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57 Prove that $\lim_{n\to\infty}[1+(x/n)]^n = e^x$ by letting $h = x/n$ and using Theorem (6.32)(i).

c 58 Graph, on the same coordinate axes, $y = 2^{-x}$ and $y = \log_2 x$.

(a) Estimate the x-coordinate of the point of intersection of the graphs.

(b) If the region R bounded by the graphs and the line $x = 1$ is revolved about the x-axis, set up an integral that can be used to approximate the volume of the resulting solid.

(c) Use Simpson's rule, with $n = 2$, to approximate the integral in part (b).

6.6 SEPARABLE DIFFERENTIAL EQUATIONS AND LAWS OF GROWTH AND DECAY

Suppose that a physical quantity varies with time and that the magnitude of the quantity at time t is given by $q(t)$, where q is differentiable and $q(t) > 0$ for every t. The derivative $q'(t)$ is the rate of change of $q(t)$ with respect to time. In many applications, this rate of change is directly proportional to the magnitude of the quantity at time t—that is,

$$q'(t) = cq(t)$$

for some constant c. The number of bacteria in certain cultures behaves in this way. If the number of bacteria $q(t)$ is small, then the rate of increase $q'(t)$ is small; however, as the number of bacteria increases, the *rate of increase* also increases. The decay of a radioactive substance obeys a similar law: As the amount of matter decreases, the rate of decay—that is, the amount of radiation—also decreases. As a final illustration, suppose an electrical condenser is allowed to discharge. If the charge on the condenser is large at the outset, the rate of discharge is also large, but as the charge weakens, the condenser discharges less rapidly.

In applied problems, the equation $q'(t) = cq(t)$ is often expressed in terms of differentials. Thus, if $y = q(t)$, we may write

$$\frac{dy}{dt} = cy, \quad \text{or} \quad dy = cy\,dt.$$

Dividing both sides of the last equation by y, we obtain

$$\frac{1}{y}\,dy = c\,dt.$$

Since it is possible to **separate the variables** y and t—in the sense that they can be placed on opposite sides of the equals sign—the differential equation $dy/dt = cy$ is a **separable differential equation**. We will study such equations in more detail later in the text and will show that solutions can be found by integrating both sides of the "separated" equation $(1/y)\,dy = c\,dt$. Thus,

$$\int \frac{1}{y}\,dy = \int c\,dt$$

and, assuming $y > 0$,

$$\ln y = ct + d$$

for some constant d. It follows that

$$y = e^{ct+d} = e^d e^{ct}.$$

If y_0 denotes the initial value of y (that is, the value corresponding to $t = 0$), then letting $t = 0$ in the last equation gives us

$$y_0 = e^d e^0 = e^d,$$

and hence the solution $y = e^d e^{ct}$ may be written

$$y = y_0 e^{ct}.$$

We have proved the following theorem.

Theorem 6.33

Let y be a differentiable function of t such that $y > 0$ for every t, and let y_0 be the value of y at $t = 0$. If $dy/dt = cy$ for some constant c, then

$$y = y_0 e^{ct}.$$

The preceding theorem states that *if the rate of change of* $y = q(t)$ *with respect to* t *is directly proportional to* y, *then* y *may be expressed in terms of an exponential function.* If y increases with t, the formula $y = y_0 e^{ct}$ is a **law of growth** ($c > 0$), and if y decreases, it is a **law of decay** ($c < 0$).

EXAMPLE 1 The number of bacteria in a culture increases from 600 to 1800 in 2 hr. Assuming that the rate of increase is directly proportional to the number of bacteria present, find

(a) a formula for the number of bacteria at time t

(b) the number of bacteria at the end of 4 hr

SOLUTION

(a) Let $y = q(t)$ denote the number of bacteria after t hours. Thus, $y_0 = q(0) = 600$ and $q(2) = 1800$. By hypothesis,

$$\frac{dy}{dt} = cy.$$

Following exactly the same steps used in the proof of Theorem (6.33), we obtain

$$y = y_0 e^{ct} = 600e^{ct}.$$

Since $y = 1800$ when $t = 2$, we obtain the following equivalent equations:

$$1800 = 600e^{2c}, \qquad 3 = e^{2c}, \qquad e^c = 3^{1/2}.$$

Substituting for e^c in $y = 600e^{ct}$ gives us

$$y = 600(3^{1/2})^t, \quad \text{or} \quad y = 600(3)^{t/2}.$$

(b) Letting $t = 4$ in $y = 600(3)^{t/2}$ yields

$$y = 600(3)^{4/2} = 600(9) = 5400.$$

EXAMPLE ■ 2 Radium decays exponentially and has a half-life of approximately 1600 yr—that is, given any quantity, one half of it will disintegrate in 1600 yr.

(a) Find a formula for the amount y remaining from 50 mg of pure radium after t years.

(b) When will the amount remaining be 20 mg?

SOLUTION

(a) If we let $y = q(t)$, then

$$y_0 = q(0) = 50 \quad \text{and} \quad q(1600) = \tfrac{1}{2}(50) = 25.$$

Since $dy/dt = cy$ for some c, it follows from Theorem (6.33) that

$$y = 50e^{ct}.$$

Since $y = 25$ when $t = 1600$,

$$25 = 50e^{1600c}, \quad \text{or} \quad e^{1600c} = \tfrac{1}{2}.$$

Hence,

$$e^c = (\tfrac{1}{2})^{1/1600} = 2^{-1/1600}.$$

Substituting for e^c in $y = 50e^{ct}$ gives us

$$y = 50(2^{-1/1600})^t, \quad \text{or} \quad y = 50(2)^{-t/1600}.$$

(b) Using $y = 50(2)^{-t/1600}$, we see that the value of t at which $y = 20$ is a solution of the equation

$$20 = 50(2)^{-t/1600}, \quad \text{or} \quad 2^{t/1600} = \tfrac{5}{2}.$$

Taking the natural logarithm of each side, we obtain

$$\frac{t}{1600}\ln 2 = \ln\frac{5}{2},$$

or

$$t = \frac{1600 \ln \frac{5}{2}}{\ln 2} \approx 2115 \text{ yr}.$$

EXAMPLE ■ 3 According to Newton's law of cooling, the rate at which an object cools is directly proportional to the difference in temperature between the object and the surrounding medium. If an object cools from 125 °F to 100 °F in half an hour when surrounded by air at a temperature of 75 °F, find its temperature at the end of the next half hour.

SOLUTION Let y denote the temperature of the object after t hours of cooling. Since the temperature of the surrounding medium is 75°, the difference in temperature is $y - 75$, and therefore, by Newton's law of cooling,

$$\frac{dy}{dt} = c(y - 75)$$

for some constant c. We separate variables and integrate as follows:

$$\frac{1}{y-75}\,dy = c\,dt$$

$$\int \frac{1}{y-75}\,dy = \int c\,dt$$

$$\ln(y - 75) = ct + b$$

for some constant b. The last equation is equivalent to

$$y - 75 = e^{ct+b} = e^b e^{ct}.$$

Since $y = 125$ when $t = 0$,

$$125 - 75 = e^b e^0 = e^b, \quad \text{or} \quad e^b = 50.$$

Hence,

$$y - 75 = 50e^{ct}, \quad \text{or} \quad y = 50e^{ct} + 75.$$

Using the fact that $y = 100$ when $t = \frac{1}{2}$ leads to the following equivalent equations:

$$100 = 50e^{c/2} + 75, \qquad e^{c/2} = \tfrac{25}{50} = \tfrac{1}{2}, \qquad e^c = \tfrac{1}{4}$$

Substituting $\frac{1}{4}$ for e^c in $y = 50e^{ct} + 75$ gives us a formula for the temperature after t hours:

$$y = 50(\tfrac{1}{4})^t + 75$$

In particular, if $t = 1$,

$$y = 50(\tfrac{1}{4}) + 75 = 87.5\,°\text{F}.$$

In biology, a function G is sometimes used, as follows, to estimate the size of a quantity at time t:

$$G(t) = ke^{(-Ae^{-Bt})}$$

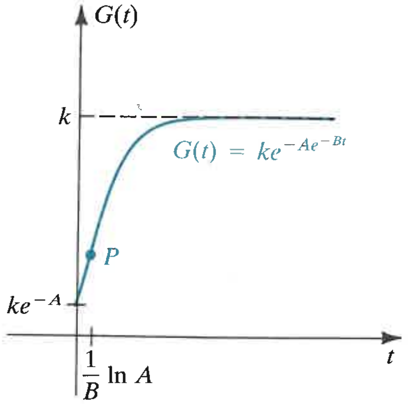
for positive constants k, A, and B. The function G is called a **Gompertz growth function**. It is always positive and increasing, but has a limit as t increases without bound. The graph of G is called a **Gompertz growth curve**.

EXAMPLE ■ 4 Discuss and sketch the graph of the Gompertz growth function G.

SOLUTION We first observe that the y-intercept is $G(0) = ke^{-A}$ and that $G(t) > 0$ for every t. Differentiating twice, we obtain

$$\begin{aligned} G'(t) &= ke^{(-Ae^{-Bt})}(-Ae^{-Bt})' \\ &= ABke^{(-Bt-Ae^{-Bt})} \\ G''(t) &= ABke^{(-Bt-Ae^{-Bt})}(-Bt - Ae^{-Bt})' \\ &= ABk(-B + ABe^{-Bt})e^{-Bt-Ae^{-Bt}}. \end{aligned}$$

Figure 6.24



Since $G'(t) > 0$ for every t, the function G is increasing on $[0, \infty)$. The second derivative $G''(t)$ is zero if

$$-B + ABe^{-Bt} = 0, \quad \text{or} \quad e^{Bt} = A.$$

Solving the last equation for t gives us $t = (1/B)\ln A$, which is a critical number for the function G'. We leave it as an exercise to show that at this time the rate of growth G' has a maximum value Bk/e. We can also show that

$$\lim_{t\to\infty} G'(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} G(t) = k.$$

Hence, as t increases without bound, the rate of growth approaches 0 and the graph of G has a horizontal asymptote $y = k$. A typical graph is sketched in Figure 6.24. The point P on the graph, corresponding to $t = (1/B)\ln A$, is a point of inflection, and the concavity changes from upward to downward at P.

In the next example, we consider a physical quantity that increases to a maximum value and then decreases asymptotically to 0.

EXAMPLE ■ 5 When uranium disintegrates into lead, one step in the process is the radioactive decay of radium into radon gas. Radon gas enters homes by diffusing through the soil into basements, where it presents a health hazard if inhaled. If a quantity Q of radium is present initially, then the amount of radon gas present after t years is given by

$$A(t) = \frac{c_1 Q}{c_2 - c_1}(e^{-c_1 t} - e^{-c_2 t}),$$

where $c_1 = \frac{1}{1600}\ln 2$ and $c_2 = \frac{1}{0.0105}\ln 2$ are the *decay constants* for radium and radon gas, respectively.

(a) Find the amount of radon gas present initially and after an extended period of time.

(b) Use a graphing utility to graph $A(t)$.

(c) Determine the maximum amount A_M of radon gas and when that amount is reached.

(d) After the maximum amount A_M has been reached, estimate how long it would take the radon gas to decrease to 90% of the maximum.

SOLUTION

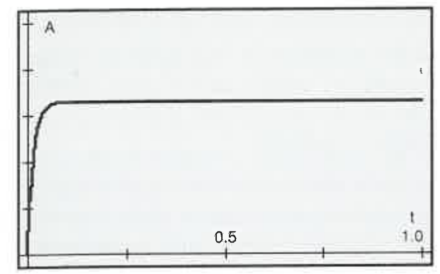
(a) The initial amount of radon gas is

$$A(0) = \frac{c_1 Q}{c_2 - c_1}(e^0 - e^0) = 0.$$

If we let t increase without bound, then

$$\lim_{t\to\infty} A(t) = \frac{c_1 Q}{c_2 - c_1}\lim_{t\to\infty}(e^{-c_1 t} - e^{-c_2 t})$$
$$= \frac{c_1 Q}{c_2 - c_1}(0 - 0) = 0.$$

Figure 6.25
$A(t) = \frac{c_1 Q}{c_2 - c_1}(e^{-c_1 t} - e^{-c_2 t})$
$0 \le t \le 1, 0 \le A \le 10^{-5}Q$



Hence, over a long period of time, the amount of radon gas decreases to 0.

(b) We use a graphing utility to obtain Figure 6.25, which illustrates the graph of $A(t)$. In the viewing window, $0 \le t \le 1$, it appears that the radon gas rises fairly quickly to its maximum level and then levels off or perhaps decreases very, very slowly. In part (a), we concluded that the amount of radon gas would eventually decrease to 0 but that is not evident from the graph in Figure 6.25. In parts (c) and (d), we do further analysis to determine how quickly the maximum is reached and how slowly the gas disappears.

(c) To find the critical numbers of A, we differentiate, obtaining

$$A'(t) = \frac{c_1 Q}{c_2 - c_1}\left[-\left(c_1 e^{-c_1 t} + c_2 e^{-c_2 t}\right)\right].$$

Thus, $A'(t) = 0$ if

$$c_1 e^{-c_1 t} = c_2 e^{-c_2 t}, \quad \text{or} \quad e^{(c_2 - c_1)t} = \frac{c_2}{c_1}.$$

It follows that

$$(c_2 - c_1)t = \ln\frac{c_2}{c_1},$$

or

$$t = \frac{\ln(c_2/c_1)}{c_2 - c_1}.$$

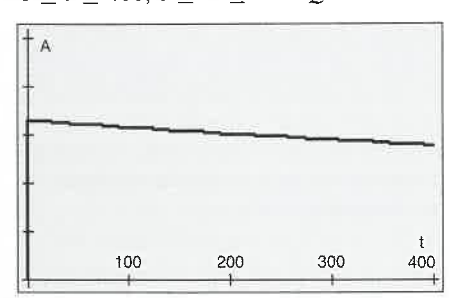
This value of t yields the maximum value of A. Substituting this value for t into the function, we find (after a fair amount of algebraic manipulation) that the maximum value is

$$A_M = A\left(\frac{\ln(c_2/c_1)}{c_2 - c_1}\right) = \left(\frac{c_1}{c_2}\right)^{c_2/(c_2 - c_1)} Q.$$

For the given values of the constants c_1 and c_2, these two numbers are approximately

$$t_M \approx 0.181 \text{ years} \approx 66 \text{ days} \quad \text{and} \quad A_M \approx (6.562)10^{-6}Q.$$

Figure 6.26
$A(t) = \frac{c_1 Q}{c_2 - c_1}(e^{-c_1 t} - e^{-c_2 t})$
$0 \le t \le 400, 0 \le A \le 10^{-5}Q$



(d) To find the value $t_1 > t_M$ that yields $A(t_1) = 0.90A_M$, we first divide both sides by Q so that we can work numerically. Using the solving routine on a computational device (or Newton's method), we find that the solution to

$$\frac{c_1}{c_2 - c_1}(e^{-c_1 t_1} - e^{-c_2 t_1}) = (0.9)(6.562)10^{-6}Q$$

for the given values of c_1 and c_2 is $t_1 \approx 243$ years. (See Figure 6.26 for a viewing window displaying this part of the graph of the function.) The fact that the radon decreases to 0 over a long period of time may not be very comforting to a homeowner since the decrease takes place so slowly.

EXERCISES 6.6

1 The number of bacteria in a culture increases from 5000 to 15,000 in 10 hr. Assuming that the rate of increase is proportional to the number of bacteria present, find a formula for the number of bacteria in the culture at any time t. Estimate the number at the end of 20 hr. When will the number be 50,000?

2 The polonium isotope ^{210}Po has a half-life of approximately 140 days. If a sample weighs 20 mg initially, how much remains after t days? Approximately how much will be left after two weeks?

3 If the temperature is constant, then the rate of change of barometric pressure p with respect to altitude h is proportional to p. If $p = 30$ in. at sea level and $p = 29$ in. when $h = 1000$ ft, find the pressure at an altitude of 5000 ft.

4 The population of a city is increasing at the rate of 5% per year. If the present population is 500,000 and the rate of increase is proportional to the number of people, what will the population be in 10 yr?

5 Agronomists use the assumption that a quarter acre of land is required to provide food for one person and estimate that there are 10 billion acres of tillable land in the world. Hence a maximum population of 40 billion people can be sustained if no other food source is available. The world population at the beginning of 1993 was approximately 5.5 billion. Assuming that the population increases at a rate of 2% per year and the rate of increase is proportional to the number of people, when will the maximum population be reached?

6 A metal plate that has been heated cools from 180 °F to 150 °F in 20 min when surrounded by air at a temperature of 60 °F. Use Newton's law of cooling (see Example 3) to approximate its temperature at the end of 1 hr of cooling. When will the temperature be 100 °F?

7 An outdoor thermometer registers a temperature of 40 °F. Five minutes after it is brought into a room where the temperature is 70 °F, the thermometer registers 60 °F. When will it register 65 °F?

8 The rate at which salt dissolves in water is directly proportional to the amount that remains undissolved. If 10 lb of salt is placed in a container of water and 4 lb dissolves in 20 min, how long will it take for two more pounds to dissolve?

9 According to Kirchhoff's first law for electrical circuits, $V = RI + L(dI/dt)$, where the constants V, R, and L denote the electromotive force, the resistance, and the inductance, respectively, and I denotes the current at time t. If the electromotive force is terminated at time $t = 0$ and if the current is I_0 at the instant of removal, prove that $I = I_0 e^{-Rt/L}$.

10 A physicist finds that an unknown radioactive substance registers 2000 counts per minute on a Geiger counter. Ten days later, the substance registers 1500 counts per minute. Approximate its half-life.

11 The air pressure P (in atmospheres) at an elevation of z meters above sea level is a solution of the differential equation $dP/dz = -9.81\rho(z)$, where $\rho(z)$ is the density of air at elevation z. Assuming that air obeys the ideal gas law, this differential equation can be rewritten as $dP/dz = -0.0342P/T$, where T is the temperature (in °K) at elevation z. If $T = 288 - 0.01z$ and if the pressure is 1 atmosphere at sea level, express P as a function of z.

12 During the first month of growth for crops such as maize, cotton, and soybeans, the rate of growth (in grams per day) is proportional to the present weight W. For a species of cotton, $dW/dt = 0.21W$. Predict the weight of a plant at the end of the month ($t = 30$) if the plant weighs 70 mg at the beginning of the month.

13 Radioactive strontium-90, ^{90}Sr, with a half-life of 29 yr, can cause bone cancer in humans. The substance is carried by acid rain, soaks into the ground, and is passed through the food chain. The radioactivity level in a particular field is estimated to be 2.5 times the safe level S. For approximately how many years will this field be contaminated?

14 The radioactive tracer ^{51}Cr, with a half-life of 27.8 days, can be used in medical testing to locate the position of a placenta in a pregnant woman. Often the tracer must be ordered from a medical supply lab. If 35 units are needed for a test and delivery from the lab requires 2 days, estimate the minimum number of units that should be ordered.

15 Veterinarians use sodium pentobarbital to anesthetize animals. Suppose that to anesthetize a dog, 30 mg is required for each kilogram of body weight. If sodium pentobarbital is eliminated exponentially from the bloodstream and half is eliminated in 4 hr, approximate the single dose that will anesthetize a 20-kg dog for 45 min.

16 In the study of lung physiology, the following differential equation is used to describe the transport of a substance across a capillary wall:

$$\frac{dh}{dt} = -\frac{V}{Q}\left(\frac{h}{k+h}\right),$$

where h is the hormone concentration in the bloodstream, t is time, V is the maximum transport rate, Q is the volume of the capillary, and k is a constant that measures the affinity between the hormones and enzymes that assist with the transport process. Find the general solution of the differential equation.

17 A space probe is shot upward from the earth. If air resistance is disregarded, a differential equation for the velocity after burnout is $v(dv/dy) = -ky^{-2}$, where y is the distance from the center of the earth and k is a positive constant. If y_0 is the distance from the center of the earth at burnout and v_0 is the corresponding velocity, express v as a function of y.

18 At high temperatures, nitrogen dioxide, NO_2, decomposes into NO and O_2. If $y(t)$ is the concentration of NO_2 (in moles per liter), then, at 600 °K, $y(t)$ changes according to the *second-order reaction law* $dy/dt = -0.05y^2$ for time t in seconds. Express y in terms of t and the initial concentration y_0.

19 The technique of carbon-14 dating is used to determine the age of archeological or geological specimens. This method is based on the fact that the unstable isotope carbon-14 (^{14}C) is present in the CO_2 in the atmosphere. Plants take in carbon from the atmosphere; when they die, the ^{14}C that has accumulated begins to decay, with a half-life of approximately 5700 yr. By measuring the amount of ^{14}C that remains in a specimen, it is possible to approximate when the organism died. Suppose that a bone fossil contains 20% as much ^{14}C as an equal amount of carbon in present-day bone. Approximate the age of the bone.

20 Refer to Exercise 19. The hydrogen isotope ^3_1H, which has a half-life of 12.3 yr, is produced in the atmosphere by cosmic rays and is brought to earth by rain. If the wood siding of an old house contains 10% as much ^3_1H as the siding on a similar new house, approximate the age of the old house.

