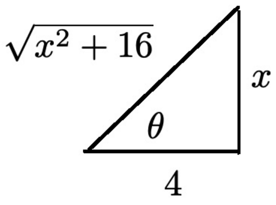
MATH 122: Calculus II

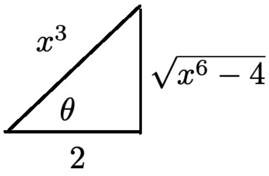
Some Notes on Assignment 13

I: Section 6.7: 51, 60, 69

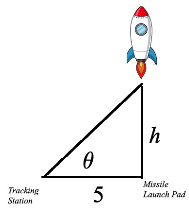
Exercise 51: Let $\tan\theta = \frac{x}{4}$ (See triangle below). The $x = 4\tan\theta$ so $dx = 4\sec^2\theta\, d\theta$ and $\sec\theta = \frac{\sqrt{x^2+16}}{4}$ so $x^2 + 16 = 16\sec^2\theta$, Thus $\int \frac{1}{x^2+16}\, dx = \int \frac{1}{16\sec^2\theta}(4\sec^2\theta)\, d\theta = \int \frac{1}{4}\, d\theta = \frac{\theta}{4} + C = \frac{1}{4}\arctan(x/4) + C$ and $\int_0^4 \frac{1}{x^2+16}\, dx = \frac{1}{4}[\arctan(1) - \arctan(0)] = \frac{1}{4}[\frac{\pi}{4} - 0] = \frac{1}{4}\,\frac{\pi}{4} = \frac{\pi}{16}$



Exercise 51



Exercise 60



Exercise 69

Exercise 60: Consider a right triangle with hypotenuse x^3 and horizontal side 2; then the vertical side is $\sqrt{x^6-4}$, a key ingredient of the integrand. The simplest ratio in the triangle involving x is $\frac{x^3}{2}$ which is $\sec\theta$. See picture above, noting also that $\sqrt{x^6-4} = 2\tan\theta$

Thus $x^3 = 2\sec\theta$ so $3x^2\, dx = 2\sec\theta\tan\theta\, d\theta$.

We can write $dx = \frac{2\sec\theta\tan\theta}{3x^2}\, d\theta$ and $\frac{1}{x}\, dx$ as $\frac{2\sec\theta\tan\theta}{3x^3}\, d\theta = \frac{2\sec\theta\tan\theta}{3(2\sec\theta)}\, d\theta = \frac{\tan\theta}{3}\, d\theta$

Hence $\int \frac{1}{x\sqrt{x^6-4}}\, dx = \int \frac{\tan\theta}{3(2\tan\theta)}\, d\theta = \int \frac{1}{6}\, d\theta = \frac{1}{6}\theta + C = \frac{1}{6}\,\text{arcsec}\left(\frac{x^3}{2}\right) + C$

Exercise 69: (See Figure above): Let $h(t)$ be height of missile t seconds after firing and θ the angle of elevation. We are looking for $h'(t)$ at the instant $\theta = 30°= \pi/6$ radians. We need to switch to radian measure to use classic differentiation formulas for the trigonometric functions). Note that $2°= \pi/90$ radians. One relation between h and θ that is true at every second is $\tan\theta = h/5$ so $h(t) = 5\tan\theta(t)$. Differentiating with respect to t yields $h'(t) = 5(\sec^2\theta)(\theta'(t))$. Since $\sec(\pi/6) = 2/\sqrt{3}$, at the given instant, we have $h' = 5\left(\frac{4}{3}\right)\left(\frac{\pi}{90}\right) = \frac{2\pi}{27} mi/sec.$

II: Section 6.9: 28, 36, 42

Exercise 28: $\displaystyle\lim_{x\to 1} \frac{2x^3 - 5x^2 + 6x - 3}{x^3 - 2x^2 + x - 1} = \frac{2-5+6-3}{1-2+1-1} = \frac{0}{-1} = 0$ (l'Hôpitals Rule does **not** apply!)

Exercise 36: Let $u = \frac{1}{x}$. Then $\frac{e^{-1/x}}{x} = \frac{e^{-u}}{1/u} = \frac{u}{e^u}$ Then

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x} = \lim_{u\to\infty} \frac{u}{e^u} = \text{(Using l'Hôpital on } \frac{\infty}{\infty} \text{ form)} \lim_{u\to\infty} \frac{1}{e^u} = 0$$

Exercise 42: (a) $\displaystyle\lim_{t\to\infty} \frac{K}{1+ce^{-rt}} = \frac{K}{1+(c)(0)} = \frac{K}{1+0} = K$ (l'Hôpitals Rule does not apply here)

(b) $y(t) = \frac{K}{1+ce^{-rt}} = \frac{K}{1+\frac{K-y(0)}{y(0)}e^{-rt}} = \frac{y(0)K}{y(0)+[K-y(0)]e^{-rt}}$ is a $\frac{\infty}{\infty}$ form as $K\to\infty$ so we can apply l'Hôpital's Rule:

$$\lim_{K\to\infty} y(t) = \lim_{K\to\infty} \frac{y(0)K}{y(0)+[K-y(0)]e^{-rt}} = \lim_{K\to\infty} \frac{y(0)}{0+e^{-rt}} = y(0)e^{rt}$$

The population will grow exponentially if K is unbounded; If K is bounded, population will approach carrying capacity over time.

III: Section 7.1: 1, 7, 13

Integration By Parts Formula: $\int u\, dv = uv - \int v\, du$

Exercise 1: Let $u = x$ and $dv = e^{-x}dx$. Then $du = 1dx$, $v = -e^{-x}$. Thus $\int xe^{-x}\, dx = -xe^{-x} - \int -e^{-x}\, dx + C = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C$.

Exercise 7: Let $u = x$ and $dv = \sec x\tan x\, dx$. Then $du = 1\, dx$ and $v = \sec x$.

Thus $\int x\ \sec xx \tan x\, dx = x\ \sec x - \int \sec x\, dx = x\ \sec x - \ln(|\sec x + \tan x|) + C$.

Exercise 13: Let $u = \ln x$ and $dv = \sqrt{x}\, dx$ so $du = \frac{1}{x}\, dx$ and $v = \frac{2}{3}x^{3/2}$.

Thus $\int \sqrt{x}\ \ln x\, dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{1}{x}\frac{2}{3}x^{3/2}\, dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\, dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$.