest half of the population owns 3/4 of the total resources in each society.

II: Page 242: 2, 9, 21, 33, 46, 52, 64, 80

2: Rationalize Numerator:

$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{5-7x}-\sqrt{5-7a}}{x-a} = \frac{\sqrt{5-7x}-\sqrt{5-7a}}{x-a}\left(\frac{\sqrt{5-7x}+\sqrt{5-7a}}{\sqrt{5-7x}+\sqrt{5-7a}}\right)$$

$$= \frac{(5-7x)-(5-7a)}{(x-a)(\sqrt{5-7x}+\sqrt{5-7a})} = \frac{-7(x-a)}{(x-a)(\sqrt{5-7x}+\sqrt{5-7a})} = \frac{-7}{\sqrt{5-7x}+\sqrt{5-7a}}$$

$$\text{Thus } f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{-7}{\sqrt{5-7x}+\sqrt{5-7a}} = \frac{-7}{2\sqrt{5-7a}}$$

9: $G(x) = \frac{6}{(3x^2-1)^4} = 6\left(3x^2-1\right)^{-4}$ implies $G'(x) = 6(-4)(3x^2-1)^{-5}(3x^2-1)' = -24(3x^2-1)^{-5}(6x) = -144x(3x^2-1)^{-5}$.

21: $f(x) = 6x^2 - \frac{5}{x} + \frac{2}{\sqrt[3]{x^2}} = 6x^2 + 5x^{-1} + 2x^{-2}3$ implies $f'(x) = 12x + 5x^{-2} - \frac{4}{3}x^{-5/3}$.

33: $g(r) = \sqrt{1+\cos 2r}$. Noting that $f(x) = \sqrt{x} = x^{1/2}$ has $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ and using the Chain Rule, we have $g'(r) = \frac{1}{2\sqrt{1+\cos 2r}}\,(-2\sin 2r) = \frac{-\sin 2r}{\sqrt{1+\cos 2r}}$

46: $k(\phi) = \frac{\sin\phi}{\cos\phi-\sin\phi}$. Use Quotient Rule:

$$k'(\phi) = \frac{(\cos\phi-\sin\phi)(\sin\phi)' - \sin\phi(\cos\phi-\sin\phi)'}{(\cos\phi-\sin\phi)^2} = \frac{(\cos\phi-\sin\phi)(\cos\phi-\sin\phi(-\sin\phi-\cos\phi)}{(\cos\phi-\sin\phi)^2}$$

$$= \frac{\cos^2\phi - \sin\phi\cos\phi + \sin^2\phi + \sin\phi\cos\phi}{(\cos\phi-\sin\phi)^2} = \frac{1}{(\cos\phi-\sin\phi)^2}$$

Note: There are other equivalents forms of the answer; e.g., $\frac{1}{1-2\sin\phi\cos\phi} = \frac{1}{1-\sin 2\phi}$

52: Use Implicit Differentiation to find y' given $3x^2 - xy^2 + y^{-1} = 1$.
Solution: Differentiate to obtain $6x - \left(y^2 + 2xyy'\right) - y^{-2}y' = 0$ and then solve algebraically for y': $6x - y^2 = \left[2xy + y^{-2}\right]y$ so $y' = \frac{6x-y^2}{2xy+y^{-2}}$.

64: If $f(x) = x^3 - x^2 - 5x + 2$, find (a) the x coordinates of all points on the graph of f where tangent line is parallel to the line between $A(-3,2)$ and $B(1,14)$.
(b) the value of f'' at each zero of f'.
Solution: We have $f'(x) = 3x^2 - 2x - 5$ and $f''(x) = 6x - 2$. The line between A and B has slope $\frac{14-2}{1-(-3)} = \frac{12}{4} = 3$. For (a), we want to find where $f'(x) = 3$; that is $3x^2 - 2x - 5 = 3$ or $0 = 3x^2 - 2x - 8 = (3x+4)(x-2)$ so $x = -4/3$ or $x = 2$.
For (b), observe that $f'(x) = 3x^2 - 2x - 5 = (3x-5)(x+1)$ so f' is 0 at $x = 5/3$ and $x = -1$. Finally, $f''(5/3) = 6(5/3) - 2 = 10 - 2 = 8$ and $f''(-1) = -6 - 2 = -8$.

80: Two cars are approaching the same intersection along roads that run at right angles to each other. Car A is traveling at 20 mi/hr and car B is traveling at 40 mi/hr. If, at a certain distance, A is 1/4 mile from the intersection and B is 1/2 mile from the intersection, find the rate at which they are approaching each other at that instant.
Solution: Set up a coordinate system so that the origin coincides with the intersection, car A travels along the horizontal axis and car B travels along the vertical axis. At time t, car A is at a point $(x,0)$ and car B is at $(0,y)$ where x and y are decreasing functions of t with $x'(t) = -20$ and $y'(t) = -40$ mi/hr. Let $D(t)$ be the distance between the two cars. We want to find $D'(t)$ at the instant $x = 1/4$ mile and $y = 1/2$ mile. Note that at each instant of time, we have $D^2(t) = x^2(t) + y^2(t)$. Differentiating this identity with respect to t, we have

$$2D(t)D'(t) = 2x(t)x'(t) + 2y(t)y'(t) \text{ so } D'(t) = \frac{x(t)x'(t)+y(t)y'(t)}{D(t)} = \frac{x(t)x'(t)+y(t)y'(t)}{\sqrt{x^2(t)+y^2(t)}}$$

At the given instant

$$D' = \frac{(1/4)(-20)+(1/2)(-40)}{\sqrt{(1/4)^2+(1/2)^2}} = \frac{-5-20}{\sqrt{1/16+1/4}} = \frac{-25}{\sqrt{5/4}} = -\frac{100}{\sqrt{5}} = -20\sqrt{5}mi/hr$$