21 The earth's atmosphere absorbs approximately 32% of the sun's incoming radiation. The earth also emits radiation (mostly in the form of heat), and the atmosphere absorbs approximately 93% of this outgoing radiation. This difference in absorption of incoming and outgoing radiation by the atmosphere is called the *greenhouse effect*. Changes in this balance will affect the earth's climate. Suppose I_0 is the intensity of the sun's radiation and I is the intensity of the radiation after traveling a distance x through the atmosphere. If $\rho(h)$ is the density of the atmosphere at height h, then the *optical thickness* is $f(x) = k\int_0^x \rho(h)\,dh$, where k is an absorption constant, and I is given by $I = I_0 e^{-f(x)}$. Show that $dI/dx = -k\rho(x)I$.

22 Certain learning processes may be illustrated by the graph of $f(x) = a + b(1 - e^{-cx})$ for positive constants a, b, and c. Suppose a manufacturer estimates that a new employee can produce 5 items the first day on the job. As the employee becomes more proficient, the daily production increases until a certain maximum production is reached. Suppose that on the nth day on the job, the number $f(n)$ of items produced is approximated by $f(n) = 3 + 20(1 - e^{-0.1n})$.

(a) Estimate the number of items produced on the fifth day, the ninth day, the twenty-fourth day, and the thirtieth day.

(b) Sketch the graph of f from $n = 0$ to $n = 30$. (Graphs of this type are called *learning curves* and are used frequently in education and psychology.)

(c) What happens as n increases without bound?

23 A spherical cell has volume V and surface area S. A simple model for cell growth before mitosis assumes that the rate of growth dV/dt is proportional to the surface area of the cell. Show that $dV/dt = kV^{2/3}$ for some $k > 0$, and express V as a function of t.

24 In Theorem (6.33), we assumed that the rate of change of a quantity $q(t)$ at time t is directly proportional to $q(t)$. Find $q(t)$ if its rate of change is directly proportional to $[q(t)]^2$.

25 Refer to Example 4.

(a) Verify that Bk/e is a maximum value for G'.

(b) Show that $\lim_{t\to\infty} G'(t) = 0$ and $\lim_{t\to\infty} G(t) = k$.

(c) Sketch the graph of G if $k = 10$, $A = 2$, and $B = 1$.

c 26 Graph the Gompertz growth function G on the interval $[0, 5]$ for $k = 1.1$, $A = 3.2$, and $B = 1.1$.

c **Exer. 27–31: Each function contains a constant term for which four values are given. Plot the four versions of the function in the same viewing window.**

27 $y = 10e^{ct}$ on $-5 \le t \le 8$ for $c = 0.05, 0.1, 0.2,$ and 0.4

28 $y = 10e^{-Ae^{-1.1t}}$ on $-3 \le t \le 8$ for $A = 0.8, 2.0, 3.2,$ and 4.4

29 $y = 10e^{-3.2e^{-Bt}}$ on $-3 \le t \le 8$ for $B = 0.2, 0.8, 1.1,$ and 2.0

30 $y = 10(e^{-c_1t} - e^{-c_2t})$ on $0 \le t \le 5$ for $c_1 = 1$ and $c_2 = 3, 5, 10,$ and 25

31 $y = 10(e^{-c_1t} - e^{-c_2t})$ on $0 \le t \le 5$ for $c_1 = 0.5, 1, 2,$ and 4 and $c_2 = 10$

6.7 INVERSE TRIGONOMETRIC FUNCTIONS

In this section, we discuss the inverse trigonometric functions, their derivatives, and their integrals. Since we may regard the values of the inverse trigonometric functions as angles, these functions have a broad range of applications, such as rates of change in the angle of elevation as an observer tracks a moving object, speed of rotation of a searchlight, and optimal angles to minimize energy loss in blood flows.

DEFINING THE INVERSE TRIGONOMETRIC FUNCTIONS

Figure 6.27

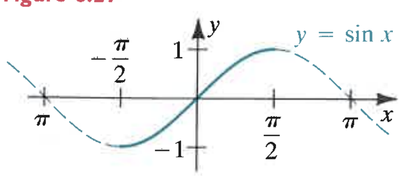
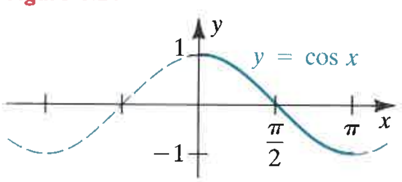


Figure 6.28



Since the trigonometric functions are not one-to-one, they do not have inverse functions (see Section 6.1). By restricting their domains, however, we may obtain one-to-one functions that have the same values as the trigonometric functions and that *do* have inverses over these restricted domains.

We consider first the graph of the sine function, whose domain is $\mathbb{R}$ and whose range is the closed interval $[-1, 1]$ (see Figure 6.27). The sine function is not one-to-one, since a horizontal line such as $y = \frac{1}{2}$ intersects the graph at more than one point. If we restrict the domain to $[-\pi/2, \pi/2]$, then, as the solid portion of the graph in Figure 6.27 illustrates, we obtain an increasing function that takes on every value of the sine function exactly once. This new function, with domain $[-\pi/2, \pi/2]$ and range $[-1, 1]$, is continuous and increasing and hence, by Theorem (6.6), has an inverse function that is continuous and increasing. The inverse function has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

If we restrict the domain of the cosine function to the interval $[0, \pi]$, as shown in the solid portion of the graph in Figure 6.28, we obtain a one-to-one continuous decreasing function that has a continuous decreasing inverse function. The inverse cosine function has domain $[-1, 1]$ and range $[0, \pi]$.

We formalize this discussion in the following definition.

Definition 6.34

> The **inverse sine function**, denoted $\sin^{-1}$, is defined by
>
> $$y = \sin^{-1} x \quad \text{if and only if} \quad x = \sin y$$
>
> for $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$.
>
> The **inverse cosine function**, denoted $\cos^{-1}$, is defined by
>
> $$y = \cos^{-1} x \quad \text{if and only if} \quad x = \cos y$$
>
> for $-1 \le x \le 1$ and $0 \le y \le \pi$.

The inverse sine and inverse cosine functions are also called the **arcsine function** (denoted $\arcsin x$) and **arccosine function** (denoted $\arccos x$), respectively. The -1 in $\sin^{-1}$ and $\cos^{-1}$ is not regarded as an exponent, but rather as a means of denoting an inverse function. We may read the notation $y = \sin^{-1} x$ as *y is the inverse sine of x* and the notation $y = \cos^{-1} x$ as *y is the inverse cosine of x*. The equations $x = \sin y$ and $x = \cos y$ in the definition allow us to regard y as an angle. Thus, we often read the inverse functions as *y is the angle whose sine is x* or *y is the angle whose cosine is x*. Note that

$$-\pi/2 \le \sin^{-1} x \le \pi/2 \quad \text{and} \quad 0 \le \cos^{-1} x \le \pi.$$

ILLUSTRATION

- If $y = \sin^{-1} \frac{1}{2}$, then $\sin y = \frac{1}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Hence, $y = \frac{\pi}{6}$.
- If $y = \arcsin\left(-\frac{1}{2}\right)$, then $\sin y = -\frac{1}{2}$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Hence, $y = -\frac{\pi}{6}$.
- If $y = \cos^{-1} \frac{1}{2}$, then $\cos y = \frac{1}{2}$ and $0 \le y \le \pi$. Hence, $y = \frac{\pi}{3}$.
- If $y = \arccos\left(-\frac{1}{2}\right)$, then $\cos y = -\frac{1}{2}$ and $0 \le y \le \pi$. Hence, $y = \frac{2\pi}{3}$.

Since the graphs of a function f and its inverse f^{-1} are reflections of each other through the line $y = x$, we can sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ by reflecting the solid portions of the graphs in Figures 6.27 and 6.28. The graphs of the inverse sine and inverse cosine functions are shown in Figures 6.29 and 6.30. We can also use the equations $x = \sin y$ with $-\pi/2 \le y \le \pi/2$ and $x = \cos y$ with $0 \le y \le \pi$ to find points on the graphs of the inverse functions.

On a calculator, the inverse sine function may be approximated by using a single [SIN^{-1}] or [ASIN] key, if available, or a two-stroke combination [INV] [SIN]. The inverse cosine function is implemented in an analogous fashion. Be sure to set the calculator to *radian mode*. For example, [SIN^{-1}] 0.85 yields the approximate result 1.01598529 radians, but if the calculator is set in degrees, the result is 58.21°.

We can proceed in a similar manner to find an inverse for the tangent function. If we restrict the domain of the tangent to the open interval $(-\pi/2, \pi/2)$, we obtain a continuous increasing function (see Figure 6.31 on the following page). We use this *new* function to define the *inverse tangent function*.

Figure 6.29

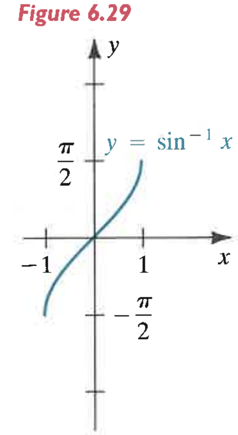
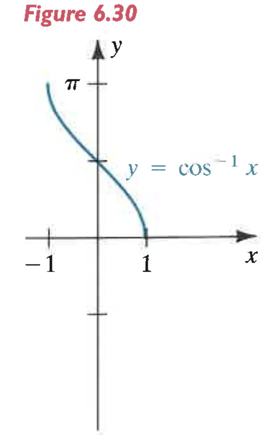


Figure 6.30



Definition 6.35

The **inverse tangent function**, or **arctangent function**, denoted by $\tan^{-1}$, or arctan, is defined by

$$y = \tan^{-1} x = \arctan x \quad \text{if and only if} \quad x = \tan y$$

for every x and $-\pi/2 < y < \pi/2$.

The domain of the arctangent function is $\mathbb{R}$ and the range is the open interval $(-\pi/2, \pi/2)$. We can obtain the graph of $y = \arctan x$ shown in Figure 6.32 by reflecting the graph in Figure 6.31 through the line $y = x$.

Figure 6.31

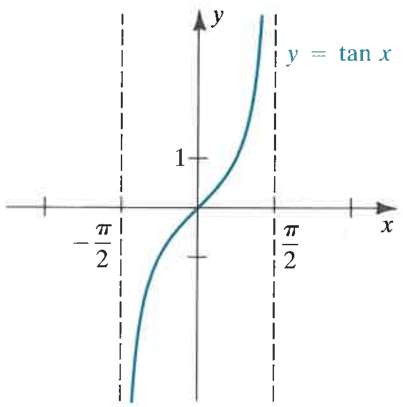
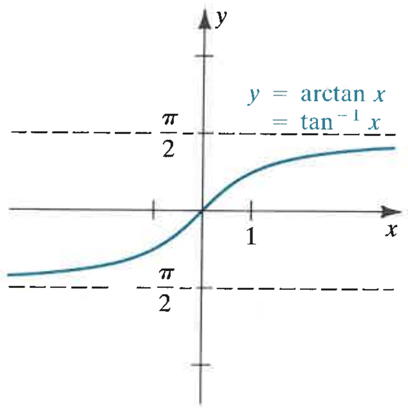


Figure 6.32



ILLUSTRATION

- If $y = \arctan(-1)$, then $\tan y = -1$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$. Hence, $y = -\dfrac{\pi}{4}$.
- If $y = \arctan(\sqrt{3})$, then $\tan y = \sqrt{3}$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$. Hence, $y = \dfrac{\pi}{3}$.

The relationships $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ that hold for any inverse function f^{-1} give us the following properties.

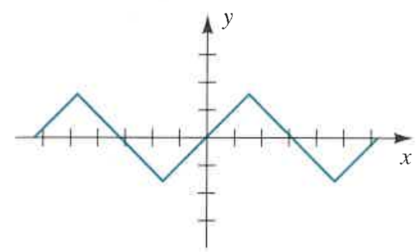
Properties of Inverse Trigonometric Functions 6.36

(i)	$\sin(\sin^{-1} x) = \sin(\arcsin x) = x$	if $-1 \le x \le 1$
(ii)	$\sin^{-1}(\sin x) = \arcsin(\sin x) = x$	if $-\pi/2 \le x \le \pi/2$
(iii)	$\cos(\cos^{-1} x) = \cos(\arccos x) = x$	if $-1 \le x \le 1$
(iv)	$\cos^{-1}(\cos x) = \arccos(\cos x) = x$	if $0 \le x \le \pi$
(v)	$\tan(\tan^{-1} x) = \tan(\arctan x) = x$	for every x
(vi)	$\tan^{-1}(\tan x) = \arctan(\tan x) = x$	if $-\pi/2 < x < \pi/2$

ILLUSTRATION

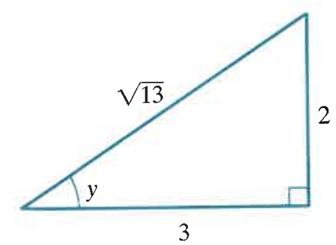
- $\sin(\sin^{-1} \frac{1}{2}) = \frac{1}{2}$ since $-1 \le \frac{1}{2} \le 1$
- $\arcsin\left(\sin \dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ since $-\dfrac{\pi}{2} \le \dfrac{\pi}{4} \le \dfrac{\pi}{2}$
- $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = -\frac{1}{2}$ since $-1 \le -\frac{1}{2} \le 1$
- $\arccos\left(\cos \dfrac{2\pi}{3}\right) = \dfrac{2\pi}{3}$ since $0 \le \dfrac{2\pi}{3} \le \pi$
- $\tan(\tan^{-1} 1000) = 1000$ by (6.36)(v)
- $\tan^{-1}\left(\tan \dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ since $-\dfrac{\pi}{2} < \dfrac{\pi}{4} < \dfrac{\pi}{2}$
- $\cos^{-1}\left[\cos\left(-\dfrac{\pi}{4}\right)\right] = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$
- $\arctan(\tan \pi) = \arctan 0 = 0$
- $\sin^{-1}\left(\sin \dfrac{2\pi}{3}\right) = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

Figure 6.33
$f(x) = \sin^{-1}(\sin x)$



Be careful when using (6.36). In the final part of the preceding illustration, for example, $2\pi/3$ is *not* between $-\pi/2$ and $\pi/2$, and hence we cannot use (6.36)(ii). Instead, we use properties of reference angles (page 45) to first evaluate $\sin(2\pi/3)$ and then find $\sin^{-1}(\sqrt{3}/2)$. As we see in this illustration, in general, $\sin^{-1}(\sin x) \neq x$. The function $f(x) = \sin^{-1}(\sin x)$ is defined and continuous for all real numbers x and has an interesting graph, part of which is shown in Figure 6.33.

Figure 6.34

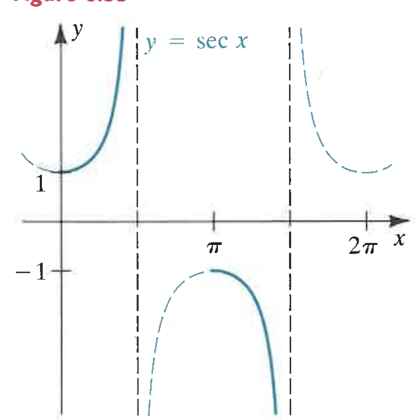


EXAMPLE ■ 1 Find the exact value of $\sec(\arctan \frac{2}{3})$.

SOLUTION If we let $y = \arctan \frac{2}{3}$, then $\tan y = \frac{2}{3}$. We wish to find $\sec y$. Since $-\pi/2 < \arctan x < \pi/2$ for every x and $\tan y > 0$, it follows that $0 < y < \pi/2$. Thus, we may regard y as the radian measure of an angle of a right triangle such that $\tan y = \frac{2}{3}$, as illustrated in Figure 6.34. By the Pythagorean theorem, the hypotenuse is $\sqrt{3^2 + 2^2} = \sqrt{13}$. Referring to the triangle, we obtain

$$\sec\left(\arctan \frac{2}{3}\right) = \sec y = \frac{\sqrt{13}}{3}.$$

Figure 6.35

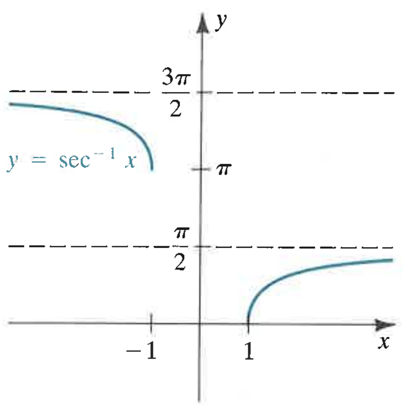


If we consider the graph of $y = \sec x$, there are many ways to restrict x so that we obtain a one-to-one function that takes on every value of the secant function. There is no universal agreement on how this should be done. It is convenient to restrict x to the intervals $[0, \pi/2)$ and $[\pi, 3\pi/2)$, as indicated by the solid portion of the graph of $y = \sec x$ in Figure 6.35, rather than to the "more natural" intervals $[0, \pi/2)$ and $(\pi/2, \pi]$, because the differentiation formula for the inverse secant is simpler. We show in the next section that $(d/dx)(\sec^{-1} x) = 1/(x\sqrt{x^2-1})$. Thus, the slope of the tangent line to the graph of $y = \sec^{-1} x$ is negative if $x < -1$ or positive if $x > 1$. For the more natural intervals, the slope is always positive, and we would have $(d/dx)(\sec^{-1} x) = 1/(|x|\sqrt{x^2-1})$.

Definition 6.37

> The **inverse secant function**, or **arcsecant function**, denoted by $\sec^{-1}$, or arcsec, is defined by
>
> $$y = \sec^{-1} x = \operatorname{arcsec} x \quad \text{if and only if} \quad x = \sec y$$
>
> for $|x| \geq 1$ and y in $[0, \pi/2)$ or in $[\pi, 3\pi/2)$.

Figure 6.36



The graph of $y = \sec^{-1} x$ is sketched in Figure 6.36.

The inverse cotangent function, $\cot^{-1}$, and the inverse cosecant function, $\csc^{-1}$, can be defined in similar fashion (see Exercises 25–26).

Most calculators and many computer applications do not provide for the direct evaluation of the secant function or the inverse secant function. We evaluate $\sec x$ by computing the reciprocal of $\cos x$, but there is no simple way to evaluate the inverse secant function. The next example suggests a procedure.

EXAMPLE ▪ 2 Use a calculator to approximate $\operatorname{arcsec}(-14.3)$.

SOLUTION From the graph of the inverse secant function in Figure 6.36, we see that $\operatorname{arcsec}(-14.3)$ will lie between π and $3\pi/2$. If we let $y = \operatorname{arcsec}(-14.3)$, then $\sec y = -14.3$ and $\cos y = -(1/14.3)$. Since the range of the inverse cosine function is $[0, \pi]$ with $\pi/2 < \arccos x \leq \pi$ when $x < 0$, a calculator provides the approximation

$$\tilde{y} = \cos^{-1}(-1/14.3) \approx 1.64078352.$$

To find the desired answer in the interval $[\pi, 3\pi/2)$, we treat $\tilde{y}$ as a reference angle and compute the answer as $y = \pi + (\pi - \tilde{y}) \approx 4.64240179$.

EXAMPLE ▪ 3 If $-1 \leq x \leq 1$, rewrite $\cos(\sin^{-1} x)$ as an algebraic expression in x.

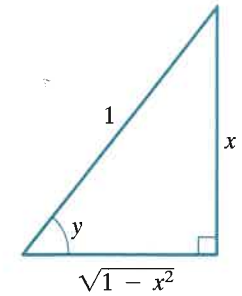
SOLUTION Let

$$y = \sin^{-1} x, \quad \text{or, equivalently,} \quad \sin y = x.$$

We wish to express $\cos y$ in terms of x. Since $-\pi/2 \leq y \leq \pi/2$, it follows that $\cos y \geq 0$, and hence

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Figure 6.37



Consequently, $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

The last identity can also be seen geometrically if $0 < x < 1$. In this case, $0 < y < \pi/2$, and we may regard y as the radian measure of an angle of a right triangle such that $\sin y = x$, as illustrated in Figure 6.37. (The side of length $\sqrt{1-x^2}$ is found by using the Pythagorean theorem.) Referring to the triangle, we have

$$\cos(\sin^{-1} x) = \cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

DIFFERENTIATING AND INTEGRATING INVERSE TRIGONOMETRIC FUNCTIONS

We consider next the derivatives and integrals of the inverse trigonometric functions and integrals that result in inverse trigonometric functions. We concentrate on the inverse sine, cosine, tangent, and secant functions. The next two theorems provide formulas with $u = g(x)$ differentiable and x restricted to values for which the indicated expressions have meaning. You may find it surprising to learn that although we use trigonometric functions to define inverse trigonometric functions, their derivatives are *algebraic* functions.

Theorem 6.38

> **(i)** $$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
>
> **(ii)** $$\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
>
> **(iii)** $$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}$$
>
> **(iv)** $$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

PROOF We shall consider only the special case $u = x$, since the formulas for $u = g(x)$ may then be obtained by applying the chain rule.

