MATH 122: Calculus II

Some Notes on Assignment 12

I: Section 6.6: 21, 22

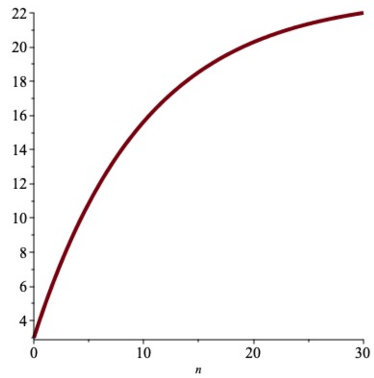
Exercise 21: $I(x) = I_0 e^{-f(x)}$ has $I'(x) = I_0 e^{-f(x)} \times (-f(x))' = -I(x)(f'(x))$ but $f(x) = k\int_0^x p(h)\,dh$ so the Fundamental Theorem of Calculus yields $f'(x) = kp(x)$ and thus $I'(x) = -kp(x)I(x)$

Exercise 22: $f(n) = 3 + 20\left(1 - e^{-0.1n}\right)$

(a) $f(5) \approx 10.87; f(9) \approx 14.87; f(24) \approx 21.19; f(30) \approx 22$

(b) $f'(n) = 20(-e^{-0.1n})(-0.1)) = 2e^{-0.1n}$ is positive for all n while $f''(n) = -.2e^{-0.1n}$ is negative so graph of f is increasing and concave down.

Here is the graph:



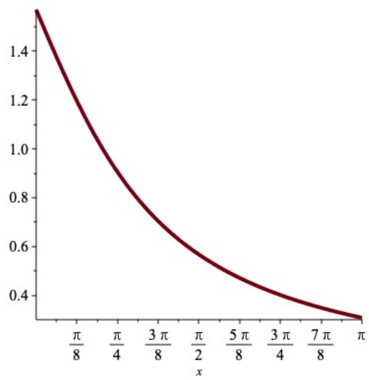
Exercise 22: Graph of $f(n) = 3 + 20\left(1 - e^{-0.1n}\right)$

(c) $f(n) = 23 - e^{-0.1n}$ and exponential term goes to 0 as n gets large so $f(n)$ approaches 23

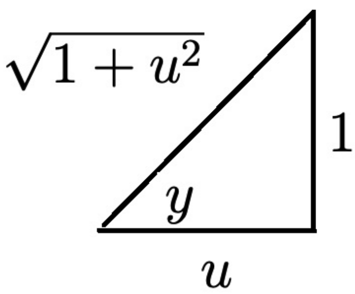
II: Section 6.7: 27, 30, 41

Exercise 27: (a) $y = \operatorname{arccot} x$ means $x = \cot y$ for any real number x and $0 < y < \pi$.

(b)



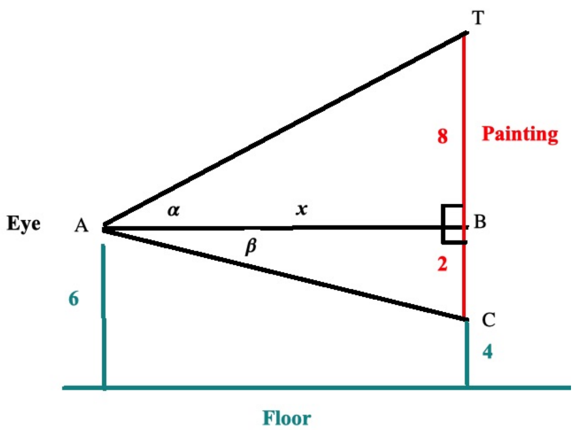
Graph of $\operatorname{arccot} x$ on $[0, \pi]$



Visualizing $\cot y = u$

(c) If $y = \operatorname{arccot} u$, then $\cot y = u$. Taking derivatives with respect to x we have $(-\csc^2 y)\, y' = u'$ so $y' = (-\sin^2 y)(u'(x)) = -\frac{1}{1+u^2} u'(x) = (-\arctan y)'$

Exercise 30: Draw lines from the critic's eyes to the top of the painting and to the bottom of the painting. See picture below. Let $x$be the distance between the critic and the painting. We let α be the angle of elevation to the painting's top and β the angle of depression to the painting's bottom.



