Math 302: Abstract Algebra
Problems 76-82 Solutions

76. An element (g, h) in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ has order 4 if $\text{lcm}(|g|, |h|)$ has order 4, i.e. if $|g| = 4$ and $|h| = 1$, 2, or 4, or if $|g| = 1$ or 2 and $|h| = 4$.

    Since we are dealing with a cyclic group $\mathbb{Z}_t$ $(t = 4)$ where 4 divides t, we know there are 2 elements of order 4 (because there are two numbers $1 \leq i \leq 4$ such that $\gcd(i, 4) = \frac{4}{4} = 1$), 1 element of order 2 (because there is one number $1 \leq i \leq 4$ such that $\gcd(i, 4) = \frac{4}{2} = 2$), and 1 element of order 1 (because there is one number $1 \leq i \leq 4$ such that $\gcd(i, 4) = \frac{4}{1} = 4$).

    Thus we have we have $2(2 + 1 + 1) + 2(1 + 1) = 12$ elements of order 4.

    The exact same reasoning works for $\mathbb{Z}_{8000000} \oplus \mathbb{Z}_{400000}$. (By the FTCG, the number of elements of order 4 in $\mathbb{Z}_{8000000}$ or $\mathbb{Z}_{400000}$ is still 2 because there are two numbers $1 \leq i \leq 4$ such that $\gcd(i, 4) = \frac{4}{4} = 1$, and so on.) In general then, the number of elements of order 4 in $\mathbb{Z}_{4m} \oplus \mathbb{Z}_{4n}$ is 12.

77. The subgroup of rotations in D_n is an abelian group. Similarly, any two-element subgroup generated by a flip in D_n is abelian. However, D_n cannot be an external direct product of the subgroup of rotations and a two-element subgroup generated by a flip. The reason is that the external direct product of two abelian groups would be abelian (this follows directly from the definition of group multiplication in an external direct product) but D_n is not abelian.

78. Notice that the groups $\mathbb{Z}_{12}$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ have the same size so it is possible to have an isomorphism between them. We know that 1 is a generator of $\mathbb{Z}_{12}$. If by the work above, if ψ is an isomorphism, then ψ is completely determined by the value of $\psi(1)$.

    On the other hand, since $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ is cyclic, any isomorphism ψ must map $1 \in \mathbb{Z}_{12}$ to a generator of $\mathbb{Z}_4 \oplus \mathbb{Z}_3$. Thus, to determine the number of isomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_3$, we must simply determine the number of generators of $\mathbb{Z}_4 \oplus \mathbb{Z}_3$.

    An element $(g, h) \in \mathbb{Z}_4 \oplus \mathbb{Z}_3$ will be a generator if it has order 12, i.e. if $|g| = 4$ and $|h| = 3$. There are 2 elements of order 4 in $\mathbb{Z}_4$ and 2 elements of order 3 in $\mathbb{Z}_3$. Thus, $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ has $2 \cdot 2 = 4$ generators, so there are 4 isomorphisms $\mathbb{Z}_{12} \to \mathbb{Z}_4 \oplus \mathbb{Z}_3$.

79. Since $\text{lcm}(|2|, |3|) = \text{lcm}(3, 5) = 15$, $(2, 3)$ is a generator of $\mathbb{Z}_3 \oplus \mathbb{Z}_5$. In $\mathbb{Z}_{15}$, $1 = 8 \cdot 2$ (i.e. 2^8 written in additive notation). Since φ is an isomorphism, it would be the case

that $\varphi(8 \cdot (2,3)) = 8 \cdot \varphi((2,3)) = 8 \cdot 2 = 1$. So the element we are looking for is $8 \cdot (2,3) = (8 \cdot 2 \bmod 3, 8 \cdot 3 \bmod 5) = (1,4)$.

80. If H has index 2 in G, then there are exactly 2 distinct left (resp. right) cosets of H in G. One of these cosets must equal H itself. The other coset (either left or right) is the complement of H in G. Call this set S.

    We know that $H = aH$ (resp. $H = Ha$) if and only if $a \in H$. Thus, if $a \in H$, we have $aH = H = Ha$, and if $a \notin H$, we have that $aH = S = Ha$. In either case, we now know that for all $a \in G$, $aH = Ha$, so by definition, H is normal in G.

81. No. For example, take $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \in H$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$.

    Then $BAB^{-1} = \begin{bmatrix} 4 & -1 \\ 9 & -2 \end{bmatrix} \notin H$.

82. We have proven that for any $a \in G$, the map $\varphi_a : G \to G$ defined by $\varphi_a(x) = axa^{-1}$ is an automorphism of G. (It's the inner automorphism determined by a.) Suppose that N is a characteristic subgroup of G. Then by definition $\varphi(N) = N$ for *any* automorphism. So let $a \in G$ and let $\varphi = \varphi_a$. Then $\varphi_a(N) = N$, i.e. $aNa^{-1} = N$. This holds for any $a \in G$, so N is normal in G.