If we let $f(x) = \sin x$ and $g(x) = \sin^{-1} x$ in Theorem (6.7), then it follows that the inverse sine function g is differentiable if $|x| < 1$. We shall use implicit differentiation to find $g'(x)$. First note that the equations

$$y = \sin^{-1} x \quad \text{and} \quad \sin y = x$$

are equivalent if $-1 < x < 1$ and $-\pi/2 < y < \pi/2$. Differentiating $\sin y = x$ implicitly, we have

$$\cos y \frac{dy}{dx} = 1$$

and hence $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{dy}{dx} = \dfrac{1}{\cos y}.$

Since $-\pi/2 < y < \pi/2$, $\cos y$ is positive and, therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Thus, $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$

for $|x| < 1$. The inverse sine function is not differentiable at ± 1. This fact is evident from Figure 6.29, since vertical tangent lines occur at the endpoints of the graph.

The formula for $(d/dx)(\cos^{-1} x)$ can be obtained in similar fashion.

It follows from Theorem (6.7) that the inverse tangent function is differentiable at every real number. Let us consider the equivalent equations

$$y = \tan^{-1} x \quad \text{and} \quad \tan y = x$$

for $-\pi/2 < y < \pi/2$. Differentiating $\tan y = x$ implicitly, we have

$$\sec^2 y \frac{dy}{dx} = 1.$$

Consequently,

$$\frac{d}{dx}(\tan^{-1} x) = \frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

Using the fact that $\sec^2 y = 1 + \tan^2 y = 1 + x^2$ gives us

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}.$$

Finally, consider the equivalent equations

$$y = \sec^{-1} x \quad \text{and} \quad \sec y = x$$

for y in either $(0, \pi/2)$ or $(\pi, 3\pi/2)$. Differentiating $\sec y = x$ implicitly yields

$$\sec y \tan y \frac{dy}{dx} = 1.$$

Since $0 < y < \pi/2$ or $\pi < y < 3\pi/2$, it follows that $\sec y \tan y \neq 0$ and, hence,

$$\frac{d}{dx}(\sec^{-1} x) = \frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$

Using the fact that $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{x^2 - 1}$, we obtain

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2 - 1}}$$

for $|x| > 1$. The inverse secant function is not differentiable at $x = \pm 1$. Note that the graph has vertical tangent lines at the points with these x-coordinates (see Figure 6.36).

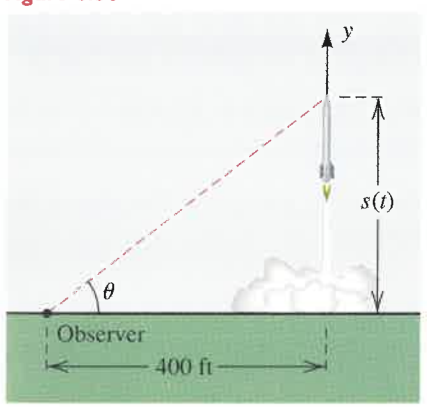
ILLUSTRATION

$f(x)$	$f'(x)$
$\sin^{-1} 3x$	$\dfrac{1}{\sqrt{1 - (3x)^2}} \dfrac{d}{dx}(3x) = \dfrac{3}{\sqrt{1 - 9x^2}}$
$\arccos(\ln x)$	$-\dfrac{1}{\sqrt{1 - (\ln x)^2}} \dfrac{d}{dx}(\ln x) = -\dfrac{1}{x\sqrt{1 - (\ln x)^2}}$
$\tan^{-1} e^{2x}$	$\dfrac{1}{1 + (e^{2x})^2} \dfrac{d}{dx}(e^{2x}) = \dfrac{2e^{2x}}{1 + e^{4x}}$
$\operatorname{arcsec}(x^2)$	$\dfrac{1}{x^2\sqrt{(x^2)^2 - 1}} \dfrac{d}{dx}(x^2) = \dfrac{2}{x\sqrt{x^4 - 1}}$

The next example illustrates an application involving derivatives of the inverse trigonometric functions. Exercises 66 and 70 demonstrate other important applications.

EXAMPLE 4 A rocket is fired directly upward with initial velocity 0 and burns fuel at a rate that produces a constant acceleration of 50 ft/sec^2 for $0 \le t \le 5$, with time t in seconds. As illustrated in Figure 6.38, an observer 400 ft from the launching pad visually follows the flight of the rocket.

Figure 6.38



(a) Express the angle of elevation θ of the rocket as a function of t.

(b) The observer perceives the rocket to be rising fastest when $d\theta/dt$ is largest. (Of course, this is an illusion, since the velocity is steadily increasing.) Determine the height of the rocket at the moment of perceived maximum velocity.

SOLUTION

(a) Let $s(t)$ denote the height of the rocket at time t (see Figure 6.38). The fact that the acceleration is always 50 gives us the differential equation

$$s''(t) = 50,$$

subject to the initial conditions $s'(0) = 0$ and $s(0) = 0$. Integrating with respect to t, we obtain

$$\int s''(t)\,dt = \int 50\,dt$$
$$s'(t) = 50t + C$$

for some constant C. Substituting $t = 0$ and using $s'(0) = 0$ gives us $0 = 50(0) + C$, or $C = 0$. Hence,

$$s'(t) = 50t.$$

Integrating again, we have

$$\int s'(t)\,dt = \int 50t\,dt$$
$$s(t) = 25t^2 + D$$

for some constant D. If we substitute $t = 0$ and use $s(0) = 0$, we obtain $0 = 25(0) + D$, or $D = 0$. Hence,

$$s(t) = 25t^2.$$

Referring to Figure 6.38, with $s(t) = 25t^2$, we find

$$\tan\theta = \frac{25t^2}{400} = \frac{t^2}{16}, \quad \text{or} \quad \theta = \arctan\frac{t^2}{16}.$$

(b) By Theorem (6.38), the rate of change of θ with respect to t is

$$\frac{d\theta}{dt} = \frac{1}{1 + (t^2/16)^2}\left(\frac{2t}{16}\right) = \frac{32t}{256 + t^4}.$$

Since we wish to find the maximum value of $d\theta/dt$, we begin by finding the critical numbers of $d\theta/dt$. Using the quotient rule, we obtain

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \frac{(256 + t^4)(32) - 32t(4t^3)}{(256 + t^4)^2} = \frac{32(256 - 3t^4)}{(256 + t^4)^2}.$$

Considering $d^2\theta/dt^2 = 0$ gives us the critical number $t = \sqrt[4]{256/3}$. It follows from the first (or second) derivative test that $d\theta/dt$ has a maximum value at $t = \sqrt[4]{256/3} \approx 3.04$ sec. The height of the rocket at this time is

$$s(\sqrt[4]{256/3}) = 25(\sqrt[4]{256/3})^2 = 25\sqrt{256/3} \approx 230.9 \text{ ft}.$$

We may use differentiation formulas (i), (ii), and (iv) of Theorem (6.38) to obtain the following integration formulas:

(1) $\displaystyle\int \frac{1}{\sqrt{1 - u^2}}\,du = \sin^{-1} u + C$

(2) $\displaystyle\int \frac{1}{1 + u^2}\,du = \tan^{-1} u + C$

(3) $\displaystyle\int \frac{1}{u\sqrt{u^2 - 1}}\,du = \sec^{-1} u + C$

These formulas can be generalized for $a > 0$ as follows.

Theorem 6.39

(i) $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \sin^{-1}\frac{u}{a} + C$

(ii) $\displaystyle\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

(iii) $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

PROOF Let us prove (ii). As usual, it is sufficient to consider the case $u = x$. We begin by writing

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a^2}\int \frac{1}{1 + (x/a)^2}\,dx.$$

Next we make the substitution $v = x/a$, $dv = (1/a)\,dx$. Introducing the factor $1/a$ in the integrand, compensating by multiplying the integral by a, and using formula (2), preceding this theorem, gives us the following:

$$\begin{aligned}\int \frac{1}{a^2 + x^2}\,dx &= \frac{1}{a}\int \frac{1}{1 + (x/a)^2}\cdot\frac{1}{a}\,dx \\ &= \frac{1}{a}\int \frac{1}{1 + v^2}\,dv \\ &= \frac{1}{a}\tan^{-1} v + C \\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C\end{aligned}$$

The remaining formulas may be proved in similar fashion. ■

In Example 5 of Section 4.7, we obtained numerical approximations for the value of the definite integral $\int_0^1 [4/(1 + x^2)]\,dx$. We can now use our knowledge of the inverse tangent function to show that the exact value of this integral is π.

EXAMPLE ■ 5 Evaluate $\displaystyle\int_0^1 \frac{4}{1 + x^2}\,dx$.

SOLUTION Using (6.39)(ii), we have

$$\begin{aligned}\int_0^1 \frac{4}{1 + x^2}\,dx &= 4\int_0^1 \frac{1}{1 + x^2}\,dx \\ &= 4\big[\arctan x\big]_0^1 \\ &= 4(\arctan 1 - \arctan 0) \\ &= 4\left(\frac{\pi}{4} - 0\right) = \pi.\end{aligned}$$

EXAMPLE ■ 6 Evaluate $\displaystyle\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\,dx$.

SOLUTION The integral may be written as in Theorem (6.39)(i) by letting $a = 1$ and using the substitution

$$u = e^{2x}, \qquad du = 2e^{2x}\,dx.$$

We introduce a factor 2 in the integrand and proceed as follows:

$$\begin{aligned}\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-(e^{2x})^2}}2e^{2x}\,dx\\ &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du\\ &= \tfrac{1}{2}\sin^{-1}u + C\\ &= \tfrac{1}{2}\sin^{-1}e^{2x} + C\end{aligned}$$

EXAMPLE ■ 7 Evaluate $\displaystyle\int \frac{x^2}{5+x^6}\,dx$.

SOLUTION The integral may be written as in Theorem (6.39)(ii) by letting $a^2 = 5$ and using the substitution

$$u = x^3, \qquad du = 3x^2\,dx.$$

We introduce a factor 3 in the integrand and proceed as follows:

$$\begin{aligned}\int \frac{x^2}{5+x^6}\,dx &= \frac{1}{3}\int \frac{1}{5+(x^3)^2}3x^2\,dx\\ &= \frac{1}{3}\int \frac{1}{(\sqrt{5})^2+u^2}\,du\\ &= \frac{1}{3}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + C\\ &= \frac{\sqrt{5}}{15}\tan^{-1}\frac{x^3}{\sqrt{5}} + C\end{aligned}$$

EXAMPLE ■ 8 Evaluate $\displaystyle\int \frac{1}{x\sqrt{x^4-9}}\,dx$.

SOLUTION The integral may be written as in Theorem (6.39)(iii) by letting $a^2 = 9$ and using the substitution

$$u = x^2, \qquad du = 2x\,dx.$$

We introduce $2x$ in the integrand by multiplying the numerator and the denominator by $2x$ and then proceed as follows:

$$\begin{aligned}\int \frac{1}{x\sqrt{x^4-9}}\,dx &= \int \frac{1}{2x\cdot x\sqrt{(x^2)^2-3^2}}2x\,dx\\ &= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-3^2}}\,du\\ &= \frac{1}{2}\cdot\frac{1}{3}\sec^{-1}\frac{u}{3} + C\\ &= \frac{1}{6}\sec^{-1}\frac{x^2}{3} + C\end{aligned}$$

The formulas we developed in previous chapters for such quantities as areas, volumes of solids of revolution, arc length, and surface area may also be applied to transcendental functions. The resulting definite integrals may be evaluated by finding antiderivatives where possible or may be approximated by the numerical methods discussed in Section 4.7. In the next example, we approximate the arc length of a piece of the inverse secant function using Simpson's rule.

EXAMPLE ■ 9 Approximate the arc length of the graph of the function $f(x) = \operatorname{arcsec} x$ from $x = 2$ to $x = 3$ to four decimal places.

SOLUTION From the definition (5.14) of arc length, we know that the arc length of this graph is

$$\int_2^3 \sqrt{1+[f'(x)]^2}\,dx.$$

If $f(x) = \operatorname{arcsec} x$, then by Theorem (6.38)(iv), $f'(x) = 1/(x\sqrt{x^2-1})$, so

$$1+[f'(x)]^2 = 1+\frac{1}{x^2(x^2-1)},$$

and the arc length is

$$\int_2^3 \sqrt{1+\frac{1}{x^2(x^2-1)}}\,dx.$$

We evaluate this definite integral numerically by using Simpson's rule with $n = 10$, 20, and 40. For each of these values of n, we obtain 1.01783 as an approximation. Thus to four decimal places, the arc length of the graph of $y = \operatorname{arcsec} x$ from $x = 2$ to $x = 3$ is 1.0178.

We conclude this section with a brief look at indefinite integrals of the inverse trigonometric integrals.

***Integrals of Inverse Trigonometric Functions* 6.40**

(i) $\displaystyle\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2} + C$

(ii) $\displaystyle\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2} + C$

(iii) $\displaystyle\int \tan^{-1}u\,du = u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2) + C$

(iv) $\displaystyle\int \sec^{-1}u\,du = u\sec^{-1}u - \ln\left|u+\sqrt{u^2-1}\right| + C$

We derive these forms of the indefinite integrals using the approach demonstrated in the next example.

EXAMPLE 10

(a) Find $f'(x)$ if $f(x) = x \arcsin x$.
(b) Use the result of part (a) to find $\int \arcsin x\, dx$.

SOLUTION

(a) By the product rule (2.19),

$$\begin{aligned} f'(x) &= (x \arcsin x)' \\ &= (x)'(\arcsin x) + (x)(\arcsin x)' \\ &= 1 \arcsin x + x\frac{1}{\sqrt{1-x^2}} \\ &= \arcsin x + \frac{x}{\sqrt{1-x^2}}. \end{aligned}$$

Thus, $$(x \arcsin x)' = \arcsin x + \frac{x}{\sqrt{1-x^2}}.$$

(b) From part (a), we have

$$\arcsin x = (x \arcsin x)' - \frac{x}{\sqrt{1-x^2}}.$$

Integrating each side of this equation with respect to x gives

$$\begin{aligned} \int \arcsin x\, dx &= \int \left[(x \arcsin x)' - \frac{x}{\sqrt{1-x^2}}\right] dx \\ &= \int (x \arcsin x)'\, dx - \int \frac{x}{\sqrt{1-x^2}}\, dx \\ &= x \arcsin x + \sqrt{1-x^2} + C. \end{aligned}$$

EXERCISES 6.7

Exer. 1–11: Find the exact value of the expression, whenever it is defined.

1 (a) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ (b) $\cos^{-1}\left(-\frac{1}{2}\right)$
(c) $\tan^{-1}(-\sqrt{3})$

2 (a) $\arcsin \frac{\sqrt{3}}{2}$ (b) $\arccos \frac{\sqrt{2}}{2}$
(c) $\arctan \frac{1}{\sqrt{3}}$

3 (a) $\sin^{-1} \frac{\pi}{3}$ (b) $\cos^{-1} \frac{\pi}{2}$
(c) $\tan^{-1} 1$

4 (a) $\sin[\arcsin(-\frac{3}{10})]$ (b) $\cos(\arccos \frac{1}{2})$
(c) $\tan(\arctan 14)$

5 (a) $\sin^{-1}\left(\sin \frac{\pi}{3}\right)$ (b) $\cos^{-1}\left(\cos \frac{5\pi}{6}\right)$
(c) $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

6 (a) $\arcsin\left(\sin \frac{5\pi}{4}\right)$ (b) $\arccos\left(\cos \frac{5\pi}{4}\right)$
(c) $\arctan\left(\tan \frac{7\pi}{4}\right)$

7 (a) $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$ (b) $\cos^{-1}\left(\cos \frac{4\pi}{3}\right)$
(c) $\tan^{-1}\left(\tan \frac{7\pi}{6}\right)$

8 (a) $\sin[\cos^{-1}(-\frac{1}{2})]$ (b) $\cos(\tan^{-1} 1)$
(c) $\tan[\sin^{-1}(-1)]$

9 (a) $\sin(\tan^{-1} \sqrt{3})$ (b) $\cos(\sin^{-1} 1)$
(c) $\tan(\cos^{-1} 0)$

10 (a) $\cot(\sin^{-1} \frac{2}{3})$ (b) $\sec[\tan^{-1}(-\frac{3}{5})]$
(c) $\csc[\cos^{-1}(-\frac{1}{4})]$

11 (a) $\cot[\sin^{-1}(-\frac{2}{5})]$ (b) $\sec(\tan^{-1} \frac{7}{4})$
(c) $\csc(\cos^{-1} \frac{1}{5})$

12 If $-1 \le x \le 1$, is it always possible to find the value of $\sin^{-1}(\sin^{-1} x)$ by pressing the calculator key sequence [INV] [SIN] twice? If not, determine the permissible values of x.

c **Exer. 13–16: Find a four-decimal-place approximation of the expression, whenever it is defined.**

13 (a) $\sin^{-1}(-0.931)$ (b) $\tan^{-1}(0.278)$

14 (a) $\cos^{-1}(-0.265)$ (b) $\sec^{-1}(15.4)$

15 (a) $\sec[\sin^{-1}(-0.582) + \tan^{-1}(0.304)]$
(b) $\cos[\sin^{-1}(0.179) + \tan^{-1}(-1.89)]$

16 (a) $\tan[\sin^{-1}(0.783) + \sec^{-1}(8.54)]$
(b) $\sin[\cos^{-1}(0.496) + \tan^{-1}(6.12)]$

Exer. 17–20: Rewrite as an algebraic expression in x for $x > 0$.

17 $\sin(\tan^{-1} x)$ **18** $\tan(\arccos x)$

19 $\sec\left(\sin^{-1} \frac{x}{3}\right)$ **20** $\cot\left(\sin^{-1} \frac{1}{x}\right)$

Exer. 21–26: Sketch the graph of the equation.

21 $y = \sin^{-1} 2x$ **22** $y = \frac{1}{2} \sin^{-1} x$

23 $y = \cos^{-1} \frac{1}{2}x$ **24** $y = 2\cos^{-1} x$

25 $y = \cos(2 \arccos x)$ **26** $y = \cos(3 \cos^{-1} x)$

27 (a) Define $\cot^{-1}$ by restricting the domain of the cotangent function to the interval $(0, \pi)$.
(b) Sketch the graph of $y = \cot^{-1} x$.
(c) Show that

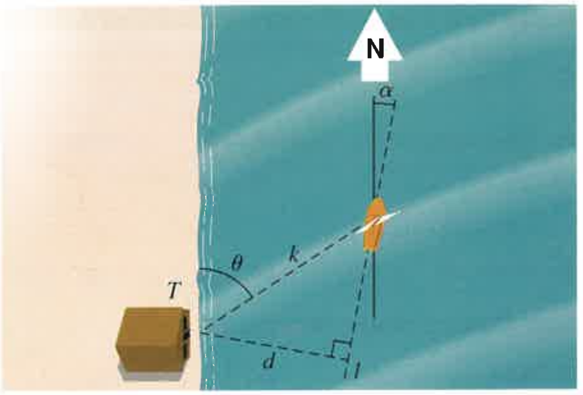
$$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2}\frac{du}{dx} = -\frac{d}{dx}(\tan^{-1} u).$$

28 (a) Define $\csc^{-1}$ by restricting the domain of the cosecant function to $[-\pi/2, 0) \cup (0, \pi/2]$.
(b) Sketch the graph of $y = \csc^{-1} x$.
(c) Show that

$$\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}.$$

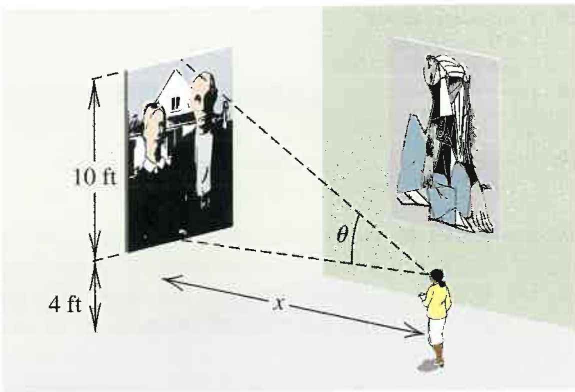
29 As shown in the figure, a sailboat is following a straight-line course l. The shortest distance from a tracking station T to the course is d miles. As the boat sails, the tracking station records its distance k from T and its direction θ with respect to T. Angle α specifies the direction of the sailboat.
(a) Express α in terms of d, k, and θ.
(b) Estimate α to the nearest degree if $d = 50$ mi, $k = 210$ mi, and $\theta = 53.4°$.

Exercise 29



30 An art critic whose eye level is 6 ft above the floor views a painting that is 10 ft in height and is mounted 4 ft above the floor, as shown in the figure.
(a) If the critic is standing x feet from the wall, express the viewing angle θ in terms of x.

Exercise 30



(b) Use the addition formula for the tangent to show that

$$\theta = \tan^{-1}\left(\frac{10x}{x^2 - 16}\right).$$

(c) For what value of x is $\theta = 45°$?

Exer. 31–48: Find $f'(x)$ if $f(x)$ is the given expression.

31 $\sin^{-1}\sqrt{x}$ **32** $\sin^{-1}\frac{1}{3}x$

33 $\tan^{-1}(3x - 5)$ **34** $\tan^{-1}(x^2)$

35 $e^{-x}\operatorname{arcsec} e^{-x}$ **36** $\sqrt{\operatorname{arcsec} 3x}$

37 $\ln \arctan(x^2)$ **38** $\arcsin \ln x$

39 $(1 + \cos^{-1} 3x)^3$ **40** $\cos^{-1}\cos e^x$

41 $\cos(x^{-1}) + (\cos x)^{-1} + \cos^{-1} x$

42 $x \arccos\sqrt{4x + 1}$

43 $3^{\arcsin(x^3)}$

44 $\left(\frac{1}{x} - \arcsin\frac{1}{x}\right)^4$

45 $\dfrac{\arctan x}{x^2 + 1}$

46 $(\sin 2x)(\sin^{-1} 2x)$

47 $\sqrt{x}\sec^{-1}\sqrt{x}$ **48** $(\tan x)^{\arctan x}$

Exer. 49–50: Find y'.

