MATH 122: Calculus II

Some Notes on Assignment 10

I: Section 6.4: 45, 48, 52

Exercise 45 : $\int_0^3 f(x)\,dx = \int_0^3 \frac{cx}{x^2+4}\,dx = \frac{c}{2}\left[\ln(x^2+4)\right]_0^3 = \frac{c}{2}(\ln 13 - \ln 4) = \frac{c}{2}\ln\left(\frac{13}{4}\right)$ so $c = \frac{2}{\ln(13/4)}$

Exercise 48 (a) Value $= v(t) = 500e^{.07t}$ which is 1000 when $e^{.07t} = \frac{1000}{500} = 2$ so $.07t = \ln 2$. Hence $t = \frac{\ln 2}{.07} = \frac{100\ln 2}{7} \approx 9.9$ years
(b) We want T so that $v'(T) = 50$; that is, $(500)(.07)e^{.07T} = 50$ which we can write as $35 = e^{7T/100} = 50$, giving $7T/100 = \ln(50/35) = \ln(10/7)$ so $T = (\frac{100}{7})\ln(10/7) \approx 5.1$ years.

Exercise 52: Let $C(t)$ be the amount consumed t years in the future. Then $C'(t) = R(t) = 6.5e^{.02t} = 6.5e^{t/50}$ and hence C is the integral of R. We seek the value T where $C(T) = \int_0^T 6.5e^{t/50}\,dt = 50$. Thus $50 = (6.5)(50)\left[e^{t/50}\right]_0^T = (6.5)(50)\left(e^{T/50} - 1\right)$. Thus $\frac{1}{6.5} = e^{T/50} - 1$ so $e^{T/50} = 1 + \frac{1}{6.5} = \frac{7.5}{6.5} = \frac{15}{13}$. Solving for T : $\frac{T}{50} = \ln\frac{15}{13}$ so $T = 50\ln\frac{15}{13} \approx 7.16$ years.

II: Section 6.5: 24, 31, 38

Exercise 24: (a) π^π is a constant so derivative is 0. (b) Power rule gives $(x^4)' = 4xx^3$. (c) $(x^\pi)' = \pi x^{\pi-1}$ (d) $(\pi^x)' = \ln\pi(\pi^x)$. (e) Let $y = x^{2x}$ so $\ln y = 2x\ln x$ and hence $\frac{1}{y}y' = 2\ln x + 2x\frac{1}{x} = 2\ln x + 2$. Thus $(x^{2x})' = x^{2x}\,(2\ln x + 2)$

Exercise 31: Let $u = -2x$ so $dx = -\frac{1}{2}du$. Then $\int 5^{-2x}\,dx = -\frac{1}{2}\int 5^u\,du = (-\frac{1}{2})\frac{1}{\ln 5}5^{-2x} + C$

Exercise 38: Let $u = 3^x + 4$ so $du = (\ln 3)3^x\,dxx$. Then $\int \frac{3^x}{\sqrt{3^x+4}}\,dx = \frac{1}{\ln 3}\int\frac{1}{\sqrt{u}}\,du = \frac{1}{\ln 3}2\sqrt{u} + C = \frac{1}{\ln 3}2\sqrt{3^x+4} + C$

III: Section 6.6: 1, 5, 7

Exercise 1: If $q(t)$ is the number of bacteria at time t, then $q(t) = 5000e^{ct}$ for some constant c. Since $q(10) = 15000$, we have $5000e^{10c} = 15000$ so $e^{10c} = 3$ which gives $10c = \ln 3$ and $c = \frac{\ln 3}{10}$ and $q(t) = 5000e^{\frac{\ln 3}{10}t}$. The number of bacteria after 20 hours is $q(20) = 5000e^{\frac{\ln 3}{10}20} = 5000e^{2\ln 3} = 5000e^{\ln 9} = 9(5000) = 45,000$.
The number of bacteria reaches 50,000 at a time T such that $50,000 = q(T) = 5000e^{\frac{\ln 3}{10}T}$ so $e^{\frac{\ln 3}{10}T} = \frac{50000}{5000} = 10$. Taking logarithms, we have $\frac{\ln 3}{10}T = \ln 10$ so $T = 10\frac{\ln 10}{\ln 3} \approx 20.96$ hours.

Exercise 5: The population P at time t is $P(t) = P_0e^{rt}$ where $P_0 = 5.5$ and $r = .02 = 1/50$. We need to find T so $P(T) = 40$. Thus $5.5e^{T/50} = 40$ so $T = 50\ln(40/5.5) \approx 99.2$ years after January 1, 1993 which is mid-March 2092.

Exercise 7: (See the discussion of Example on page of our text). Let T be the temperature of the thermometer t minutes after it is brought into the 70 degree room. Then we are given $T(0) = 40, T(5) = 60$, and $T'(t) = c(T-70)$. Thus $\int\frac{T'(t)}{T-70}\,dt = \int c\,dt$ so $\ln(T-70) = ct + C$ and $T - 70 = Ce^{ct}$ for some constant C. Using $T = 40$ when $t = 0 : 40 - 70 = Ce^0$ so $C = -30, T = 70 - 30e^{ct}$. Now use $T(5) = 60 : 60 = 70 - 30e^{5c}$ implies $e^{5c} = 10/30 = 1/3$ so $5c = \ln 1/3 = -\ln 3$ or $c = \frac{ln 3}{5}$. Hence $T(t) = 70 - 30e^{-(t/5)\ln 3} = 70 - 30\left(\frac{1}{3}\right)^{t/5}$.
Temperature reaches 65 when $65 = 70 - 30\left(\frac{1}{3}\right)^{t/5}$ so $30\left(\frac{1}{3}\right)^{t/5} = 5$ and $\left(\frac{1}{3}\right)^{t/5} = \frac{1}{6}$ Solving for t, we have

$$t = 5\,\frac{\ln(1/6)}{\ln(1/3)} \approx 8.15 \text{ minutes}$$