MATH 122: Calculus II

Class 15: October 6, 2025

- 7 PM – ?
- One Sheet of Notes
- No Books, Calculators, Computers, Smart Phones, etc.

Section C	11:15 AM		Section D	1:10 PM
Last Name	*Room*		*Last Name*	*Room*
A – L	Warner 104		A – P	Warner 105
M – Z	Warner 011		Q – Z	Warner 104

Today

Indeterminate Forms and l'Hôpital's Rule

Section 6.9 of Our Text

Limit Rules

Suppose a and b are **numbers** with

$$\lim_{x \to c} f(x) = a \text{ and } \lim_{x \to c} g(x) = b$$

Then

$$\lim_{x \to c} f(x) + g(x) = a + b$$

$$\lim_{x \to c} f(x) - g(x) = a - b$$

$$\lim_{x \to c} f(x)g(x) = ab$$

But what about

$$\lim_{x \to c} \frac{f(x)}{g(x)}?$$

Indeterminate Forms

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

Assume

$$\lim_{x \to c} f(x) = a \text{ and } \lim_{x \to c} g(x) = b$$

$$\text{If } b \neq 0, \text{ then } \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{a}{b}$$

Determinate Forms

$$\lim_{x\to c} \frac{f(x)}{g(x)} \text{ with } \lim_{x\to c} f(x) = a \text{ and } \lim_{x\to c} g(x) = b$$

$$\text{If } b \neq 0, \text{ then } \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{a}{b}$$

$$\text{If } b = 0 \text{ and } a \neq 0, \text{ then } \lim_{x\to c} \frac{f(x)}{g(x)}$$ **DOES NOT EXIST**

Indeterminate Forms

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

BASIC FORMS: $\frac{0}{0}$ and $\frac{\infty}{\infty}$

OTHER FORMS: $0 \times \infty, 0^0, \infty^0, 1^\infty, \infty - \infty$

Begin with $\frac{0}{0}$ Form

Examples:

$$\lim_{x \to 0} \frac{\sin x}{x}, \ \lim_{x \to 3} \frac{x^2 - 9}{x - 3}, \ \lim_{x \to 0} \frac{x^2}{x}, \ \lim_{x \to 0} \frac{x^2}{x^3}, \ \lim_{x \to 0} \frac{2x}{3x}$$

l'Hôpital's Rule

Suppose that f and g are differentiable on an open interval (a, b) containing c, except possibly at c itself.

If $f(x)/g(x)$ has the indeterminate form $0/0$ or ∞/∞ at $x = c$ and if $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided either

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists or } \lim_{x \to c} \frac{f'(x)}{g'(x)} = \infty$$

Cauchy's Formula (Generalized Mean Value Theorem)

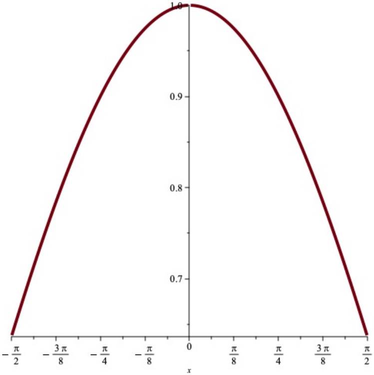
If f and g are continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) \neq 0$ for every x in (a, b), then there is a number w in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(w)}{g'(w)}$$

Augustin-Louis Cauchy (21 August 1789 – 23 May 1857)

Applying l'Hôpital's Rule

$$\lim_{x\to 0}\frac{\sin x}{x} =_{l'H}= \lim_{x\to 0}\frac{(\sin x)'}{x'} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$$



$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

Method I: Use Algebra

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3} (x+3) = 6$$

Method II: Via l'Hôpital

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} =_{\ell H} \lim_{x \to 3} \frac{2x}{1} = 6$$

$$=$$

$$l'h$$

$$=_{(l'h)}$$

$$\ell' h$$

Example: Find $\lim_{x \to 0} \dfrac{10e^x + 10e^{-x} - 20}{1 - \cos 3x}$

The quotient has the indeterminate form 0/0.
By l'Hôpital:

$$\lim_{x \to 0} \frac{10e^x + 10e^{-x} - 20}{1 - \cos 3x} =_{\ell H} \lim_{x \to 0} \frac{10e^x - 10e^{-x}}{3 \sin 3x}$$

But this new quotient is also of the form 0/0.
What to do? Apply l'Hôpital Again!

$$\lim_{x \to 0} \frac{10e^x - 10e^{-x}}{3 \sin 3x} =_{\ell H} \lim_{x \to 0} \frac{10e^x + 10e^{-x}}{9 \cos 3x} = \frac{20}{9}$$

Another Example

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} \left(\frac{\infty}{\infty}\right) =_{\ell H} \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x} \left(\frac{0}{0}\right)$$

$$=_{\ell H} \lim_{x\to 0^+} \frac{-2\sin x \cos x}{1} = 0$$

Indeterminate Form $0 \times \infty$

$$A = \lim_{x \to 0^+} (7x \; \ln x)$$

Rewrite as

$$A = \lim_{x \to 0^+} \frac{7 \ln x}{1/x} \text{ has the form } \left(\frac{\infty}{\infty}\right)$$

Apply l'Hôpital:

$$A =_{\ell H} \lim_{x \to 0^+} \frac{7/x}{-1/x^2} = \lim_{x \to 0^+} -7x = 0$$

Why is $\infty - \infty$ Indeterminate?

$$\lim_{x\to\infty}(x - x) = 0$$

$$\lim_{x\to\infty}(2x - x) = \infty$$

$$\lim_{x\to\infty}(x - 2x) = -\infty$$

$$\lim_{x\to\infty}((x + 2) - x) = 2$$

Example of $\infty - \infty$ Limit

$$\lim_{x\to\infty}\left(\sqrt{x}-\sqrt{x-10^6}\right)$$

$$=\lim_{x\to\infty}\left(\sqrt{x}-\sqrt{x-10^6}\right)\frac{\sqrt{x}+\sqrt{x-10^6}}{\sqrt{x}+\sqrt{x-10^6}}$$

$$=\lim_{x\to\infty}\frac{x-(x-10^6)}{\sqrt{x}+\sqrt{x-10^6}}$$

$$=\lim_{x\to\infty}\frac{10^6}{\sqrt{x}+\sqrt{x-10^6}}=0$$

Example of 0^0 Limit

$$\lim_{x\to 0} x^{7x}$$

Let $y = x^x$ so $\ln y = 7x \ln x$. Then

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} 7x \ln x \,(0 \times \infty) = 0$$

Since $\lim_{x\to 0} \ln y = 0$, $\lim_{x\to 0} y = e^0 = 1$


WARNING

WARNINGS!

l'Hôpital's Rule: Quotient of the Derivatives NOT Derivative of Quotient

When Limit is Determinate, l'Hôpital Will Give Wrong Answer

$$\lim_{x \to 1} \frac{x+1}{x+2} = \frac{2}{3}$$

BUT

$$\lim_{x \to 1} \frac{(x+1)'}{(x+2)'} = \lim_{x \to 1} \frac{1}{1} = 1$$

Next Topic

Techniques of Integration
Chapter 7 in Text