49 $x^2 + x\sin^{-1} y = ye^x$ **50** $\ln(x + y) = \tan^{-1} xy$

Exer. 51–62: Evaluate the integral.

51 **(a)** $\displaystyle\int \frac{1}{x^2 + 16}\,dx$ **(b)** $\displaystyle\int_0^4 \frac{1}{x^2 + 16}\,dx$

52 **(a)** $\displaystyle\int \frac{e^x}{1 + e^{2x}}\,dx$ **(b)** $\displaystyle\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$

53 **(a)** $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\,dx$ **(b)** $\displaystyle\int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1 - x^4}}\,dx$

54 $\displaystyle\int \frac{\sin x}{\cos^2 x + 1}\,dx$ **55** $\displaystyle\int \frac{1}{\sqrt{x}(1 + x)}\,dx$

56 $\displaystyle\int \frac{\cos x}{\sqrt{9 - \sin^2 x}}\,dx$ **57** $\displaystyle\int \frac{e^x}{\sqrt{16 - e^{2x}}}\,dx$

58 $\displaystyle\int \frac{\sec x \tan x}{1 + \sec^2 x}\,dx$ **59** $\displaystyle\int \frac{x}{x^2 + 9}\,dx$

60 $\displaystyle\int \frac{1}{x\sqrt{x^6 - 4}}\,dx$ **61** $\displaystyle\int \frac{1}{\sqrt{e^{2x} - 25}}\,dx$

62 $\displaystyle\int \frac{1}{x\sqrt{x - 1}}\,dx$

63 The floor of a storage shed has the shape of a right triangle. The sides opposite and adjacent to an acute angle θ of the triangle are measured as 10 ft and 7 ft, respectively, with a possible error of ± 0.5 in. in the 10-ft measurement. Use the differential of an inverse trigonometric function to approximate the error in the calculated value of θ.

64 Use differentials to approximate the arc length of the graph of $y = \tan^{-1} x$ from $A(0, 0)$ to $B(0.1, \tan^{-1} 0.1)$.

65 An airplane at a constant altitude of 5 mi and a speed of 500 mi/hr is flying in a direction away from an observer on the ground. Use inverse trigonometric functions to find the rate at which the angle of elevation is changing when the airplane flies over a point 2 mi from the observer.

66 A searchlight located $\frac{1}{8}$ mi from the nearest point P on a straight road is trained on an automobile traveling on the road at a rate of 50 mi/hr. Use inverse trigonometric functions to find the rate at which the searchlight is rotating when the car is $\frac{1}{4}$ mi from P.

67 A billboard 20 ft high is located on top of a building, with its lower edge 60 ft above the level of a viewer's eye. Use inverse trigonometric functions to find how far from a point directly below the sign a viewer should stand to maximize the angle between the lines of sight of the top and bottom of the billboard (see Example 9 of Section 3.6).

68 The velocity, at time t, of a point moving on a coordinate line is $(1 + t^2)^{-1}$ ft/sec. If the point is at the origin at $t = 0$, find its position at the instant that the acceleration and the velocity have the same absolute value.

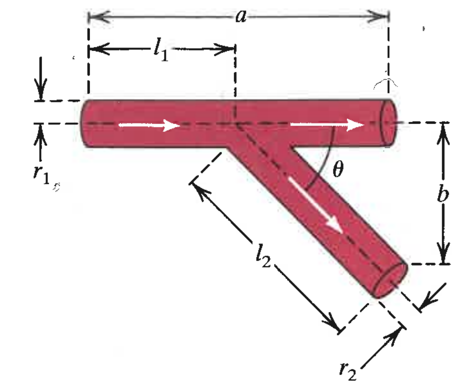
69 A missile is fired vertically from a point that is 5 mi from a tracking station and at the same elevation. For the first 20 sec of flight, its angle of elevation changes at a constant rate of 2° per second. Use inverse trigonometric functions to find the velocity of the missile when the angle of elevation is 30°.

70 Blood flowing through a blood vessel causes a loss of energy due to friction. According to *Poiseuille's law*, this energy loss E is given by $E = kl/r^4$, where r is the radius of the blood vessel, l is the length, and k is a constant. Suppose that a blood vessel of radius r_2 and length l_2 branches off, at an angle θ, from a blood vessel of radius r_1 and length l_1, as illustrated in the figure on the following page, where the white arrows indicate the direction of blood flow. The energy loss is then the sum of the individual energy losses—that is,

$$E = \frac{kl_1}{r_1^4} + \frac{kl_2}{r_2^4}.$$

Express l_1 and l_2 in terms of a, b, and θ, and find the angle that minimizes the energy loss.

Exercise 70



Exer. 71–74: (a) Verify the correctness by differentiation. (b) Derive the formulas using the approach illustrated in Example 10.

71 Formula (i) of (6.40) **72** Formula (ii) of (6.40)

73 Formula (iii) of (6.40)

74 Formula (iv) of (6.40) (Hint for part (b): Verify first that if $g(x) = \ln|x + \sqrt{x^2 - 1}|$, then $g'(x) = 1/\sqrt{x^2 - 1}$.)

Exer. 75–78: Evaluate the integral.

75 $\displaystyle\int \sin^{-1} 2x\,dx$ **76** $\displaystyle\int \cos^{-1}\tfrac{1}{3}x\,dx$

77 $\displaystyle\int x\tan^{-1}(x^2)\,dx$ **78** $\displaystyle\int \frac{\sec^{-1}\sqrt{x}}{\sqrt{x}}\,dx$

c **Exer. 79–82: Approximate the arc length of the graph of the function between A and B. Use Simpson's rule or numerical integration provided on a calculator or a computer to ensure at least four correct decimal places.**

79 $y = \arcsin x$; $A(0, 0)$, $B\left(\frac{1}{2}, \frac{\pi}{6}\right)$

80 $y = \arccos x$; $A\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$, $B\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

81 $y = \arctan x$; $A(0, 0)$, $B\left(\sqrt{3}, \frac{\pi}{3}\right)$

82 $y = \arctan x$; $A(-5, -\arctan 5)$, $B(5, \arctan 5)$

c **Exer. 83–84: Approximate the surface area generated if the graph of the function between A and B is revolved about the x-axis. Use Simpson's rule or numerical integration provided on a calculator or a computer to ensure at least four correct decimal places.**

83 $y = 4\arctan(x^2)$; $A(0, 0)$, $B(1, \pi)$

84 $y = \operatorname{arcsec} x$; $A(2, \operatorname{arcsec} 2)$, $B(10, \operatorname{arcsec} 10)$

6.8 HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

The hyperbolic functions and their inverses, which we investigate in this section, are used to solve a variety of problems in the physical sciences and engineering.

HYPERBOLIC FUNCTIONS

Many of the advanced applications of calculus involve the exponential expressions

$$\frac{e^x - e^{-x}}{2} \quad \text{and} \quad \frac{e^x + e^{-x}}{2},$$

which define the hyperbolic functions. The properties of these expressions are similar in many ways to those of $\sin x$ and $\cos x$. Later in our discussion, we shall see why they are called the *hyperbolic sine* and the *hyperbolic cosine* of x.

Definition 6.41

The **hyperbolic sine function**, denoted by **sinh**, and the **hyperbolic cosine function**, denoted by **cosh**, are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

for every real number x.

We pronounce $\sinh x$ and $\cosh x$ as *sinch x* and *kosh x*, respectively.

The graph of $y = \cosh x$ may be found by **addition of y-coordinates**. Noting that $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$, we first sketch the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same coordinate plane, as shown with dashes in Figure 6.39. We then add the y-coordinates of points on these graphs to obtain the graph of $y = \cosh x$. Note that the range of cosh is $[1, \infty)$.

We may find the graph of $y = \sinh x$ by adding y-coordinates of the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$, as shown in Figure 6.40.

Figure 6.39

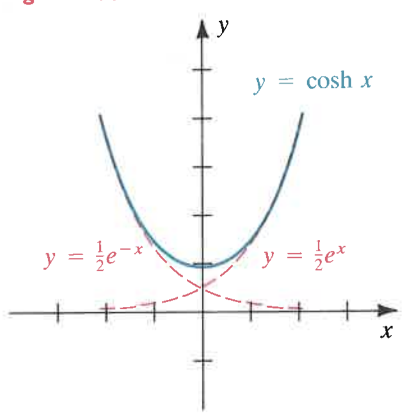
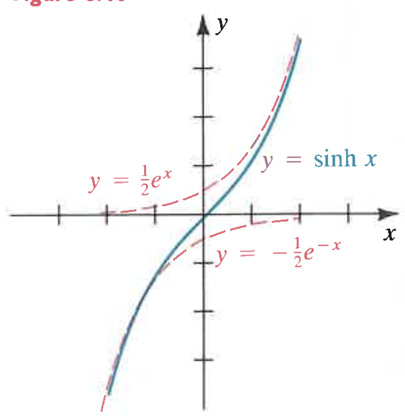


Figure 6.40



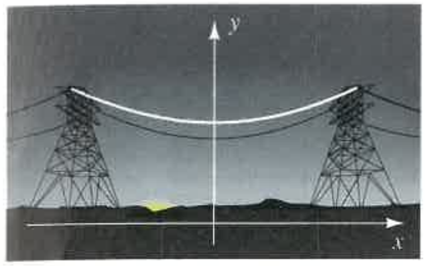
Some scientific calculators have keys that can be used to find values of sinh and cosh directly. We can also substitute numbers for x in Definition (6.41), as in the following illustration.

ILLUSTRATION

- $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.0179$
- $\cosh 0.5 = \dfrac{e^{0.5} + e^{-0.5}}{2} \approx 1.1276$

The hyperbolic cosine function can be used to describe the shape of a uniform flexible cable, or chain, whose ends are supported from the same height. As illustrated in Figure 6.41, telephone or power lines may be strung between poles in this manner. The shape of the cable appears to be a parabola, but is actually a **catenary** (after the Latin word for *chain*). If we introduce a coordinate system, as in Figure 6.41, we will later show that an equation corresponding to the shape of the cable is $y = a\cosh(x/a)$ for some real number a.

Figure 6.41



The hyperbolic cosine function also occurs in the analysis of motion in a resisting medium. If an object is dropped from a given height and if air resistance is disregarded, then the distance y that it falls in t seconds is $y = \frac{1}{2}gt^2$, where g is a gravitational constant. However, air resistance cannot always be disregarded. As the velocity of the object increases, air resistance may significantly affect its motion. For example, if the air resistance is directly proportional to the square of the velocity, then the distance y that the object falls in t seconds is given by

$$y = A\ln(\cosh Bt)$$

for constants A and B (see Exercise 42). Another application is given in Example 2 of this section.

Many identities similar to those for trigonometric functions hold for the hyperbolic sine and cosine functions. For example, if $\cosh^2 x$ and $\sinh^2 x$ denote $(\cosh x)^2$ and $(\sinh x)^2$, respectively, we have the following identity.

Theorem 6.42

$$\cosh^2 x - \sinh^2 x = 1$$

PROOF By Definition (6.41),

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\ &= \tfrac{4}{4} = 1.\end{aligned}$$

Theorem (6.42) is analogous to the identity $\cos^2 x + \sin^2 x = 1$. Other hyperbolic identities are stated in the exercises. To verify an identity, it is sufficient to express the hyperbolic functions in terms of exponential functions and show that one side of the equation can be transformed into the other, as illustrated in the proof of Theorem (6.42). The hyperbolic identities are similar to (but not always the same as) certain trigonometric identities—differences usually involve signs of terms.

If t is a real number, there is an interesting geometric relationship between the points $P(\cos t, \sin t)$ and $Q(\cosh t, \sinh t)$ in a coordinate plane. Let us consider the graphs of $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$, sketched in

Figure 6.42 $x^2 + y^2 = 1$

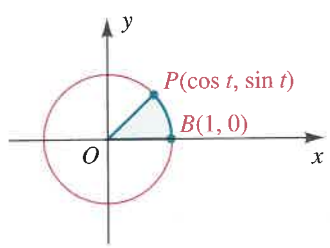
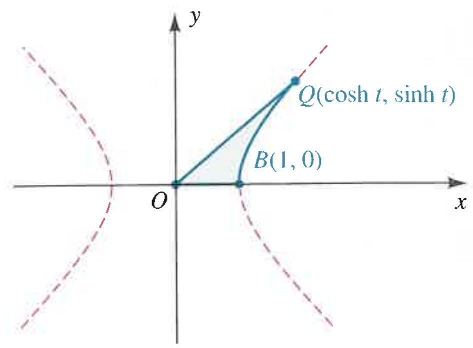


Figure 6.43 $x^2 - y^2 = 1$



Figures 6.42 and 6.43. The graph in Figure 6.42 is the unit circle with center at the origin. The graph in Figure 6.43 is a *hyperbola*. (Hyperbolas and their properties are discussed in the Precalculus Review Chapter.) Note first that since $\cos^2 t + \sin^2 t = 1$, the point $P(\cos t, \sin t)$ is on the circle $x^2 + y^2 = 1$. Next, by Theorem (6.42), $\cosh^2 t - \sinh^2 t = 1$, and hence the point $Q(\cosh t, \sinh t)$ is on the hyperbola $x^2 - y^2 = 1$. These are the reasons for referring to cos and sin as *circular* functions and to cosh and sinh as *hyperbolic* functions.

The graphs in Figures 6.42 and 6.43 are related in another way. If $0 < t < \pi/2$, then t is the radian measure of angle POB, shown in Figure 6.42. The area A of the shaded circular sector is $A = \frac{1}{2}(1)^2 t = \frac{1}{2}t$, and hence $t = 2A$. Similarly, if $Q(\cosh t, \sinh t)$ is the point in Figure 6.43, then $t = 2A$ for the area A of the shaded hyperbolic sector (see Exercise 33).

The impressive analogies between the trigonometric and the hyperbolic sine and cosine functions motivate us to define hyperbolic functions that correspond to the four remaining trigonometric functions. The **hyperbolic tangent, hyperbolic cotangent, hyperbolic secant,** and **hyperbolic cosecant functions,** denoted by tanh, coth, sech, and csch, respectively, are defined as follows.

Definition 6.43

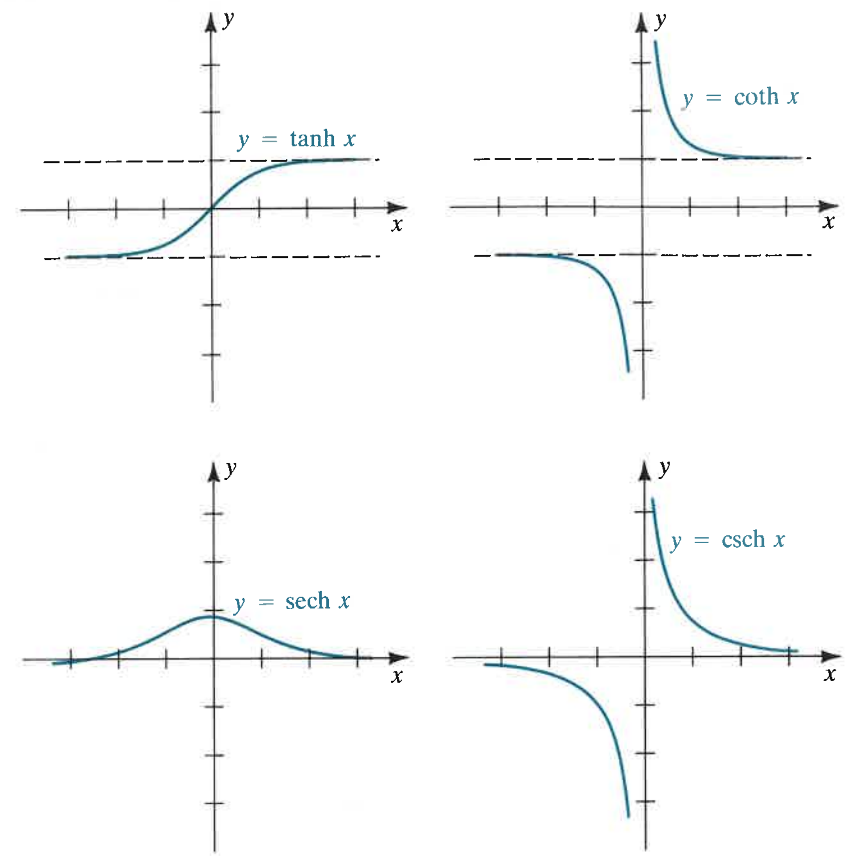
(i) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(ii) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$

(iii) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(iv) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}, \quad x \neq 0$

Figure 6.44



We pronounce the four function values in the preceding definition as *tansh x, cotansh x, setch x,* and *cosetch x*. Their graphs are sketched in Figure 6.44.

If we divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ (see (6.42)) by $\cosh^2 x$, we obtain

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

Using the definitions of $\tanh x$ and $\operatorname{sech} x$ gives us (i) of the next theorem. Formula (ii) may be obtained by dividing both sides of (6.42) by $\sinh^2 x$.

Theorem 6.44

(i) $1 - \tanh^2 x = \operatorname{sech}^2 x$ **(ii)** $\coth^2 x - 1 = \operatorname{csch}^2 x$

Note the similarities and differences between (6.44) and the analogous trigonometric identities.

Derivative formulas for the hyperbolic functions are listed in the next theorem, where $u = g(x)$ and g is differentiable.

Theorem 6.45

(i) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(ii) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(iv) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

(v) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

(vi) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

PROOF As usual, we consider only the case $u = x$. Since $(d/dx)(e^x) = e^x$ and $(d/dx)(e^{-x}) = -e^{-x}$,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

To differentiate $\tanh x$, we apply the quotient rule as follows:

$$\begin{aligned}\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\ &= \frac{\cosh x(d/dx)(\sinh x) - \sinh x(d/dx)(\cosh x)}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} \\ &= \operatorname{sech}^2 x\end{aligned}$$

The remaining formulas can be proved in similar fashion. ■

EXAMPLE 1 If $f(x) = \cosh(x^2 + 1)$, find $f'(x)$.

SOLUTION Applying Theorem (6.45)(i), with $u = x^2 + 1$, we obtain

$$\begin{aligned}f'(x) &= \sinh(x^2 + 1) \cdot \frac{d}{dx}(x^2 + 1) \\ &= 2x \sinh(x^2 + 1).\end{aligned}$$

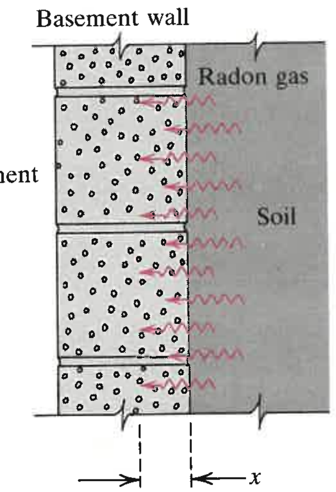
EXAMPLE 2 Radon gas can readily diffuse through solid materials such as brick and cement. If the direction of diffusion in a basement wall is perpendicular to the surface, as illustrated in Figure 6.45, then the radon concentration $f(x)$ (in joules/cm^3) in the air-filled pores within the wall at a distance x from the outside surface can be approximated by

$$f(x) = A\sinh(qx) + B\cosh(qx) + k,$$

where the constant q depends on the porosity of the wall, the half-life of radon, and a diffusion coefficient; the constant k is the maximum radon concentration in the air-filled pores; and A and B are constants that depend on initial conditions. Show that $y = f(x)$ is a solution of the *diffusion equation*

$$\frac{d^2y}{dx^2} - q^2 y + q^2 k = 0.$$

Figure 6.45



SOLUTION Differentiating $y = f(x)$ twice gives us

$$\frac{dy}{dx} = qA\cosh(qx) + qB\sinh(qx)$$

and

$$\frac{d^2y}{dx^2} = q^2 A\sinh(qx) + q^2 B\cosh(qx).$$

Since $y = A\sinh(qx) + B\cosh(qx) + k$, we have

$$q^2 y = q^2 A\sinh(qx) + q^2 B\cosh(qx) + q^2 k.$$

Subtracting the expressions for d^2y/dx^2 and $q^2 y$ yields

$$\frac{d^2y}{dx^2} - q^2 y = -q^2 k$$

and hence

$$\frac{d^2y}{dx^2} - q^2 y + q^2 k = 0.$$

The integration formulas that correspond to the derivative formulas in Theorem (6.45) are as follows.

Theorem 6.46

(i) $\int \sinh u\,du = \cosh u + C$

(ii) $\int \cosh u\,du = \sinh u + C$

(iii) $\int \operatorname{sech}^2 u\,du = \tanh u + C$

(iv) $\int \operatorname{csch}^2 u\,du = -\coth u + C$

(v) $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

(vi) $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

EXAMPLE 3 Evaluate $\int x^2 \sinh x^3\,dx$.

