Math 302 - Abstract Algebra
Sample Exam 2

This exam is closed-book and closed-notebook. Justify all of your work on these problems. Please write your answers on separate, unused paper. When you are finished, please write and sign the honor pledge at the top of this page, and staple it together with your work.

1. Let G be a group and for $g \in G$, let $L_g : G \to G$ be the permutation given by $L_g(x) = gx$. Let $\overline{G} = \{L_g \mid g \in G\}$. Prove that G is isomorphic to $\overline{G}$. (Note: you can assume that $\overline{G}$ is a group. You do not need to prove this.)

   (Here, I am asking you to give the proof of this theorem that we learned in class.)

2. Prove that for $n \geq 3$, $Z(S_n) = \{\varepsilon\}$.

   (Here, I am asking you to give the proof of this problem that we did in homework.)

3. Suppose that G is a group of order 25. Prove that G is cyclic or that $g^5 = e$ for all $g \in G$.

4. Let H be a subgroup of G. Prove that if H is normal in G, then for all $a, b \in G$, $ab \in H$ implies that $ba \in H$.

   (For extra practice you could also show the converse: if for all $a, b \in G$, $ab \in H$ implies $ba \in H$, then H is normal in G.)

5. Prove that the additive group $\mathbb{R}$ of all real numbers is not isomorphic to the multiplicative group $\mathbb{R}^*$ of nonzero real numbers.

   Hint: If there were an isomorphism $\varphi : \mathbb{R} \to \mathbb{R}^*$, then $\varphi(k) = -1$ for some k. Use this fact to arrive at a contradiction.

6. Does there exist an isomorphism $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \to \mathbb{Z}_{15}$? If not, why not? if so, how many?