MATH 122: Calculus II

Some Notes on Assignment 14

I: Section 6.9: 49, 51, 60, 82

We use $=^{LH}$ to indicate that the equality follows from applying l'Hôpital's Rule.

Exercise 49: $x \ln x$ is of the indeterminate form $0 \cdot \infty$ as $x \to 0^+$. Rewrite expression as $\frac{\ln x}{1/x}$ which is of form 0/0 and we can use l'Hôpital's Rule:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} =^{LH} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

Exercise 51: $(x^2-1)e^{-x^2}$ is of the indeterminate form $\infty \cdot 0$ as $x \to \infty$. Rewrite the expression as $\frac{x^2-1}{e^{xx^2}}$ which then is an $\frac{\infty}{\infty}$ form and we can use l'Hôpital's Rule:

$$\lim_{x\to\infty}(x^2-1)(e^{-x^2}) = \lim_{x\to\infty}\frac{x^2-1}{e^{x^2}} =^{LH} \lim_{x\to\infty}\frac{2x}{2x\, e^{x^2}} = \lim_{x\to\infty}\frac{1}{e^{x^2}} = 0$$

Exercise 60: $y = x^{1/x}$ is of indeterminate form ∞^0 as $x \to \infty$. Note that $\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$ is an 0/0 form. We can use l'Hôpital's Rule on $\ln y$:

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{x} =^{LH} \lim_{x\to\infty}\frac{1/x}{1} = \lim_{x\to\infty}\frac{1}{x} = 0. \text{ Since } \ln y \to 0, \text{ we have } y \to 1$$

Exercise 82: $y = (1+\frac{r}{m})^{mt}$ is an indeterminate 1^∞ form as $m \to \infty$. We"ll work with $\ln y = mt\ln\left(1+\frac{r}{m}\right) = \frac{\ln\left(1+\frac{r}{m}\right)}{1/mt}$ which is an 0/0 form on which we can use l'Hôpital:

$$\lim_{m\to\infty}\ln y = \lim_{m\to\infty}\frac{\ln\left(1+\frac{r}{m}\right)}{1/mt} =^{LH} \lim_{m\to\infty}\frac{\frac{-r/m^2}{(1+r/m)}}{\frac{-1}{tm^2}} = \lim_{m\to\infty}\frac{rt}{(1+\frac{r}{m})} = rt$$

II: Section 7.1: 19, 24, 31

Exercise 19: Let $u = \ln\cos x$ and $dv = \sin x\, dx$ so $du = -\frac{\sin x}{\cos x}\, dx$, $v = -\cos x$. Then

$\int \sin x \ln\cos x dx = (\ln\cos x)(-\cos x) - \int(-\cos x)(-\frac{\sin x}{\cos x})dx = (\ln\cos x)(-\cos x) - \int \sin x dx = (\ln\cos x)(-\cos x) + \cos x + C$

Exercise 24: Let $I = \int \sin\ln x\, dx$ and $J = \int \cos\ln x\, dx$. Use integration by parts with $u = \sin\ln x, dv = dx$ on I and $U = \cos\ln x, dV = dx$ on J. We obtain $I = x\sin\ln x - \int \frac{\cos\ln x}{x}x\ dx = x\sin\ln x - \int \cos\ln x\ dx = x\sin\ln x - J$

but $J = x\cos\ln x - \int \frac{-\sin\ln x}{x}x\ dx = x\cos\ln x + \int \sin\ln x\ dx = x\cos\ln x + I$

Thus $I = x\sin\ln x - x\cos\ln x - I$ so $2I = x\sin\ln x - x\cos\ln x$ and $I = \frac{1}{2}\left(x\sin\ln x - x\cos\ln x\right) + C$.

Exercise 31: Recall Example 3 on Page 632 where we used integration by parts to show that $\int x\ \ln x\ dx = x\ln x - x$ and write $\int(\ln x)^2\ dx$ as $\int(\ln x)(\ln x)\ dx$. Let $u = \ln x$ and $dv = \ln x\ dx$ so $u = \frac{1}{x}\ dx$ and $v = x\ln x - x$.

Then $\int(\ln x)^2\ dx = \ln x(x\ln x - x) - \int(x\ln x - x)\frac{1}{x}\ dx = \ln x(x\ln x - x) - \int(\ln x - 1)\ dx = \ln x(x\ln x - x) - \int \ln x + x + C = \ln x(x\ln x - x) - x\ln x + x + x + C = x(\ln x)^2 - 2x\ln x + 2x + C$

III: Section 7.2: 1, 5, 9

Exercise 1: $\cos^3 x = (\cos^2 x)(\cos x) = (1-\sin^2 x)(\cos x)$ so let $u = \sin x$. Then $du = \cos x\ dx$ and $\int \cos^3\ dx = \int(1-u^2)\ du = u - \frac{u^3}{3} + C = \sin x + \frac{\sin^3 x}{3} + C$

Exercise 5: $\sin^3 x\cos^2 x = \sin^2 x\cos^2 x\sin x = (1-\cos^2 x)\cos^2 x\sin x = (cos^2 x - \cos^4 x)\sin x$. Let $u = \cos x$ so $du = -\sin x\ dx$

Then $\int \sin^3 x\cos^2 x\ dx = \int(u^2-u^4)(-1)\ du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$.

Exercise 9: $\tan^3 x\sec^4 x = \tan^3 x\sec^2 x\sec^2 x = \tan^3 x(1+\tan^2 x)\sec^2 x = (\tan^3 x + \tan^5 x)\sec^2 x$. Let $u = \tan x$ so $du = \sec^2 x\ dx$.

Then $\int \tan^3 x\sec^4 x\ dx = \int(u^3+u^5)\ du = \frac{u^4}{4} + \frac{u^6}{6} + C = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$.

Note: You can also do this problem by rewriting everything in terms of sine and cosine.