SOLUTION If we let $u = x^3$, then $du = 3x^2\,dx$ and

$$\int x^2 \sinh x^3\,dx = \tfrac{1}{3}\int (\sinh x^3)3x^2\,dx$$
$$= \tfrac{1}{3}\int \sinh u\,du = \tfrac{1}{3}\cosh u + C = \tfrac{1}{3}\cosh x^3 + C.$$

INVERSE HYPERBOLIC FUNCTIONS

We now investigate the inverses of the hyperbolic functions, which frequently occur in evaluating certain types of integrals. We will also see how an inverse hyperbolic function is used in the derivation of the equation for a hanging cable.

The hyperbolic sine function is continuous and increasing for every x and hence, by Theorem (6.6), has a continuous, increasing inverse function, denoted by $\sinh^{-1}$. Since $\sinh x$ is defined in terms of e^x, we might expect that $\sinh^{-1}$ can be expressed in terms of the inverse, ln, of the natural exponential function. The first formula of the next theorem shows that this is the case.

Theorem 6.47

(i) $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

(ii) $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), \quad x \ge 1$

(iii) $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}, \quad |x| < 1$

(iv) $\operatorname{sech}^{-1} x = \ln\dfrac{1+\sqrt{1-x^2}}{x}, \quad 0 < x \le 1$

PROOF To prove (i), we begin by noting that

$$y = \sinh^{-1} x \quad \text{if and only if} \quad x = \sinh y.$$

The equation $x = \sinh y$ can be used to find an explicit form for $\sinh^{-1} x$. Thus, if

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

then $$e^y - 2x - e^{-y} = 0.$$

Multiplying both sides by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

Applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}, \quad \text{or} \quad e^y = x \pm \sqrt{x^2+1}.$$

Since $x - \sqrt{x^2+1} < 0$ and e^y is never negative, we must have

$$e^y = x + \sqrt{x^2+1}.$$

The equivalent logarithmic form is

$$y = \ln(x + \sqrt{x^2+1});$$

that is, $$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

Formulas (ii)–(iv) are obtained in similar fashion. As with trigonometric functions, some inverse functions exist only if the domain is restricted. For example, if the domain of cosh is restricted to the set of nonnegative real numbers, then the resulting function is continuous and increasing, and its inverse function $\cosh^{-1}$ is defined by

$$y = \cosh^{-1} x \quad \text{if and only if} \quad \cosh y = x, \quad y \ge 0.$$

Employing the process used for $\sinh^{-1} x$ leads us to (ii). Similarly,

$$y = \tanh^{-1} x \quad \text{if and only if} \quad \tanh y = x \quad \text{for} \quad |x| < 1.$$

Using Definition (6.43), we may write $\tanh y = x$ as

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x.$$

Solving for y gives us (iii).

Finally, if we restrict the domain of sech to nonnegative numbers, the result is a one-to-one function, and we define

$$y = \operatorname{sech}^{-1} x \quad \text{if and only if} \quad \operatorname{sech} y = x, \quad y \ge 0.$$

Again, introducing the exponential form leads to (iv). ■

In the next theorem, $u = g(x)$, where g is differentiable and x is suitably restricted.

Theorem 6.48

(i) $$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

(ii) $$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$$

(iii) $$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$$

(iv) $$\frac{d}{dx}(\operatorname{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

PROOF By Theorem (6.47)(i),

$$\begin{aligned}\frac{d}{dx}(\sinh^{-1} x) &= \frac{d}{dx}(\ln(x+\sqrt{x^2+1}))\\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})\sqrt{x^2+1}}\\ &= \frac{1}{\sqrt{x^2+1}}.\end{aligned}$$

This formula can be extended to $(d/dx)(\sinh^{-1} u)$ by applying the chain rule. The remaining formulas can be proved in similar fashion.

EXAMPLE 4 If $y = \sinh^{-1}(\tan x)$, find dy/dx.

SOLUTION Using Theorem (6.48)(i) with $u = \tan x$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{\tan^2 x+1}}\frac{d}{dx}\tan x = \frac{1}{\sqrt{\sec^2 x}}\sec^2 x\\ &= \frac{1}{|\sec x|}|\sec x|^2 = |\sec x|.\end{aligned}$$

The following theorem may be verified by differentiating the right-hand side of each formula.

Theorem 6.49

(i) $$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \sinh^{-1}\frac{u}{a} + C, \quad a > 0$$

(ii) $$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \cosh^{-1}\frac{u}{a} + C, \quad 0 < a < u$$

(iii) $$\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C, \quad |u| < a$$

(iv) $$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|u|}{a} + C, \quad 0 < |u| < a$$

If we use Theorem (6.47), then each of the integration formulas in the preceding theorem can be expressed in terms of the natural logarithm function. To illustrate,

$$\begin{aligned}\int \frac{1}{\sqrt{a^2+u^2}}\,du &= \sinh^{-1}\frac{u}{a} + C\\ &= \ln\left(\frac{u}{a}+\sqrt{\left(\frac{u}{a}\right)^2+1}\right) + C.\end{aligned}$$

We can show that if $a > 0$, then the last formula can be written as

$$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln(u+\sqrt{a^2+u^2}) + D,$$

where D is a constant. In Section 7.3, we shall discuss another method for evaluating the integrals in Theorem (6.49).

EXAMPLE 5 Evaluate $\displaystyle\int \frac{1}{\sqrt{25+9x^2}}\,dx$.

SOLUTION We may express the integral as in Theorem (6.49)(i), by using the substitution

$$u = 3x, \qquad du = 3\,dx.$$

Since du contains the factor 3, we adjust the integrand by multiplying by 3 and then compensate by multiplying the integral by $\frac{1}{3}$ before substituting:

$$\begin{aligned}\int \frac{1}{\sqrt{25+9x^2}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+(3x)^2}}3\,dx\\ &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+u^2}}\,du\\ &= \frac{1}{3}\sinh^{-1}\frac{u}{5} + C\\ &= \frac{1}{3}\sinh^{-1}\frac{3x}{5} + C\end{aligned}$$

EXAMPLE 6 Evaluate $\int \frac{e^x}{16-e^{2x}}\,dx$.

SOLUTION Substituting $u = e^x$, $du = e^x\,dx$ and applying Theorem (6.49)(iii) with $a = 4$, we have

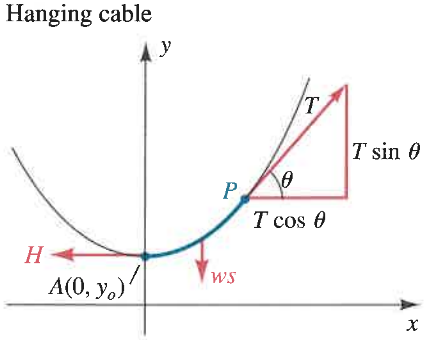
$$\begin{aligned}\int \frac{e^x}{16-e^{2x}}\,dx &= \int \frac{1}{4^2-(e^x)^2}e^x\,dx\\ &= \int \frac{1}{4^2-u^2}\,du\\ &= \frac{1}{4}\tanh^{-1}\frac{u}{4}+C\\ &= \frac{1}{4}\tanh^{-1}\frac{e^x}{4}+C\end{aligned}$$

for $|u| < a$ (that is, $e^x < 4$).

We now consider how the hyperbolic cosine and the inverse hyperbolic sine functions are used in describing the shape of the curve along which a hanging cable lies. We first derive a differential equation for the function whose graph is the curve, and then we solve the differential equation.

Figure 6.41 shows a hanging cable in the form of a power line strung between two towers. A section of the cable is shown in Figure 6.46, where we have set up a coordinate system with the vertical y-axis running through the lowest point $A(0, y_0)$ of the cable. Consider a section of the cable running upward from A to a point P. Figure 6.46 also shows the forces acting on the cable: There is a horizontal tension H at the point A, a tangential tension T at the point P, and a downward gravitational force ws.

Figure 6.46
Hanging cable



The tangential tension can be resolved in a horizontal component $T\cos\theta$ and a vertical component $T\sin\theta$, where θ is the angle that the tangent line to the cable at P makes with the horizontal. (This angle is also the angle of inclination of the tangent line.) Thus, the derivative dy/dx at P is equal to $\tan\theta$. The force due to gravity is equal to the weight of the section of the cable, expressed as ws, where w is the weight per unit length and s is the length of the section.

Since the cable is not moving, the forces acting on any section of it must cancel out. Since the cable is not moving to the right or the left, the magnitude of the horizontal tension at point A equals the magnitude of the horizontal tension at point P:

$$T\cos\theta = H$$

But the cable is also stationary in the vertical direction, so the magnitude of the gravitational force equals the vertical tension at P:

$$T\sin\theta = ws$$

We can now write

$$\frac{ws}{H} = \frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{dy}{dx},$$

or simply, $$\frac{dy}{dx} = \frac{ws}{H}.$$

If we differentiate this equation with respect to x and use Theorem (5.17), we obtain

$$\frac{d^2y}{dx^2} = \frac{w}{H}\frac{ds}{dx} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Letting $a = w/H$ gives

$$\frac{d^2y}{dx^2} = a\sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

for some constant a, as the differential equation satisfied by the equation $y = f(x)$ of the curve formed by a hanging cable. We can now solve the differential equation to find an explicit expression for the function f.

EXAMPLE 7 Solve the differential equation

$$\frac{d^2y}{dx^2} = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

to find an explicit formula for the curve of a hanging cable.

SOLUTION The differential equation

$$\frac{d^2y}{dx^2} = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

is a second-order differential equation, because it involves the second derivative of y with respect to x. We first reduce it to a first-order differential equation by the substitution

$$z = \frac{dy}{dx}$$

so that $$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{dz}{dx}.$$

This result converts the original differential equation to a first-order equation

$$\frac{dz}{dx} = a\sqrt{1+z^2}$$

in which the variables separate. Dividing each side by $\sqrt{1+z^2}$ and integrating, we obtain

$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \int a\,dx.$$

The integrand on the left-hand side of the equation, $1/\sqrt{1+z^2}$, is the derivative of the inverse hyperbolic sine of z. By Theorem (6.49)(i), we have

$$\sinh^{-1} z = ax + C \quad\text{and hence}\quad z = \sinh(ax+C).$$

Since $z = dy/dx$, the last equation becomes

$$\frac{dy}{dx} = \sinh(ax + C).$$

Because $(0, y_0)$ is the minimum point on the curve, the tangent line to the curve at $(0, y_0)$ is horizontal. Thus, at $x = 0$, $dy/dx = 0$, and

$$0 = \sinh(a \cdot 0 + C) = \sinh C.$$

Therefore, $\quad C = 0 \quad$ and $\quad \dfrac{dy}{dx} = \sinh ax.$

Thus, $\quad y = \displaystyle\int \sinh ax\, dx = \frac{1}{a}(\cosh ax) + C.$

To find the constant of integration, we first use the fact that $y = y_0$ at $x = 0$:

$$y_0 = \frac{1}{a}(\cosh 0) + C = \frac{1}{a}(1) + C = \frac{1}{a} + C$$

Hence, if we choose our coordinate system so that $y_0 = 1/a$, we have $C = 0$ and the equation for the hanging cable is

$$y = \frac{1}{a}(\cosh ax).$$

Note that if the coordinate system has already been established in such a way that $y_0 \neq 1/a$, then the equation for the catenary has the more general form

$$y = \left(y_0 - \frac{1}{a}\right) + \frac{1}{a}\cosh ax.$$

Another commonly used form for the equation of the catenary is

$$y = b + a\cosh\left(\frac{x}{a}\right),$$

where a and b are constants and the lowest point on the curve occurs at $x = 0$.

EXAMPLE ■ 8 A cable television line hangs between two 30-ft poles that are 36 ft apart. At its lowest point, the cable is 16 ft above the level ground. Determine the height of the cable above a point on the ground that is 6 ft from the poles.

SOLUTION We use the form

$$y = b + a\cosh\left(\frac{x}{a}\right)$$

and determine first the values of the constants a and b. Since the lowest point occurs at $x = 0$, we have

$$16 = b + a\cosh\left(\frac{0}{a}\right) = b + a\cosh 0 = b + a,$$

so that $\quad b = 16 - a.$

We also have $y = 30$ when $x = 18$ since the poles are 36 ft apart. Thus,

$$30 = b + a\cosh\left(\frac{18}{a}\right),$$

or $\quad b = 30 - a\cosh\left(\dfrac{18}{a}\right)$

Equating the two expressions for b, we have

$$16 - a = 30 - a\cosh\left(\frac{18}{a}\right)$$

or, equivalently,

$$a - a\cosh\left(\frac{18}{a}\right) + 14 = 0.$$

We use Newton's method to solve for a, obtaining $a \approx 13.42$. So $b = 16 - a \approx 2.58$, and the equation for the catenary becomes

$$y = 2.58 + 13.42\cosh\left(\frac{x}{13.42}\right).$$

At a point 6 ft from one of the poles, we have $x = \pm 12$. When $x = \pm 12$, $y = 2.58 + 13.42\cosh(\pm 12/13.42) \approx 21.73$. Thus, at a point 12 ft from the lowest point on the cable television line, the height of the cable is approximately 21.73 ft.

The analysis we have seen for hanging cables also applies to the Gateway Arch to the West in St. Louis. All the internal forces are in equilibrium when a cable hangs freely. There are no transverse forces pushing the cable out of shape. Constructing an arch in the shape of an inverted hyperbolic cosine creates a structure for which there are also no transverse forces that might cause the arch to collapse. This inherent stability of the inverted catenary, along with its beauty, led Saarinen to choose it for his design of the Gateway Arch.

As with other functions that we have studied, we can gain an understanding of compositions of functions that use inverse hyperbolic functions as components by combining the techniques of calculus with the graphs that a graphing utility can display. The next example illustrates this process.

EXAMPLE ■ 9 For the function $f(x) = \ln[\sinh^{-1}(x^2 + 1)]$,

(a) determine the domain of the function f

(b) find the derivative f'

(c) use a graphing utility to plot both the function and its derivative in the viewing window $-5 \le x \le 5, -1 \le y \le 1.5$

SOLUTION

(a) The function f is a composition of functions, requiring that we first add 1 to the square of x, then compute an inverse hyperbolic sine, and finally determine the natural logarithm of the resulting number. Since $x^2 + 1$ and the inverse hyperbolic sine are defined for all real numbers, the only step that may cause difficulty in computing $f(x)$ is that the natural logarithm is defined only for positive values.

We note first that by Theorem 6.47(i),

$$\sinh^{-1}(x^2+1) = \ln\left[(x^2+1) + \sqrt{(x^2+1)^2+1}\right] = \ln u,$$

where $u = x^2 + 1 + \sqrt{(x^2+1)^2+1}$. Now u is strictly positive and has its minimum value $1 + \sqrt{2}$ when $x = 0$. Hence,

$$\sinh^{-1}(x^2+1) \geq \ln(1+\sqrt{2}) \approx \ln 2.4142136 \approx 0.8813736.$$

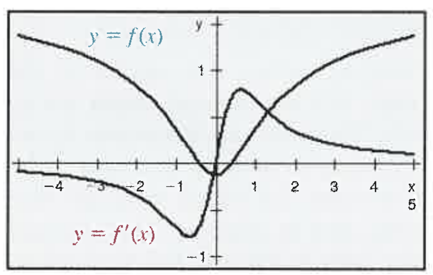
Since $\sinh^{-1}(x^2+1)$ is always positive, $f(x) = \ln(\sinh^{-1}(x^2+1))$ is defined for all real numbers x. Thus, the domain of f consists of all real numbers.

(b) We use the chain rule twice to find the derivative:

$$f'(x) = \frac{[\sinh^{-1}(x^2+1)]'}{\sinh^{-1}(x^2+1)} = \frac{\dfrac{1}{\sqrt{(x^2+1)^2+1}}(x^2+1)'}{\sinh^{-1}(x^2+1)}$$

$$= \frac{2x}{\sqrt{(x^2+1)^2+1}\,\sinh^{-1}(x^2+1)}$$

Figure 6.47
$-5 \leq x \leq 5,\ -1 \leq y \leq 1.5$



(c) We use a graphing utility to plot f and f' in the specified viewing window, as shown in Figure 6.47. From the figure, it appears that the graph of f is symmetric about the y-axis and the graph of f' is symmetric about the origin. We can confirm these observations by substituting into the expressions for the function and its derivative to find that $f(-x) = f(x)$ and that $f'(-x) = -f'(x)$. Using the trace feature, we find that f' has a maximum of 0.6580 at approximately $x = 0.7477$, where the graph of f has a point of inflection. By symmetry, there is also a point of inflection for f at $x = -0.7477$, where f' has a minimum.

EXERCISES 6.8

c **Exer. 1–2: Approximate to four decimal places.**

1 **(a)** $\sinh 4$ **(b)** $\cosh \ln 4$ **(c)** $\tanh(-3)$ **(d)** $\coth 10$ **(e)** $\operatorname{sech} 2$ **(f)** $\operatorname{csch}(-1)$

2 **(a)** $\sinh \ln 4$ **(b)** $\cosh 4$ **(c)** $\tanh 3$ **(d)** $\coth(-10)$ **(e)** $\operatorname{sech}(-2)$ **(f)** $\operatorname{csch} 1$

Exer. 3–14: Find $f'(x)$ if $f(x)$ is the given expression.

3 $\sinh 5x$

4 $\sinh(x^2+1)$

5 $\cosh(x^3)$

6 $\cosh^3 x$

7 $\sqrt{x}\tanh\sqrt{x}$

8 $\arctan\tanh x$

9 $\coth(1/x)$

10 $\coth x/\cot x$

11 $\dfrac{\operatorname{sech}(x^2)}{x^2+1}$

12 $\sqrt{\operatorname{sech} 5x}$

13 $\operatorname{csch}^2 6x$

14 $x\operatorname{csch} e^{4x}$

c **Exer. 15–18: (a) Find the domain of the function. (b) Find $f'(x)$. (c) Plot f and f' in the indicated viewing window.**

15 $f(x) = \cosh\sqrt{4x^2+3}$; $-3 \leq x \leq 3,\ -25 \leq y \leq 50$

16 $f(x) = \dfrac{1+\cosh x}{1-\cosh x}$; $-8 \leq x \leq 8,\ -5 \leq y \leq 2$

17 $f(x) = \dfrac{1}{\tanh x + 1}$; $-3 \leq x \leq 3,\ -25 \leq y \leq 50$

18 $f(x) = \ln|\tanh x|$; $-2 \leq x \leq 2,\ -10 \leq y \leq 10$

Exer. 19–30: Evaluate the integral.

19 $\int x^2\cosh(x^3)\,dx$

20 $\int \dfrac{1}{\operatorname{sech} 7x}\,dx$

21 $\int \dfrac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$

22 $\int x\sinh(2x^2)\,dx$

23 $\int \dfrac{1}{\cosh^2 3x}\,dx$

24 $\int \operatorname{sech}^2(5x)\,dx$

25 $\int \operatorname{csch}^2(\tfrac{1}{2}x)\,dx$

26 $\int (\sinh 4x)^{-2}\,dx$

27 $\int \tanh 3x\operatorname{sech} 3x\,dx$

28 $\int \sinh x\operatorname{sech}^2 x\,dx$

29 $\int \cosh x\operatorname{csch}^2 x\,dx$

30 $\int \coth 6x\operatorname{csch} 6x\,dx$

31 Find the points on the graph of $y = \sinh x$ at which the tangent line has slope 2.

32 Find the arc length of the graph of $y = \cosh x$ from $(0, 1)$ to $(1, \cosh 1)$.

33 If A is the region shown in Figure 6.43, prove that $t = 2A$.

34 The region bounded by the graphs of $y = \cosh x$, $x = -1$, $x = 1$, and $y = 0$ is revolved about the x-axis. Find the volume of the resulting solid.

35 The Gateway Arch to the West in St. Louis has the shape of an inverted catenary (see figure). Rising 630 ft at its center and stretching 630 ft across its base, the shape of the arch can be approximated by

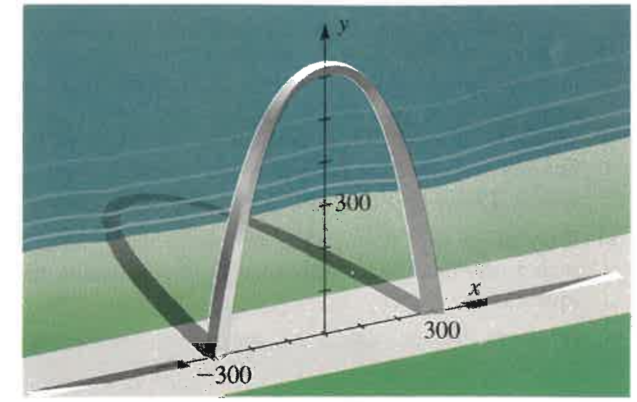
$$y = -127.7\cosh(x/127.7) + 757.7$$

for $-315 \leq x \leq 315$.

(a) Approximate the total open area under the arch.

(b) Approximate the total length of the arch.

