The Second Isomorphism Theorem states that

$$K/K\cap N \approx KN/N.$$

$$\begin{tikzcd} & G \arrow[d, dash] & \\ & KN \arrow[dl, dash] \arrow[dr, dash] & \\ K & & N \end{tikzcd}$$

$\boxed{N \lhd G}$

which implicitly assumes KN/N is a group.

Since $N \lhd G$, this is true, as we show here. (In fact, when $N \lhd G$, $KN < G$, and since $N \lhd G$, $N \lhd KN$, so KN/N is a group.)

To see that $N \lhd G \Rightarrow KN < G$, let

$$KN = \{kn \mid k \in K,\ n \in N\}.$$

- First, since $e \in K$ and $e \in N$, letting $k=e$, $n=e$, we have $ee=e\in KN$
- Next, suppose $k_1n_1, k_2n_2 \in KN$. We NTS $k_1n_1k_2n_2 \in KN$, ie that it can be expressed as kn for some $k \in K$, $n \in N$.
  But $N \lhd G$ so, $Nk_2 = k_2N$, thus there exists $n_3 \in N$ st. $n_1k_2 = k_2n_3$. So $k_1n_1k_2n_2 = \underbrace{k_1k_2}_{\in K}\underbrace{n_3n_2}_{\in N} \in KN$.
- Finally, suppose $k_1n_1 \in KN$. Then in G, $(k_1n_1)^{-1} = n_1^{-1}k_1^{-1}$. We must show that $n_1^{-1}k_1^{-1}$ can be expressed as kn for some $k \in K$, $n \in N$.

But again, since $N \lhd G$, $Nk_1^{-1} = k_1^{-1}N$ so there exists $n_4 \in N$ st. $n_1^{-1}k_1^{-1} = \underbrace{k_1^{-1}}_{\in K}\underbrace{n_4}_{\in N} \in KN$.

Thus, under the condition that $N \lhd G$, KN is a subgroup of G. Further, $K < KN$ and $N < KN$

↳ $k \in K = ke \in KN$

↳ $n \in N = en \in KN$

and since $N \lhd G$, $N \lhd KN$ so KN/N is a group.