Math 302: Abstract Algebra
Sample Exam 2 Solutions

1. (Note: we are assuming here that $L_g$ is a permutation of the elements of $G$, so we don't have to prove this part.)

   To show that $G$ and $\overline{G}$ are isomorphic, define a map $\varphi : G \to \bar{G}$ by

   $$\varphi(g) = L_g.$$

   We must check that $\varphi$ is 1-1, onto, and operation-preserving.

   To see that $\varphi$ is 1-1, suppose that $\varphi(g) = \varphi(h)$, i.e. $L_g = L_h$ as functions. We must show that $g = h$. To say that $L_g = L_h$ as functions means that $L_g(x) = L_h(x)$ for all $x \in G$. In particular, $L_g(e) = L_h(e)$, so $ge = he$, so $g = h$. Thus $\varphi$ is 1-1.

   Since $\bar{G}$ is defined to be the set $\{L_g \,|\, g \in G\}$, we have directly that $\varphi$ is onto. Indeed, for any $L_g \in \bar{G}$, $L_g = \varphi(g)$.

   Finally, we show that $\varphi(g)\varphi(h) = \varphi(gh)$. To do this, we must show that as functions, $L_g L_h = L_{gh}$. For any $x \in G$

   $$L_g L_h(x) = L_g(hx) = g(hx) = (gh)x = L_{gh}(x).$$

   Since $x$ was arbitrary, we conclude that as functions $L_g L_h = L_{gh}$.

   Therefore, $\varphi : G \to \overline{G}$ is an isomorphism, so $G$ is isomorphic to $\overline{G}$.

2. Let $n \geq 3$ and suppose that $\alpha$ is any nonidentity element in $S_n$.

   Since $\alpha$ is not the identity, we know that $\alpha$ must *not* fix at least one element $a$ in $\{1, 2, \ldots, n\}$. Thus, if we express $\alpha$ in disjoint cycle notation, it will contain a cycle of the form $(ab\cdots)$. (Note that $\alpha$ isn't necessarily comprised of a single cycle. It's just that it must contain a cycle of this form.)

   We would to show that our generic nonidentity element $\alpha$ is not contained in the center $Z(S_n)$. We can do this if we can produce an element $\beta \in S_n$ for which $\alpha\beta \neq \beta\alpha$.

   Since $n \geq 3$, we have a third element $c \in \{1, 2, \ldots, n\}$ which is not equal to either $a$ or $b$. Consider $\beta = (ac)$.

   Then we have

   $$\beta\alpha(a) = \beta(b) = b$$

   since $\beta$ fixes $b$.

On the other hand we have

$$\alpha\beta(a) = \alpha(c) = x$$

for some $x \in \{1, 2, \ldots, n\}$. Importantly, $x \neq b$. This is because $\alpha$ is a permutation, so it is 1-1. If $\alpha(c) = b$ then this would say $\alpha(c) = \alpha(a)$, and since $\alpha$ is 1-1, we would have $c = a$, a contradition.

Therefore we have shown that $\beta\alpha(a) \neq \alpha\beta(a)$, so as functions $\beta\alpha \neq \alpha\beta$. Since we have an element $\beta$ that does not commute with $\alpha$, $\alpha$ cannot be in $Z(S_n)$. Since $\alpha \in S_n$ was an arbitrary nonidentity element, we conclude that $Z(S_n) = \{\varepsilon\}$.

3. The order of any element $g$ in $G$ must divide the order of $G$. Since $|G| = 25$, the only possible orders for $g$ are 1, 5, or 25. Note that the only element that can have order 1 is the identity.

   If there is an element $g$ such that $|g| = 25$, then $\langle g \rangle = G$ and $G$ is cyclic.

   If there is not an element $g$ such that $|g| = 25$, then all nonidentity elements must have order 5. In this case for all nonidentity elements, $g^5 = e$. But it's also the case that $e^5 = e$, so we have shown that in the case that $G$ is not cyclic, $g^5 = e$ for all $g \in G$.

4. Suppose that $H$ is normal in $G$ and $ab \in H$. We must show that $ba \in H$.

   Using the litmus test, if $ab \in H$, this means that $a^{-1}H = bH$.

   But $H$ is normal so we have that $a^{-1}H = Ha^{-1}$ and $bH = Hb$. Thus $Ha^{-1} = Hb$.

   But using the litmus test (on the right), we conclude that $H = Hba$, i.e. $ba \in H$, as desired.

5. For contradiction, suppose that there is an isomorphism $\varphi : \mathbb{R} \to \mathbb{R}^*$.

   Since $\varphi$ is an onto map, there must be some $k$ such that $\varphi(k) = -1$.

   Note that $k \neq 0$ because an isomorphism will always map the identity to the identity, but while 0 is the identity in $\mathbb{R}$, $-1$ is not the identity in $\mathbb{R}^*$.

   Since $k \neq 0$, it is also the case that $2k \neq 0$. But then we have

   $$\varphi(2k) = (\varphi(k))^2 = (-1)^2 = 1.$$

   This is impossible because $\varphi$ is 1-1 and we know that $\varphi(0) = 1$ but $0 \neq 2k$. Thus, we have a contradiction, so $\mathbb{R}$ cannot be isomorphic to $\mathbb{R}^*$.

6. Yes. Since $\gcd(3, 5) = 1$, $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ is cyclic of order 15. Thus is it is isomorphic to $\mathbb{Z}_{15}$.

   To determine how many isomorphism there are, we first note that for any cyclic group $G = \langle a \rangle$, and for any operation-preserving map, once we know the value of $\varphi(a)$, we know the value of $\varphi(g)$ for any $g \in G$. This follows from the fact that $g = a^k$ for some $k$ so

$$\varphi(g) = \varphi(a^k) = (\varphi(a))^k.$$

   Furthermore, any isomorphism must map a generator to a generator.

   So, in order to count the number of isomorphism $\varphi : \mathbb{Z}_3 \oplus \mathbb{Z}_5 \to \mathbb{Z}_{15}$, if we let $(1, 1)$ be a generator of $\mathbb{Z}_3 \oplus \mathbb{Z}_5$, we simply need to count the number of generators in $\mathbb{Z}_{15}$ that $(1, 1)$ could map to.

   But we know that $i \in \{0, 1, \ldots, 14\}$ generates $\mathbb{Z}_{15}$ if and only if $\gcd(i, 15) = 1$. So $i = 1, 2, 4, 7, 8, 11, 13, 14$, and there are 8 possible isomorphisms.