Exercise 35



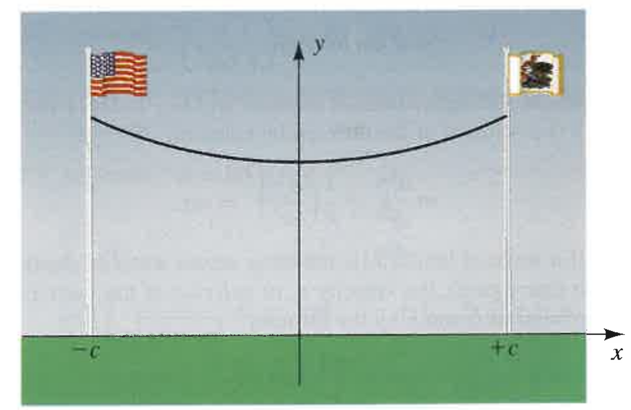
36 A uniform flexible cable supported by poles at $x = -c$ and $x = c$ takes the shape of the graph of the equation $y = b + a\cosh(x/a)$ for $-c \leq x \leq c$ (see figure).

(a) Find the height of the cable on the poles at each end.

(b) Find the height of the cable at its lowest point.

(c) Find the arc length of the cable hanging between the two poles.

Exercise 36



c **Exer. 37–40: Refer to Exercise 36.**

37 A power line is strung between two 21-ft poles that are 33 ft apart. At its lowest point, the cable is 16 ft above level ground. Find the arc length of the cable hanging between the two poles.

38 A rope 12 ft long is hung between two 5-ft high poles that are 10 ft apart. How high will the rope be off the level ground at its lowest point?

39 Two children pick up a 15-ft rope to play jump rope. Each child grasps the rope 6 in. from an end and holds the rope 3.5 ft above level ground. The two move together until the rope just touches the ground hanging

between their hands before they start to swing the rope. How far apart will they be?

40 A telephone line is to be strung across a city street between two 25-ft poles that are 30 ft apart. To allow large trucks to pass under the line, the lowest point should be at least 19 ft high. Find the arc length of the line between the two poles if it has a lowest point of exactly 19 ft.

41 If an object falls through the air toward the ground in such a way that the air resistance is proportional to the square of the velocity,

(a) show that position y of the object satisfies the differential equation

$$y'' = g - \alpha(y')^2$$

(b) make the change of variable

$$z = y'$$

as in Example 7 and solve the differential equation in part (a)

42 If a steel ball of mass m is released into water and the force of resistance is directly proportional to the square of the velocity, then the distance y that the ball travels in t seconds is given by

$$y = km \ln \cosh\left(\sqrt{\frac{g}{km}}\, t\right),$$

where g is a gravitational constant and $k > 0$. Show that y is a solution of the differential equation

$$m\frac{d^2y}{dt^2} + \frac{1}{k}\left(\frac{dy}{dt}\right)^2 = mg.$$

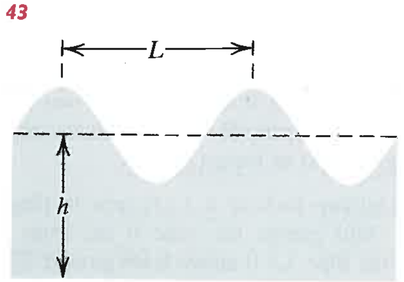
43 If a wave of length L is traveling across water of depth h (see figure), the velocity v, or *celerity*, of the wave is related to L and h by the formula

$$v^2 = \frac{gL}{2\pi} \tanh \frac{2\pi h}{L},$$

where g is a gravitational constant.

(a) Find $\lim_{h\to\infty} v^2$ and conclude that $v \approx \sqrt{gL/(2\pi)}$ in deep water.

Exercise 43



(b) If $x \approx 0$ and f is a continuous function, then, by the mean value theorem (3.12), $f(x) - f(0) \approx f'(0)\,x$. Use this fact to show that $v \approx \sqrt{gh}$ if $h/L \approx 0$. Conclude that wave velocity is independent of wave length in shallow water.

44 A soap bubble formed by two parallel concentric rings is shown in the figure. If the rings are not too far apart, it can be shown that the function f whose graph generates this surface of revolution is a solution of the differential equation $yy'' = 1 + (y')^2$, where $y = f(x)$. If A and B are positive constants, show that $y = A\cosh Bx$ is a solution if and only if $AB = 1$. Conclude that the graph is a catenary.

Exercise 44

c **45** Graph, on the same coordinate axes, $y = \tanh x$ and $y = \operatorname{sech}^2 x$ for $0 \le x \le 2$.

(a) Estimate the x-coordinate a of the point of intersection of the graphs.

(b) Use Newton's method to approximate a to three decimal places.

c **46** Graph, on the same coordinate axes, $y = \cosh^2 x$ and $y = 2$.

(a) Set up integrals for estimating the centroid of the region R bounded by the graphs.

(b) Use Simpson's rule, with $n = 2$, to approximate the coordinates of the centroid of R.

Exer. 47–58: Verify the identity.

47 $\cosh x + \sinh x = e^x$

48 $\sinh(-x) = -\sinh x$

49 $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

50 $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

51 $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

52 $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

53 $\sinh 2x = 2 \sinh x \cosh x$

54 $\cosh 2x = \cosh^2 x + \sinh^2 x$

55 $\sinh^2 \dfrac{x}{2} = \dfrac{\cosh x - 1}{2}$

56 $\cosh^2 \dfrac{x}{2} = \dfrac{\cosh x + 1}{2}$

57 $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for every positive integer n (*Hint:* Use Exercise 47.)

58 $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$ for every positive integer n

c **Exer. 59–60: Approximate to four decimal places.**

59 (a) $\sinh^{-1} 1$	(b) $\cosh^{-1} 2$
(c) $\tanh^{-1}(-\frac{1}{2})$	(d) $\operatorname{sech}^{-1}\frac{1}{2}$
60 (a) $\sinh^{-1}(-2)$	(b) $\cosh^{-1} 5$
(c) $\tanh^{-1}\frac{1}{3}$	(d) $\operatorname{sech}^{-1}\frac{3}{5}$

Exer. 61–68: Find $f'(x)$ if $f(x)$ is the given expression.

61 $\sinh^{-1} 5x$	**62** $\sinh^{-1} e^x$
63 $\cosh^{-1}\sqrt{x}$	**64** $\sqrt{\cosh^{-1} x}$
65 $\tanh^{-1}(-4x)$	**66** $\tanh^{-1}\sin 3x$
67 $\operatorname{sech}^{-1} x^2$	**68** $\operatorname{sech}^{-1}\sqrt{1-x}$

c **Exer. 69–72: (a) Find the domain of the function. (b) Find $f'(x)$. (c) Plot f and f' in the indicated viewing window.**

69 $f(x) = \ln\cosh^{-1} 4x$; $\quad 0 \le x \le 10,\ 0 \le y \le 2$

70 $f(x) = \cosh^{-1}\ln 4x$; $\quad 0 \le x \le 10,\ 0 \le y \le 2$

71 $f(x) = \tanh^{-1}(x+1)$; $\quad -2 \le x \le 0,\ -3 \le y \le 5$

72 $f(x) = \tanh^{-1} x^3$; $\quad -1 \le x \le 1,\ -3 \le y \le 5$

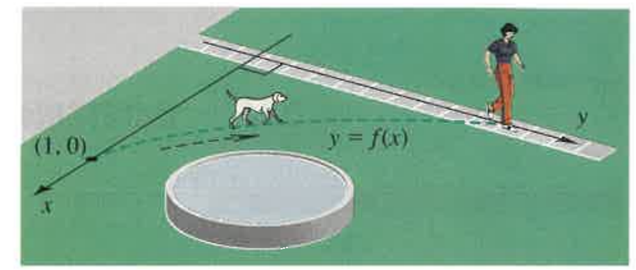
Exer. 73–80: Evaluate the integral.

73 $\displaystyle\int \frac{1}{\sqrt{81+16x^2}}\,dx$	**74** $\displaystyle\int \frac{1}{\sqrt{16x^2-9}}\,dx$
75 $\displaystyle\int \frac{1}{49-4x^2}\,dx$	**76** $\displaystyle\int \frac{\sin x}{\sqrt{1+\cos^2 x}}\,dx$
77 $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-16}}\,dx$	**78** $\displaystyle\int \frac{2}{5-3x^2}\,dx$
79 $\displaystyle\int \frac{1}{x\sqrt{9-x^4}}\,dx$	**80** $\displaystyle\int \frac{1}{\sqrt{5-e^{2x}}}\,dx$

81 A point moves along the line $x = 1$ in a coordinate plane with a velocity that is directly proportional to its distance from the origin. If the initial position of the point is (1, 0) and the initial velocity is 3 ft/sec, express the y-coordinate of the point as a function of time t (in seconds).

82 The rectangular coordinate system shown in the figure illustrates the problem of a dog seeking its master. The dog, initially at the point $(1, 0)$, sees its master at the point $(0, 0)$. The master proceeds up the y-axis at a constant speed, and the dog runs directly toward its master at all times. If the speed of the dog is twice that of the master, it can be shown that the path of the dog is given by $y = f(x)$, where y is a solution of the differential equation $2xy'' = \sqrt{1+(y')^2}$. Solve this equation by first letting $z = dy/dx$ and solving $2xz' = \sqrt{1+z^2}$, obtaining $z = \frac{1}{2}[\sqrt{x} - (1/\sqrt{x})]$. Finally, solve the equation $y' = \frac{1}{2}[\sqrt{x} - (1/\sqrt{x})]$.

Exercise 82



Exer. 83–86: Sketch the graph of the equation.

83 $y = \sinh^{-1} x$	**84** $y = \cosh^{-1} x$
85 $y = \tanh^{-1} x$	**86** $y = \operatorname{sech}^{-1} x$

Exer. 87–91: (a) Derive the formula. (b) and (c) Verify the formula.

87 (a) $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), \quad x \ge 1$

(b) $\dfrac{d}{dx}(\cosh^{-1} u) = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}, \quad u > 1$

(c) $\displaystyle\int \frac{1}{\sqrt{u^2-a^2}}\,du = \cosh^{-1}\frac{u}{a} + C, \quad 0 < a < u$

88 (a) $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}, \quad |x| < 1$

(b) $\dfrac{d}{dx}(\tanh^{-1} u) = \dfrac{1}{1-u^2}\dfrac{du}{dx}, \quad |u| < 1$

(c) $\displaystyle\int \frac{1}{a^2-u^2}\,du = \tanh^{-1}\frac{u}{a} + C, \quad |u| < a$

89 (a) $\operatorname{sech}^{-1} x = \ln\dfrac{1+\sqrt{1-x^2}}{x}, \quad 0 < x \le 1$

(b) $\dfrac{d}{dx}(\operatorname{sech}^{-1} u) = -\dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}, \quad 0 < u < 1$

(c) $\displaystyle\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|u|}{a} + C,$

$0 < |u| < a$

90 **(a)** $\coth^{-1} x = \dfrac{1}{2}\ln\dfrac{x+1}{x-1} = \tanh^{-1}\left(\dfrac{1}{x}\right), \quad |x| > 1$

(b) $\dfrac{d}{dx}(\coth^{-1} u) = \dfrac{1}{1-u^2}\dfrac{du}{dx}, \quad |u| > 1$

(c) $\displaystyle\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\coth^{-1}\frac{u}{a} + C, \quad |u| > a$

91 **(a)** $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right) = \sinh^{-1}\left(\dfrac{1}{x}\right), \quad x \neq 0$

(b) $\dfrac{d}{dx}(\operatorname{csch}^{-1} u) = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}, \quad |u| \neq 0$

(c) $\displaystyle\int \frac{1}{u\sqrt{a^2+u^2}}\,du = -\frac{1}{a}\operatorname{csch}^{-1}\frac{|u|}{a} + C, \quad u \neq 0$

6.9 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

In Chapter 1, we considered limits of quotients such as

$$\lim_{x\to 3}\frac{x^2-9}{x-3} \quad\text{and}\quad \lim_{x\to 0}\frac{\sin x}{x}.$$

In each case, taking the limits of the numerator and the denominator separately gives the undefined expression 0/0. In a limit of the form

$$\lim_{x\to 0}(\cos x - 1)^{(x^2)},$$

we obtain an undefined expression of the form 0^0 if we take the limits of the base and the exponent separately. For such *indeterminate forms*, we have used algebraic, geometric, and trigonometric methods accompanied by an ingenious manipulation to calculate limits. In this section, we develop other techniques that allow us to proceed in a more direct manner to evaluate several different types of indeterminate forms that occur in both theoretical settings and applications such as electric circuits and insulated cables.

THE FORMS 0/0 AND ∞/∞

We first consider the **indeterminate form 0/0** for limits of quotients where both the numerator and the denominator have limit 0 and the **indeterminate form** ∞/∞ where both the numerator and the denominator approach ∞ or $-\infty$. The following table displays general definitions of these forms.

Indeterminate form	Limit form: $\lim_{x\to c}\dfrac{f(x)}{g(x)}$
$\dfrac{0}{0}$	$\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$
$\dfrac{\infty}{\infty}$	$\lim_{x\to c} f(x) = \infty$ or $-\infty$ and $\lim_{x\to c} g(x) = \infty$ or $-\infty$

The main tool for investigating these indeterminate forms is *l'Hôpital's rule*. The proof of this rule makes use of the following formula, which bears the name of the French mathematician Augustin Cauchy (1789–1857). (See *Mathematicians and Their Times*, Chapter 9.)

Cauchy's Formula 6.50

If f and g are continuous on $[a, b]$ and differentiable on (a, b) and if $g'(x) \neq 0$ for every x in (a, b), then there is a number w in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(w)}{g'(w)}.$$

PROOF We first note that $g(b) - g(a) \neq 0$, because otherwise $g(a) = g(b)$ and, by Rolle's theorem (3.10), there is a number c in (a, b) such that $g'(c) = 0$, contrary to our assumption about g'.

Let us introduce a new function h as follows:

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

for every x in $[a, b]$. It follows that h is continuous on $[a, b]$ and differentiable on (a, b) and that $h(a) = h(b)$. By Rolle's theorem, there is a number w in (a, b) such that $h'(w) = 0$—that is,

$$[f(b) - f(a)]g'(w) - [g(b) - g(a)]f'(w) = 0.$$

This is equivalent to Cauchy's formula. ■

Cauchy's formula is a generalization of the mean value theorem (3.12) for if we let $g(x) = x$ in (6.50), we obtain

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(w)}{1} = f'(w).$$

The next result is the main theorem on indeterminate forms.

l'Hôpital's Rule* 6.51

Suppose that f and g are differentiable on an open interval (a, b) containing c, except possibly at c itself. If $f(x)/g(x)$ has the indeterminate form $0/0$ or ∞/∞ at $x = c$ and if $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided either

$$\lim_{x\to c}\frac{f'(x)}{g'(x)} \text{ exists} \quad\text{or}\quad \lim_{x\to c}\frac{f'(x)}{g'(x)} = \infty.$$

*G. l'Hôpital (1661–1704) was a French nobleman who published the first calculus book. The rule appeared in that book; however, it was actually discovered by John Bernoulli (1667–1748), who communicated the result to l'Hôpital in 1694. (See *Mathematicians and Their Times*, Chapter 5.)

PROOF Suppose that $f(x)/g(x)$ has the indeterminate form $0/0$ at $x = c$ and $\lim_{x\to c}[f'(x)/g'(x)] = L$ for some number L. We wish to prove that $\lim_{x\to c}[f(x)/g(x)] = L$. Let us introduce two functions F and G as follows:

$$F(x) = f(x) \quad \text{if } x \neq c \quad \text{and} \quad F(c) = 0$$
$$G(x) = g(x) \quad \text{if } x \neq c \quad \text{and} \quad G(c) = 0$$

Since
$$\lim_{x\to c} F(x) = \lim_{x\to c} f(x) = 0 = F(c),$$

the function F is continuous at c and hence is continuous *throughout* the interval (a, b). Similarly, G is continuous on (a, b). Moreover, at every $x \neq c$, we have $F'(x) = f'(x)$ and $G'(x) = g'(x)$. It follows from Cauchy's formula, applied either to the interval $[c, x]$ or to $[x, c]$, that there is a number w between c and x such that

$$\frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F'(w)}{G'(w)} = \frac{f'(w)}{g'(w)}.$$

Using the fact that $F(x) = f(x)$, $G(x) = g(x)$, and $F(c) = G(c) = 0$ gives us

$$\frac{f(x)}{g(x)} = \frac{f'(w)}{g'(w)}.$$

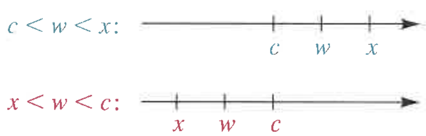
Since w is always between c and x (see Figure 6.48), it follows that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(w)}{g'(w)} = \lim_{w\to c} \frac{f'(w)}{g'(w)} = L,$$

which is what we wished to prove.

A similar argument may be given if $\lim_{x\to c}[f'(x)/g'(x)] = \infty$. The proof for the indeterminate form ∞/∞ is more difficult and may be found in texts on advanced calculus. ■

Figure 6.48



L'Hôpital's rule is sometimes used incorrectly, by applying the quotient rule to $f(x)/g(x)$. Note that (6.51) states that the derivatives of $f(x)$ and $g(x)$ are taken *separately*, after which the limit of $f'(x)/g'(x)$ is investigated.

EXAMPLE ■ 1 Find $\displaystyle\lim_{x\to 0} \frac{\cos x + 2x - 1}{3x}$.

SOLUTION Both the numerator and the denominator have the limit 0 as $x \to 0$. Hence the quotient has the indeterminate form $0/0$ at $x = 0$. By l'Hôpital's rule (6.51),

$$\lim_{x\to 0} \frac{\cos x + 2x - 1}{3x} = \lim_{x\to 0} \frac{-\sin x + 2}{3},$$

provided the limit on the right exists or equals ∞. Since

$$\lim_{x\to 0} \frac{-\sin x + 2}{3} = \frac{2}{3},$$

it follows that

$$\lim_{x\to 0} \frac{\cos x + 2x - 1}{3x} = \frac{2}{3}.$$

Sometimes it is necessary to use l'Hôpital's rule several times in the same problem, as illustrated in the next example.

EXAMPLE ■ 2 Find $\displaystyle\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$.

SOLUTION The given quotient has the indeterminate form $0/0$. By l'Hôpital's rule,

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2 \sin 2x},$$

provided the second limit exists. Because the last quotient has the indeterminate form $0/0$, we apply l'Hôpital's rule a second time, obtaining

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2 \sin 2x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{4 \cos 2x} = \frac{2}{4} = \frac{1}{2}.$$

It follows that the given limit exists and equals $\frac{1}{2}$.

L'Hôpital's rule is also valid for one-sided limits, as illustrated in the following example.

EXAMPLE ■ 3 Find $\displaystyle\lim_{x\to(\pi/2)^-} \frac{4 \tan x}{1 + \sec x}$.

SOLUTION The indeterminate form is ∞/∞. By l'Hôpital's rule,

$$\lim_{x\to(\pi/2)^-} \frac{4 \tan x}{1 + \sec x} = \lim_{x\to(\pi/2)^-} \frac{4 \sec^2 x}{\sec x \tan x} = \lim_{x\to(\pi/2)^-} \frac{4 \sec x}{\tan x}.$$

The last quotient again has the indeterminate form ∞/∞ at $x = \pi/2$; however, additional applications of l'Hôpital's rule always produce the form ∞/∞ (verify this fact). In this case, the limit may be found by using trigonometric identities to change the quotient as follows:

$$\frac{4 \sec x}{\tan x} = \frac{4/\cos x}{\sin x/\cos x} = \frac{4}{\sin x}$$

Consequently,

$$\lim_{x\to(\pi/2)^-} \frac{4 \tan x}{1 + \sec x} = \lim_{x\to(\pi/2)^-} \frac{4}{\sin x} = \frac{4}{1} = 4.$$

There is another form of l'Hôpital's rule that can be proved for $x \to \infty$ or $x \to -\infty$. Let us give a partial proof of this fact. Suppose that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

If we let $u = 1/x$ and apply l'Hôpital's rule, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{u\to 0^+} \frac{f(1/u)}{g(1/u)} = \lim_{u\to 0^+} \frac{(d/du)(f(1/u))}{(d/du)(g(1/u))}.$$

By the chain rule,

$$\frac{d}{du}(f(1/u)) = f'(1/u)(-1/u^2) \quad \text{and} \quad \frac{d}{du}(g(1/u)) = g'(1/u)(-1/u^2).$$

Substituting in the last limit and simplifying, we obtain

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{u\to 0^+} \frac{f'(1/u)}{g'(1/u)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

We shall also refer to this result as l'Hôpital's rule. The next two examples illustrate the application of the rule to the form ∞/∞.

EXAMPLE 4 Find $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$.

SOLUTION The indeterminate form is ∞/∞. By l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})}.$$

The last expression has the indeterminate form $0/0$. However, further applications of l'Hôpital's rule would again lead to $0/0$ (verify this fact). If, instead, we simplify the expression algebraically, we can find the limit as follows:

$$\lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

EXAMPLE 5 Find $\lim_{x\to\infty} \dfrac{e^{3x}}{x^2}$, if it exists.