(a) We have a right triangle ATB with $\tan\alpha = 8/x$ and another right triangle ABC with $\tan\beta = 2/x$. Note also that $\theta = \alpha + \beta = \arctan(8/x) + \arctan(2/x)$.
(b) $\tan\theta = \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-(\tan\alpha)(\tan\beta)} = \frac{\frac{8}{x}+\frac{2}{x}}{1-\frac{8}{x}\frac{2}{x}} = \frac{\frac{10}{x}}{1-\frac{16}{x^2}} = \frac{\frac{10}{x}}{\frac{x^2-16}{x^2}} = \left(\frac{10}{x}\right)\left(\frac{x^2}{x^2-16}\right) = \frac{10x}{x^2-16}$.
Note that to get a positive angle, we need $x > 4$
(c) Since the tangent of 45°=1, we need to solve $\frac{10x}{x^2-16}. = 1$; that is $x^2 - 16 = 10x$ or $x^2 - 10x - 16 = 0$. Apply the quadratic formula to obtain $x = \frac{10\pm\sqrt{100M+64}}{2} = \frac{10\pm\sqrt{164}}{2} = 5 \pm \sqrt{41}$.Since $x > 0$, we have $x = 5 + \sqrt{41} \approx 11.4$ feet.

Exercise 41: $f(x) = \cos(\frac{1}{x}) + \frac{1}{\cos x} + \arccos x = \cos(\frac{1}{x}) + \sec x + \arccos x$
so $f'(x) = -\sin(\frac{1}{x})\,(-\frac{1}{x^2}) + \sec x\tan x - \frac{1}{\sqrt{1-x^2}} = \sin(\frac{1}{x})\,(\frac{1}{x^2}) + \sec x\tan x - \frac{1}{\sqrt{1-x^2}}$

III: Section 6.9: 1, 10, 19

These 3 problems can be done with or without l'Hôpital"s Rule

Exercise 1: By l'Hôpital $(\frac{0}{0})$, we have $\lim_{x\to0}\frac{\sin x}{2x} = \lim_{x\to0}\frac{\cos x}{2} = \frac{1}{2}$

Alternatively, $\lim_{x\to0}\frac{\sin x}{2x} = \lim_{x\to0}\frac{1}{2}\frac{\sin x}{x} = \frac{1}{2}\,1(\text{Theorem }) = \frac{1}{2}$.

Exercise 10: By l'Hôpital $(\frac{0}{0})$, $\lim_{x\to\pi/2}\frac{1-\sin x}{\cos x} = \lim_{x\to\pi/2}\frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$

Alternatively, $\frac{1-\sin x}{\cos x} = \left(\frac{1-\sin x}{\cos x}\right)\left(\frac{1+\sin x}{1+\sin x}\right) = \frac{1-\sin^2 x}{(\cos x)(1+\sin x)} = \frac{\cos^2 x}{(\cos x)(1+\sin x)} = \frac{\cos x}{(1+\sin x)}$ which has limit $\frac{0}{1+1} = 0$ as $x \to \pi/2$.

Exercise 19: By 2 applications of l'Hôpital $(\frac{\infty}{\infty})$, $\lim_{x\to\infty}\frac{2x^2+3x+1}{5x^2+x+4} = \lim_{x\to\infty}\frac{4x+3}{10x+1} = \lim_{x\to\infty}\frac{4}{10} = \frac{4}{10} = \frac{2}{5}$

Alternatively, $\frac{2x^2+3x+1}{5x^2+x+4} = \frac{x^2(2+\frac{3}{x}+\frac{1}{x^2})}{x^2(5+\frac{1}{x}+\frac{4}{x^2})} = \frac{2+\frac{3}{x}+\frac{1}{x^2}}{5+\frac{1}{x}+\frac{4}{x^2}}$ which has limit $\frac{2+0+0}{5+0+0} = \frac{2}{5}$ as $x \to \infty$.