SOLUTION The indeterminate form is ∞/∞. We apply l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{e^{3x}}{x^2} = \lim_{x\to\infty} \frac{3e^{3x}}{2x}$$

The last quotient has the indeterminate form ∞/∞, so we apply l'Hôpital's rule a second time, obtaining

$$\lim_{x\to\infty} \frac{3e^{3x}}{2x} = \lim_{x\to\infty} \frac{9e^{3x}}{2} = \infty.$$

Thus, e^{3x}/x^2 has no limit, increasing without bound as $x \to \infty$.

It is extremely important to verify that a given quotient has the indeterminate form $0/0$ or ∞/∞ before using l'Hôpital's rule. If we apply the rule to a form that is not indeterminate, we may obtain an incorrect conclusion, as illustrated in the next example.

EXAMPLE 6 Find $\lim_{x\to 0} \dfrac{e^x + e^{-x}}{x^2}$, if it exists.

SOLUTION The quotient does *not* have either of the indeterminate forms, $0/0$ or ∞/∞, at $x = 0$. To investigate the limit, we write

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{x^2} = \lim_{x\to 0} (e^x + e^{-x}) \left(\frac{1}{x^2}\right).$$

Since $\quad \lim_{x\to 0} (e^x + e^{-x}) = 2 \quad$ and $\quad \lim_{x\to 0} \dfrac{1}{x^2} = \infty,$

it follows that

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{x^2} = \infty.$$

If we had overlooked the fact that the quotient does not have the indeterminate form $0/0$ or ∞/∞ at $x = 0$ and had (incorrectly) applied l'Hôpital's rule, we would have obtained

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2x}.$$

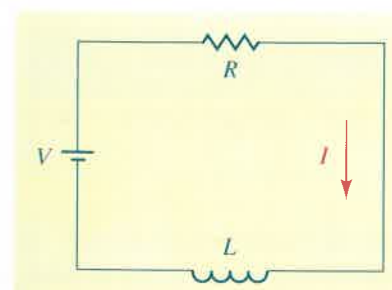
Since the last quotient has the indeterminate form $0/0$, we might have applied l'Hôpital's rule, obtaining

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{2} = \frac{1+1}{2} = 1.$$

This would have given us the (wrong) conclusion that the given limit exists and equals 1.

The next example illustrates an application of an indeterminate form in the analysis of an electrical circuit.

Figure 6.49



EXAMPLE 7 The schematic diagram in Figure 6.49 illustrates an electrical circuit consisting of an electromotive force V, a resistor R, and an inductor L. The current I at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L}).$$

When the voltage is first applied (at $t = 0$), the inductor opposes the rate of increase of current and I is small; however, as t increases, I approaches V/R.

(a) If L is the only independent variable, find $\lim_{L\to 0^+} I$.

(b) If R is the only independent variable, find $\lim_{R\to 0^+} I$.

SOLUTION

(a) If we consider V, R, and t as constants and L as a variable, then the expression for I is not indeterminate at $L = 0$. Using standard limit theorems, we obtain

$$\begin{aligned}\lim_{L\to 0^+} I &= \lim_{L\to 0^+} \frac{V}{R}\left(1 - e^{-Rt/L}\right)\\ &= \frac{V}{R}\left(1 - \lim_{L\to 0^+} e^{-Rt/L}\right)\\ &= \frac{V}{R}(1-0) = \frac{V}{R}.\end{aligned}$$

Thus, if $L \approx 0$, then the current can be approximated by Ohm's law $I = V/R$.

(b) If V, L, and t are constant and if R is a variable, then I has the indeterminate form $0/0$ at $R = 0$. Applying l'Hôpital's rule, we have

$$\begin{aligned}\lim_{R\to 0^+} I &= V \lim_{R\to 0^+} \frac{1 - e^{-Rt/L}}{R}\\ &= V \lim_{R\to 0^+} \frac{0 - e^{-Rt/L}(-t/L)}{1}\\ &= V[0 - (1)(-t/L)] = \frac{V}{L}t.\end{aligned}$$

This result may be interpreted as follows. As $R \to 0^+$, the current I is directly proportional to the time t, with the constant of proportionality V/L. Thus, at $t = 1$, the current is V/L; at $t = 2$, it is $(V/L)(2)$; at $t = 3$, it is $(V/L)(3)$; and so on.

THE FORMS $0 \cdot \infty$, 0^0, ∞^0, 1^∞, AND $\infty - \infty$

There are a number of other indeterminate forms whose limits can be found by rewriting the expressions as quotients and applying l'Hôpital's rule. We begin with products that may lead to the indeterminate form $0 \cdot \infty$, as defined in the following table.

Indeterminate form	Limit form: $\lim_{x\to c}[f(x)g(x)]$
$0 \cdot \infty$	$\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$ or $-\infty$

In the exercises, we shall also consider the indeterminate form $0 \cdot \infty$ for the case $x \to \infty$ or $x \to -\infty$. The following guidelines may be used.

Guidelines for Investigating $\lim_{x\to c} [f(x)g(x)]$ for the Form $0 \cdot \infty$ 6.52

1. Write $f(x)g(x)$ as
$$\frac{f(x)}{1/g(x)} \quad \text{or} \quad \frac{g(x)}{1/f(x)}.$$
2. Apply l'Hôpital's rule (6.51) to the resulting indeterminate form $0/0$ or ∞/∞.

The choice in guideline (1) is not arbitrary. The following example shows that using $f(x)/[1/g(x)]$ gives us the limit, whereas using $g(x)/[1/f(x)]$ leads to a more complicated expression.

EXAMPLE 8 Find $\lim_{x\to 0^+} x^2 \ln x$.

SOLUTION The indeterminate form is $0 \cdot \infty$. Applying guideline (1) of (6.52), we write

$$x^2 \ln x = \frac{\ln x}{1/x^2}.$$

Because the quotient on the right has the indeterminate form ∞/∞ at $x = 0$, we may apply l'Hôpital's rule:

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3}$$

The last quotient has the indeterminate form ∞/∞; however, further applications of l'Hôpital's rule would again lead to ∞/∞. In this case, we simplify the quotient algebraically and find the limit as follows:

$$\lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \frac{x^3}{-2x} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0$$

If, in applying guideline (1), we had rewritten the given expression as

$$x^2 \ln x = \frac{x^2}{1/\ln x} = \frac{x^2}{(\ln x)^{-1}},$$

then the resulting indeterminate form would have been $0/0$. By l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0^+} x^2 \ln x &= \lim_{x\to 0^+} \frac{x^2}{(\ln x)^{-1}}\\ &= \lim_{x\to 0^+} \frac{2x}{-(\ln x)^{-2}(1/x)}\\ &= \lim_{x\to 0^+} [-2x^2(\ln x)^2].\end{aligned}$$

The expression $-2x^2(\ln x)^2$ is more complicated than $x^2 \ln x$, so this choice in guideline (1) does *not* give us the limit.

EXAMPLE 9 Find $\lim_{x\to(\pi/2)^-}(2x-\pi)\sec x$.

SOLUTION The indeterminate form is $0\cdot\infty$. Using guideline (1) of (6.52), we begin by writing

$$(2x-\pi)\sec x=\frac{2x-\pi}{1/\sec x}=\frac{2x-\pi}{\cos x}.$$

Because the last expression has the indeterminate form $0/0$ at $x=\pi/2$, l'Hôpital's rule may be applied as follows:

$$\lim_{x\to(\pi/2)^-}\frac{2x-\pi}{\cos x}=\lim_{x\to(\pi/2)^-}\frac{2}{-\sin x}=\frac{2}{-1}=-2$$

The indeterminate forms defined in the next table may occur in investigating limits involving exponential expressions.

Indeterminate form	Limit form: $\lim_{x\to c} f(x)^{g(x)}$
0^0	$\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=0$
∞^0	$\lim_{x\to c} f(x)=\infty$ or $-\infty$ and $\lim_{x\to c} g(x)=0$
1^∞	$\lim_{x\to c} f(x)=1$ and $\lim_{x\to c} g(x)=\infty$ or $-\infty$

In exercises, we will also consider cases in which $x\to\infty$ or $x\to-\infty$. One method for investigating these forms is to consider

$$y=f(x)^{g(x)}$$

and take the natural logarithm of both sides, obtaining

$$\ln y=\ln f(x)^{g(x)}=g(x)\ln f(x).$$

If the indeterminate form for y is 0^0 or ∞^0, then the indeterminate form for $\ln y$ is $0\cdot\infty$, which may be handled using earlier methods. Similarly, if y has the form 1^∞, then the indeterminate form for $\ln y$ is $\infty\cdot 0$. It follows that

$$\text{if}\quad \lim_{x\to c}\ln y=\ln\left(\lim_{x\to c} y\right)=L,\quad\text{then}\quad \lim_{x\to c} y=e^L;$$

that is, $$\lim_{x\to c} f(x)^{g(x)}=e^L.$$

This procedure may be summarized as follows.

Guidelines for Investigating $\lim_{x\to c} f(x)^{g(x)}$ for the Forms $0^0, 1^\infty$, and ∞^0 6.53

1 Let $y=f(x)^{g(x)}$.

2 Take natural logarithms in guideline (1):

$$\ln y=\ln f(x)^{g(x)}=g(x)\ln f(x)$$

3 Investigate $\lim_{x\to c}\ln y=\lim_{x\to c}[g(x)\ln f(x)]$ and conclude the following:

(a) If $\lim_{x\to c}\ln y=L$, then $\lim_{x\to c} y=e^L$.

(b) If $\lim_{x\to c}\ln y=\infty$, then $\lim_{x\to c} y=\infty$.

(c) If $\lim_{x\to c}\ln y=-\infty$, then $\lim_{x\to c} y=0$.

A common error is to stop after showing $\lim_{x\to c}\ln y=L$ and conclude that the given expression has the limit L. Remember that *we wish to find the limit of* y. Thus, if $\ln y$ has the limit L, then y has the limit e^L. The guidelines may also be used if $x\to\infty$ or if $x\to-\infty$ or for one-sided limits.

EXAMPLE 10 Find $\lim_{x\to 0^+} x^x$.

SOLUTION The indeterminate form is 0^0. (See the discussion and graph of the function x^x in Example 5 of Section 6.5.) Using Guidelines (6.53), we proceed as follows:

Guideline 1 $y=x^x$

Guideline 2 $\ln y=x\ln x$

Guideline 3 This expression has the indeterminate form $0\cdot\infty$. We apply guideline (1) of (6.52) to write

$$x\ln x=\frac{\ln x}{1/x}.$$

Since the quotient on the right has the indeterminate form ∞/∞ at $x=0$, we may apply l'Hôpital's rule:

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$

We can evaluate this last limit by an algebraic simplification:

$$\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}\left(-\frac{x^2}{x}\right)=\lim_{x\to 0^+}(-x)=0$$

Since $\lim_{x\to 0^+}\ln y=0$, by guideline (3a) of (6.53), we have

$$\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}y=e^0=1.$$

Here we have a rigorous proof of a property of x^x that we observed from the graph in Figure 6.23.

The final indeterminate form we shall consider is defined in the following table.

Indeterminate form	Limit form: $\lim_{x\to c}[f(x)-g(x)]$
$\infty - \infty$	$\lim_{x\to c} f(x)=\infty$ and $\lim_{x\to c} g(x)=\infty$

When investigating $\infty - \infty$, we try to change the form of $f(x)-g(x)$ to a quotient or a product and then apply l'Hôpital's rule or some other method of evaluation, as illustrated in the next example.

EXAMPLE 11 Find $\lim_{x\to 0^+}\left(\dfrac{1}{e^x-1}-\dfrac{1}{x}\right)$.

SOLUTION The form is $\infty - \infty$; however, if the difference is written as a single fraction, then

$$\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}.$$

This gives us the indeterminate form 0/0. It is necessary to apply l'Hôpital's rule twice, since the first application leads to the indeterminate form 0/0. Thus,

$$\lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}=\lim_{x\to 0^+}\frac{1-e^x}{xe^x+e^x-1}$$
$$=\lim_{x\to 0^+}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}.$$

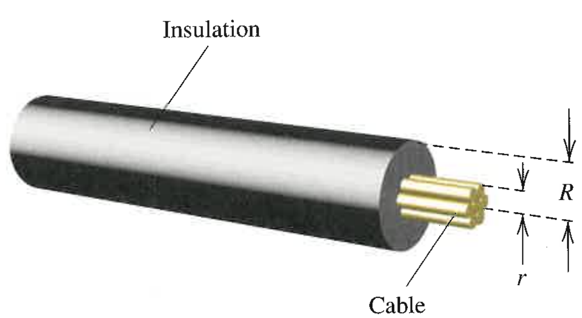
EXAMPLE 12 The velocity v of an electrical impulse in an insulated cable is given by

$$v=-k\left(\frac{r}{R}\right)^2\ln\left(\frac{r}{R}\right),$$

where k is a positive constant, r is the radius of the cable, and R is the distance from the center of the cable to the outside of the insulation, as shown in Figure 6.50. Find

(a) $\lim_{R\to r^+} v$ **(b)** $\lim_{r\to 0^+} v$

Figure 6.50



SOLUTION

(a) The limit notation implies that r is fixed and R is a variable. In this case, the expression for v is not indeterminate, and

$$\lim_{R\to r^+} v=-k\lim_{R\to r^+}\left(\frac{r}{R}\right)^2\ln\left(\frac{r}{R}\right)=-k(1)^2\ln 1=-k(0)=0.$$

(b) If R is fixed and r is a variable, then the expression for v has the indeterminate form $0\cdot\infty$ at $r=0$, and we first change the form of the expression algebraically, as follows:

$$\lim_{r\to 0^+} v=-k\lim_{r\to 0^+}\frac{\ln(r/R)}{(r/R)^{-2}}=-\frac{k}{R^2}\lim_{r\to 0^+}\frac{\ln r-\ln R}{r^{-2}}$$

The last quotient has the indeterminate form ∞/∞ at $r=0$, so we may apply l'Hôpital's rule, obtaining

$$\lim_{r\to 0^+} v=-\frac{k}{R^2}\lim_{r\to 0^+}\frac{(1/r)-0}{-2r^{-3}}$$
$$=-\frac{k}{R^2}\lim_{r\to 0^+}\left(\frac{r^2}{-2}\right)=-\frac{k}{R^2}(0)=0.$$

EXERCISES 6.9

Exer. 1–36: Find the limit, if it exists.

1 $\lim_{x\to 0}\dfrac{\sin x}{2x}$

2 $\lim_{x\to 0}\dfrac{5x}{\tan x}$

3 $\lim_{x\to 5}\dfrac{\sqrt{x-1}-2}{x^2-25}$

4 $\lim_{x\to -3}\dfrac{x^2+2x-3}{2x^2+3x-9}$

5 $\lim_{x\to 2}\dfrac{2x^2-5x+2}{5x^2-7x-6}$

6 $\lim_{x\to 1}\dfrac{x^3-3x+2}{x^2-2x-1}$

7 $\lim_{x\to 0}\dfrac{\sin x-x}{\tan x-x}$

8 $\lim_{x\to 0}\dfrac{x+1-e^x}{x^2}$

9 $\lim_{x\to 0}\dfrac{x-\sin x}{x^3}$

10 $\lim_{x\to \pi/2}\dfrac{1-\sin x}{\cos x}$

11 $\lim_{x\to \pi/2}\dfrac{1+\sin x}{\cos^2 x}$

12 $\lim_{x\to 0^+}\dfrac{\ln x}{\cot x}$

13 $\lim_{x\to (\pi/2)^-}\dfrac{2+\sec x}{3\tan x}$

14 $\lim_{x\to \infty}\dfrac{x^2}{\ln x}$

15 $\lim_{x\to 0^+}\dfrac{\ln\sin x}{\ln\sin 2x}$

16 $\lim_{x\to 0}\dfrac{e^x-e^{-x}-2\sin x}{x\sin x}$

17 $\lim_{x\to 0}\dfrac{x\cos x+e^{-x}}{x^2}$

18 $\lim_{x\to 0}\dfrac{2e^x-3x-e^{-x}}{x^2}$

19 $\lim_{x\to \infty}\dfrac{2x^2+3x+1}{5x^2+x+4}$

20 $\lim_{x\to \infty}\dfrac{e^{3x}}{\ln x}$

21 $\lim_{x\to \infty}\dfrac{x^n}{e^x}, n>0$

22 $\lim_{x\to \infty}\dfrac{e^x}{x^n}, n>0$

23 $\lim_{x\to \infty}\dfrac{x\ln x}{x+\ln x}$

24 $\lim_{x\to 2^+}\dfrac{\ln(x-1)}{(x-2)^2}$

25 $\lim_{x\to 0}\dfrac{\sin^{-1}2x}{\sin^{-1}x}$

26 $\lim_{x\to 0}\dfrac{\tan x-\sin x}{x^3\tan x}$

27 $\lim_{x\to -\infty}\dfrac{3-3^x}{5-5^x}$

28 $\lim_{x\to 1}\dfrac{2x^3-5x^2+6x-3}{x^3-2x^2+x-1}$

29 $\lim_{x\to 1}\dfrac{x^4-x^3-3x^2+5x-2}{x^4-5x^3+9x^2-7x+2}$

30 $\lim_{x\to 1}\dfrac{x^4+x^3-3x^2-x+2}{x^4-5x^3+9x^2-7x+2}$

31 $\lim_{x\to 0}\dfrac{x-\tan^{-1}x}{x\sin x}$

32 $\lim_{x\to \infty}\dfrac{x^{3/2}+5x-4}{x\ln x}$

33 $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{\tan^{-1} x}$

34 $\lim_{x\to\infty} \dfrac{2e^{3x}+\ln x}{e^{3x}+x^2}$

35 $\lim_{x\to\infty} \dfrac{x-\cos x}{x}$

36 $\lim_{x\to 0^+} \dfrac{e^{-1/x}}{x}$

c **Exer. 37–38: Predict the limit after substituting the indicated values of x for $k = 1, 2, 3,$ and 4.**

37 $\lim_{x\to 0^+} \dfrac{\ln(\tan x + \cos x)}{\sqrt{\ln(x^2+1)}}$; $x = 10^{-k}$

38 $\lim_{x\to 0} \dfrac{\tan^2(\sin^{-1} x)}{1-\cos[\ln(1+x)]}$; $x = \pm 10^{-k}$

39 An object of mass m is released from a hot-air balloon. If the force of resistance due to air is directly proportional to the velocity $v(t)$ of the object at time t, then it can be shown that

$$v(t) = (mg/k)(1 - e^{-(k/m)t}),$$

where $k > 0$ and g is a gravitational constant. Find $\lim_{k\to 0^+} v(t)$.

40 If a steel ball of mass m is released into water and the force of resistance is directly proportional to the square of the velocity, then the distance $s(t)$ that the ball travels in time t is given by

$$s(t) = (m/k)\ln\cosh(\sqrt{gk/m}\,t),$$

where $k > 0$ and g is a gravitational constant. Find $\lim_{k\to 0^+} s(t)$.

41 Refer to Definition (3.24) for simple harmonic motion. The following is an example of the phenomenon of resonance. A weight of mass m is attached to a spring suspended from a support. The weight is set in motion by moving the support up and down according to the formula $h = A\cos\omega t$, where A and ω are positive constants and t is time. If frictional forces are negligible, then the displacement s of the weight from its initial position at time t is given by

$$s = \frac{A\omega^2}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t),$$

with $\omega_0 = \sqrt{k/m}$ for some constant k and with $\omega \neq \omega_0$. Find $\lim_{\omega\to\omega_0} s$, and show that the resulting oscillations increase in magnitude.

42 The logistic model for population growth predicts the size $y(t)$ of a population at time t by means of the formula $y(t) = K/(1 + ce^{-rt})$, where r and K are positive constants and $c = [K - y(0)]/y(0)$. Ecologists call K the *carrying capacity* and interpret it as the maximum number of individuals that the environment can sustain. Find $\lim_{t\to\infty} y(t)$ and $\lim_{K\to\infty} y(t)$, and discuss the graphical significance of these limits.

43 The *sine integral* $Si(x) = \int_0^x [(\sin u)/u]\,du$ is a special function in applied mathematics. Find

(a) $\lim_{x\to 0} \dfrac{Si(x)}{x}$ **(b)** $\lim_{x\to 0} \dfrac{Si(x) - x}{x^3}$

44 The *Fresnel cosine integral* $C(x) = \int_0^x \cos u^2\,du$ is used in the analysis of the diffraction of light. Find

(a) $\lim_{x\to 0} \dfrac{C(x)}{x}$ **(b)** $\lim_{x\to 0} \dfrac{C(x) - x}{x^5}$

c **45** **(a)** Refer to Exercise 44. Use Simpson's rule, with $n = 2$, to approximate $C(x)$ for $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1.

(b) Graph C on $[0, 1]$ using the values found in part (a).

c **46** Refer to Exercise 45. Let R be the region under the graph of C from $x = 0$ to $x = 1$ and V the volume of the solid obtained by revolving R about the x-axis. Approximate V by using Simpson's rule, with $n = 2$.

47 Let $x > 0$. If $n \neq -1$, then $\int_1^x t^n\,dt = [t^{n+1}/(n+1)]_1^x$. Show that

$$\lim_{n\to -1} \int_1^x t^n\,dt = \int_1^x t^{-1}\,dt.$$

48 Find $\lim_{x\to\infty} f(x)/g(x)$ if

$$f(x) = \int_0^x e^{(t^2)}\,dt \quad\text{and}\quad g(x) = e^{(x^2)}.$$

Exer. 49–76: Find the limit, if it exists.

49 $\lim_{x\to 0^+} x\ln x$

50 $\lim_{x\to(\pi/2)^-} \tan x \ln\sin x$

51 $\lim_{x\to\infty} (x^2 - 1)e^{-x^2}$

52 $\lim_{x\to -\infty} x\tan^{-1} x$

53 $\lim_{x\to 0} e^{-x}\sin x$

54 $\lim_{x\to 0^+} \sin x\ln\sin x$

55 $\lim_{x\to\infty} x\sin\dfrac{1}{x}$

56 $\lim_{x\to 0} x\sec^2 x$

57 $\lim_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^{5x}$

58 $\lim_{x\to 0^+} (e^x + 3x)^{1/x}$

59 $\lim_{x\to 0^+} (e^x - 1)^x$

60 $\lim_{x\to\infty} x^{1/x}$

61 $\lim_{x\to(\pi/2)^-} (\tan x)^x$

62 $\lim_{x\to 0^+} (1 + 3x)^{\csc x}$

63 $\lim_{x\to 0^+} (2x + 1)^{\cot x}$

64 $\lim_{x\to\infty} \left(\dfrac{x^2}{x-1} - \dfrac{x^2}{x+1}\right)$

65 $\lim_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

66 $\lim_{x\to 1^-} (1 - x)^{\ln x}$

67 $\lim_{x\to 0^+} \left(\dfrac{1}{\sqrt{x^2+1}} - \dfrac{1}{x}\right)$

68 $\lim_{x\to 0} (\cot^2 x - \csc^2 x)$

69 $\lim_{x\to 0^+} \cot 2x\tan^{-1} x$

70 $\lim_{x\to 0^+} (1 + ax)^{b/x}$

71 $\lim_{x\to(\pi/2)^-} (1 + \cos x)^{\tan x}$

72 $\lim_{x\to -3^-} \left(\dfrac{x}{x^2 + 2x - 3} - \dfrac{4}{x+3}\right)$

73 $\lim_{x\to 0^+} (x + \cos 2x)^{\csc 3x}$

74 $\lim_{x\to\infty} (\sqrt{x^4 + 5x^2 + 3} - x^2)$

75 $\lim_{x\to\infty} (\sinh x - x)$

76 $\lim_{x\to\infty} [\ln(4x + 3) - \ln(3x + 4)]$

c **Exer. 77–78: Graph f on the given interval and use the graph to estimate $\lim_{x\to 0} f(x)$.**

77 $f(x) = (x\tan x)^{(x^2)}$; $[-1, 1]$

78 $f(x) = \left[\dfrac{\ln(x+1)}{\tan x}\right]^{1/x}$; $[-0.5, 0.5]$

Exer. 79–80: (a) Find the local extrema and discuss the behavior of $f(x)$ near $x = 0$. (b) Find horizontal asymptotes, if they exist. (c) Sketch the graph of f for $x > 0$.

79 $f(x) = x^{1/x}$

80 $f(x) = x^{\sqrt{x}}$

81 The *geometric mean* of two positive real numbers a and b is defined as $\sqrt{ab}$. Use l'Hôpital's rule to prove that

$$\sqrt{ab} = \lim_{x\to\infty} \left(\frac{a^{1/x} + b^{1/x}}{2}\right)^x.$$

82 If a sum of money P is invested at an interest rate of $100r$ percent per year, compounded m times per year, then the principal at the end of t years is given by $P(1 + rm^{-1})^{mt}$. If we regard m as a real number and let m increase without bound, then the interest is said to be *compounded continuously*. Use l'Hôpital's rule to show that, in this case, the principal after t years is Pe^{rt}.

83 Refer to Exercise 39. In the velocity formula

$$v(t) = (mg/k)(1 - e^{-(k/m)t}),$$

m represents the mass of the falling object. Find $\lim_{m\to\infty} v(t)$ and conclude that $v(t)$ is approximately proportional to time t if the mass is very large.

CHAPTER 6 REVIEW EXERCISES

Exer. 1–2: Find $f^{-1}(x)$.

1 $f(x) = 10 - 15x$

2 $f(x) = 9 - 2x^2$, $x \le 0$

Exer. 3–4: Show that the function f has an inverse function, and find $[(d/dx)(f^{-1}(x))]_{x=a}$ for the given number a.

3 $f(x) = 2x^3 - 8x + 5$, $-1 \le x \le 1$; $a = 5$

4 $f(x) = e^{3x} + 2e^x - 5$, $x \ge 0$; $a = -2$

Exer. 5–26: Find $f'(x)$ if $f(x)$ is the given expression.

5 $\ln|4 - 5x^3|^5$

6 $(1 - 2x)\ln|1 - 2x|$

7 $\ln\dfrac{(3x+2)^4\sqrt{6x-5}}{8x-7}$

8 $\dfrac{\ln x}{e^{2x}+1}$

9 $\dfrac{1}{\ln(2x^2+3)}$

10 $\dfrac{x}{\ln x}$

11 $e^{\ln(x^2+1)}$

12 $\ln(e^{4x} + 9)$

13 $10^x\log x$

14 $5^{3x} + (3x)^5$

15 $\sqrt{\ln\sqrt{x}}$

16 $(1 + \sqrt{x})^e$

17 $x^2e^{-x^2}$

18 $\sqrt{e^{3x} + e^{-3x}}$

19 $10^{\ln x}$

20 $7^{\ln|x|}$

21 $x^{\ln x}$

22 $\ln|\tan x - \sec x|$

23 $\csc e^{-2x}\cot e^{-2x}$

24 $3^{\sin 3x}$

25 $\ln\cos^4 4x$

26 $(\sin x)^{\cos x}$

Exer. 27–28: Use implicit differentiation to find y'.

27 $1 + xy = e^{xy}$

28 $\ln(x + y) + x^2 - 2y^3 = 1$

Exer. 29–30: Use logarithmic differentiation to find dy/dx.

29 $y = (x + 2)^{4/3}(x - 3)^{3/2}$

30 $y = \sqrt[3]{(3x - 1)\sqrt{2x + 5}}$

Exer. 31–54: Evaluate the integral.

31 **(a)** $\displaystyle\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx$ **(b)** $\displaystyle\int_1^4 \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx$

32 **(a)** $\displaystyle\int x4^{-x^2}\,dx$ **(b)** $\displaystyle\int_0^1 x4^{-x^2}\,dx$

33 $\displaystyle\int x\tan x^2\,dx$

34 $\displaystyle\int \cot\left(x + \frac{\pi}{6}\right)dx$

35 $\displaystyle\int x^e\,dx$

36 $\displaystyle\int \frac{1}{x - x\ln x}\,dx$

37 $\displaystyle\int \frac{(1+e^x)^2}{e^{2x}}\,dx$

38 $\displaystyle\int \frac{(e^{2x}+e^{3x})^2}{e^{5x}}\,dx$

39 $\displaystyle\int \frac{x^2}{x+2}\,dx$

40 $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

41 $\displaystyle\int \frac{e^{4/x^2}}{x^3}\,dx$

42 $\displaystyle\int \frac{x}{x^4+2x^2+1}\,dx$

43 $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

44 $\displaystyle\int (1+e^{-3x})^2\,dx$

45 $\displaystyle\int 5^x e^x\,dx$

46 $\displaystyle\int \frac{1}{x\sqrt{\log x}}\,dx$

47 $\displaystyle\int e^{-x}\sin e^{-x}\,dx$

48 $\displaystyle\int \tan x\, e^{\sec x}\sec x\,dx$

49 $\displaystyle\int \frac{\csc^2 x}{1+\cot x}\,dx$

50 $\displaystyle\int \frac{\cos 2x}{1-2\sin 2x}\,dx$

51 $\displaystyle\int e^x\tan e^x\,dx$

52 $\displaystyle\int \frac{\sec(1/x)}{x^2}\,dx$

53 $\displaystyle\int (\csc 3x+1)^2\,dx$

54 $\displaystyle\int (\cot 9x+\csc 9x)\,dx$

55 Solve the differential equation $y'' = -e^{-3x}$ subject to the conditions $y = -1$ and $y' = 2$ if $x = 0$.

56 In *seasonal population growth*, the population $q(t)$ at time t (in years) increases during the spring and summer but decreases during the fall and winter. A differential equation that is sometimes used to describe this type of growth is $q'(t)/q(t) = k\sin 2\pi t$, where $k > 0$ and $t = 0$ corresponds to the first day of spring.

(a) Show that the population $q(t)$ is seasonal.

(b) If $q_0 = q(0)$, find a formula for $q(t)$.

57 A particle moves on a coordinate line with an acceleration at time t of $e^{t/2}$ cm/sec². At $t = 0$, the particle is at the origin and its velocity is 6 cm/sec. How far does it travel during the time interval $[0, 4]$?

58 Find the local extrema of $f(x) = x^2\ln x$ for $x > 0$. Discuss concavity, find the points of inflection, and sketch the graph of f.

59 Find an equation of the tangent line to the graph of the equation $y = xe^{1/x^3} + \ln|2-x^2|$ at the point $P(1, e)$.

60 Find the area of the region bounded by the graphs of the equations $y = e^{2x}$, $y = x/(x^2+1)$, $x = 0$, and $x = 1$.

61 The region bounded by the graphs of $y = e^{4x}$, $x = -2$, $x = -3$, and $y = 0$ is revolved about the x-axis. Find the volume of the resulting solid.

62 The 1993 population estimate for India was 907 million, and the population has been increasing at a rate of about 2% per year, with the rate of increase proportional to the number of people. If t denotes the time (in years) after 1993, find a formula for $N(t)$, the population (in millions) at time t. Assuming that this rapid growth rate continues, estimate the population and the rate of population growth in the year 2000.

63 A radioactive substance has a half-life of 5 days. How long will it take for an amount A to disintegrate to the extent that only 1% of A remains?

64 The carbon-14 dating equation $T = -8310\ln x$ is used to predict the age T (in years) of a fossil in terms of the percentage $100x$ of carbon still present in the specimen (see Exercise 19, Section 6.6).

(a) If $x = 0.04$, estimate the age of the fossil to the nearest 1000 years.

(b) If the maximum error in estimating x in part (a) is ± 0.005, use differentials to approximate the maximum error in T.

65 The rate at which sugar dissolves in water is proportional to the amount that remains undissolved. Suppose that 10 lb of sugar is placed in a container of water at 1:00 P.M., and one half is dissolved at 4:00 P.M.

(a) How long will it take two more pounds to dissolve?

(b) How much of the 10 lb will be dissolved at 8:00 P.M.?

66 According to Newton's law of cooling, the rate at which an object cools is directly proportional to the difference in temperature between the object and its surrounding medium. If $f(t)$ denotes the temperature at time t, show that $f(t) = T + [f(0) - T]e^{-kt}$, where T is the temperature of the surrounding medium and k is a positive constant.

67 The bacterium *E. coli* undergoes cell division approximately every 20 min. Starting with 100,000 cells, determine the number of cells after 2 hr.

c 68 By letting $h = 0.1, 0.01$, and 0.001, predict which of the following expressions gives the best approximation of e for small values of h:

$$(1+h)^{1/h}, \quad (1+h+h^2)^{1/h}, \quad (1+h+\tfrac{1}{2}h^2)^{1/h}$$

Exer. 69–84: Find $f'(x)$ if $f(x)$ is the given expression.

69 $\arctan\sqrt{x-1}$

70 $\tan^{-1}(\ln 3x)$

71 $x^2\,\mathrm{arcsec}\,(x^2)$

72 $2^{\arctan 2x}$

73 $\ln\tan^{-1}(x^2)$

74 $\dfrac{1-x^2}{\arccos x}$

75 $\sin^{-1}\sqrt{1-x^2}$

76 $(\tan x + \tan^{-1} x)^4$

77 $\tan^{-1}(\tan^{-1} x)$

78 $e^{4x}\sec^{-1} e^{4x}$

79 $\cosh e^{-5x}$

80 $e^{-x}\sinh e^{-x}$

81 $\dfrac{\sinh x}{\cosh x - \sinh x}$

82 $\ln\tanh(5x+1)$

83 $\sinh^{-1}(x^2)$

84 $\tanh^{-1}(\tanh\sqrt[3]{x})$

Exer. 85–98: Evaluate the integral.

85 $\displaystyle\int \frac{1}{4+9x^2}\,dx$

86 $\displaystyle\int \frac{x}{4+9x^2}\,dx$

87 $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{2x}}}\,dx$

88 $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

89 $\displaystyle\int \frac{x}{\mathrm{sech}\,(x^2)}\,dx$

90 $\displaystyle\int \frac{\sinh(\ln x)}{x}\,dx$

91 $\displaystyle\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

92 $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

93 $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$

94 $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}\,dx$

95 $\displaystyle\int \frac{1}{x\sqrt{9-4x^2}}\,dx$

96 $\displaystyle\int \frac{1}{x\sqrt{4x^2-9}}\,dx$

97 $\displaystyle\int \frac{x}{\sqrt{25x^2+36}}\,dx$

98 $\displaystyle\int \frac{1}{\sqrt{25x^2+36}}\,dx$

99 Find the points on the graph of $y = \sin^{-1} 3x$ at which the tangent line is parallel to the line through $A(2, -3)$ and $B(4, 7)$.

100 Find the points of inflection, and discuss the concavity of the graph of $y = x\sin^{-1} x$.

101 Find the local extrema of $f(x) = 8\sec x + \csc x$ on the interval $(0, \pi/2)$, and describe where $f(x)$ is increasing or is decreasing on that interval.

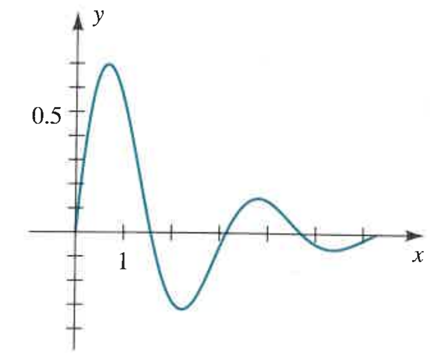
102 Find the area of the region bounded by the graphs of $y = x/(x^4+1)$, $x = 1$, and $y = 0$.

103 Damped oscillations are oscillations of decreasing magnitude that occur when frictional forces are considered. Shown in the figure is a graph of the damped oscillations given by $f(x) = e^{-x/2}\sin 2x$.

(a) Find the x-coordinates of the extrema of f for $0 \le x \le 2\pi$.

(b) Approximate the x-coordinates in part (a) to two decimal places.

Exercise 103



104 Find the arc length of the graph of $y = \ln\tanh\frac{1}{2}x$ from $x = 1$ to $x = 2$.

105 A balloon is released from level ground, 500 m away from a person who observes its vertical ascent. If the balloon rises at a constant rate of 2 m/sec, use inverse trigonometric functions to find the rate at which the angle of elevation of the observer's line of sight is changing at the instant the balloon is at a height of 100 m. (Disregard the observer's height.)

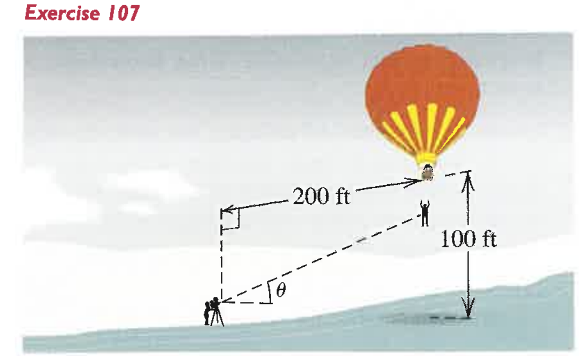
106 A square picture with sides 2 ft long is hung on a wall with the base 6 ft above the floor. A person whose eye level is 5 ft above the floor approaches the picture at a rate of 2 ft/sec. If θ is the angle between the line of sight and the top and bottom of the picture, find

(a) the rate at which θ is changing when the person is 8 ft from the wall

(b) the distance from the wall at which θ has its maximum value

107 A stunt man jumps from a hot-air balloon that is hovering at a constant altitude, 100 ft above a lake. A movie camera on shore, 200 ft from a point directly below the balloon, follows the stunt man's descent (see figure). At what rate is the angle of elevation θ of the camera changing 2 sec after the stunt man jumps? (Disregard the height of the camera.)

Exercise 107

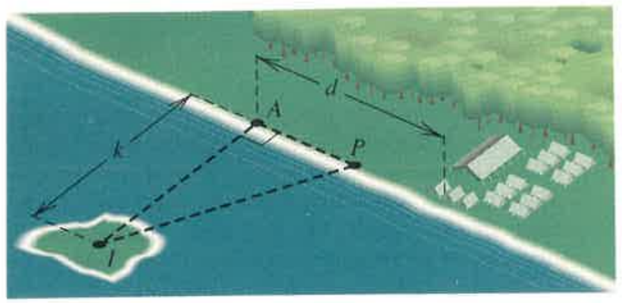


108 A person on a small island I, which is k miles from the closest point A on a straight shoreline, wishes to reach a camp that is d miles downshore from A by swimming to some point P on shore and then walking the rest of the way (see figure). Suppose the person burns c_1 calories per mile while swimming and c_2 calories per mile while walking, where $c_1 > c_2$.

(a) Find a formula for the total number c of calories burned in completing the trip.

(b) For what angle AIP does c have a minimum value?

Exercise 108



Exer. 109–120: Find the limit, if it exists.

109 $\lim_{x\to 0} \dfrac{\ln(2-x)}{1+e^{2x}}$

110 $\lim_{x\to 0} \dfrac{\sin 2x - \tan 2x}{x^2}$

111 $\lim_{x\to\infty} \dfrac{x^2+2x+3}{\ln(x+1)}$

112 $\lim_{x\to 0} \dfrac{e^{2x}-e^{-2x}-4x}{x^3}$

113 $\lim_{x\to\infty} \dfrac{x^e}{e^x}$

114 $\lim_{x\to(\pi/2)^-} \cos x \ln \cos x$

115 $\lim_{x\to\infty} (1-2e^{1/x})x$

116 $\lim_{x\to 0} (1+8x^2)^{1/x^2}$

117 $\lim_{x\to\infty} (e^x+1)^{1/x}$

118 $\lim_{x\to 0^+} \left(\dfrac{1}{\tan x} - \dfrac{1}{x}\right)$

119 $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x}$

120 $\lim_{x\to\infty} \dfrac{3^x+2x}{x^3+1}$

121 Find $\lim_{x\to\infty} f(x)/g(x)$ if $f(x) = \int_1^x (\sin t)^{2/3}\,dt$ and $g(x) = x^2$.

122 Gauss's error integral erf $(x) = (2/\sqrt{\pi})\int_0^x e^{-u^2}\,du$ is used in probability theory. It has the special property $\lim_{x\to\infty}$ erf $(x) = 1$. Find $\lim_{x\to\infty} e^{(x^2)}[1-\text{erf}(x)]$.

EXTENDED PROBLEMS AND GROUP PROJECTS

1 For each positive integer n, let

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

By the use of circumscribed and inscribed rectangles, show that

$$\ln(n+1) < s_n < 1 + \ln n.$$

Estimate the size of $s_{1,000,000}$. How large must n be to guarantee that $s_n > 100$? What happens to s_n as $n \to \infty$?

2 In our development of calculus, we took the trigonometric functions as basic and *then* defined the inverse trigonometric functions. Show that we can give a rigorous development of the ordinary trigonometric functions that reverses the process. In particular, do the following.

(a) Show that the function $f(t) = 1/(1+t^2)$ is continuous and positive for all $t > 0$.

(b) Define the function

$$A(x) = \int_0^x \frac{1}{1+t^2}\,dt.$$

Show that A is differentiable and increasing for all real numbers x. What can you say about the range of the function A?

(c) Define T as the inverse of the function A of part (b). What properties does the function T have? What is $T'(x)$? What are the similarities between $T(x)$ and $\tan x$?

(d) Define S as the inverse of the function B, where

$$B(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt \quad \text{for } -1 < x < 1,$$

and define C as the inverse of the function D, where

$$D(x) = \int_0^x \frac{-1}{\sqrt{1-t^2}}\,dt \quad \text{for } -1 < x < 1.$$

Use the fundamental theorem of calculus and other results to establish properties of the functions S and C. Show that $S'(x) = C(x)$ and $C'(x) = -S(x)$. Is it true that $S^2(x) + C^2(x) = 1$ for all x?

(e) Suppose we define the sine, cosine, and tangent functions by the formulas $\sin x = S(x)$, $\cos x = C(x)$, and $\tan x = T(x)$. Do these functions have all the same properties as the usual sine, cosine, and tangent functions? If so, what are the advantages and the disadvantages of defining the trigonometric functions in this way?

3 To what extent are exponential and logarithmic functions determined by their arithmetic properties? In particular, do the following.

(a) Show that if f is a function such that $f' = f$ and $f(x+y) = f(x)f(y)$ for all x and y, then f must be either the natural exponential function or the function that is identically zero.

(b) Prove that if f is continuous and if $f(x+y) = f(x)f(y)$, then either f is identically zero or $f(x) = [f(1)]^x$ for all x.

(c) Prove that if f is a continuous function defined on the positive real numbers such that $f(xy) = f(x) + f(y)$ for all positive numbers x and y, then f is identically zero or $f(x) = f(e)\ln x$ for all $x > 0$.