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INTRODUCTION

IN 1948, THE FINNISH-BORN AMERICAN ARCHITECT Eero Saarinen (1910–1961) submitted the winning design for a new national park, the Thomas Jefferson Westward Expansion Memorial in St. Louis. The center of his design was a great gleaming stainless-steel arch. Saarinen wanted "to create a monument which would have lasting significance and would be a landmark of our time. An absolutely simple shape . . . seemed to be the basis of the great memorials that have kept their significance and dignity over time." Saarinen designed his arch to be the purest expression of the forces within. This arch . . . is a catenary curve—the curve of a hanging chain—a curve in which the forces of thrust are continuously kept within the center of the legs of the arch. The mathematical precision seemed to enhance the timelessness of the form, but at the same time its dynamic quality seemed to link it to our own time.

To understand the mathematics of Saarinen's Gateway Arch to the West, we need to examine the natural exponential function. This function and its inverse, the natural logarithm, are perhaps the most important functions in applications of calculus to the natural world. They are examples of *transcendental functions*, the main topic of this chapter. We begin in Section 6.1 with a brief review of inverse functions and develop a formula for the derivative of an inverse function that will be useful throughout the entire chapter. Next, we employ a definite integral to introduce in Section 6.2 the *natural logarithm function*, which is then used to define in Section 6.3 the *natural exponential function* as the inverse of the natural logarithm. The natural logarithmic and exponential functions occur in many indefinite integral problems, a number of which are studied in Section 6.4. There are many other pairs of exponential and logarithmic functions; we analyze the general case in Section 6.5. After developing the theory of logarithms and exponentials, we explore in Section 6.6 a number of applications that involve these functions as solutions to first-order separable differential equations, an important modeling tool.

In Sections 6.7 and 6.8, we introduce other important transcendental functions: the inverse trigonometric functions and the hyperbolic functions and their inverses. We derive the equation for the catenary curve as an application of the hyperbolic functions. The chapter concludes with l'Hôpital's rule, which provides a direct way to evaluate limits of quotients in which both the numerator and the denominator approach 0 or both approach ∞ or $-\infty$. Such limits often occur when dealing with transcendental functions.

Transcendental functions frequently occur in the descriptions of curves that possess both aesthetic appeal and important structural properties of stability.

CHAPTER 6

Transcendental Functions

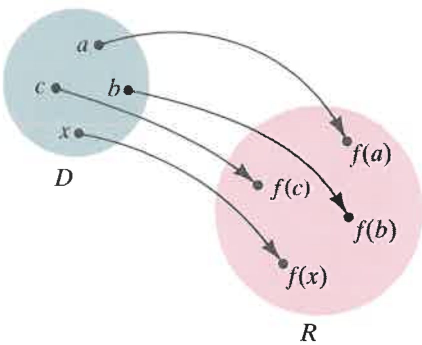
6.1 THE DERIVATIVE OF THE INVERSE FUNCTION

A function f may have the same value for different numbers in its domain. For example, if $f(x) = x^2$, then $f(2) = 4 = f(-2)$ but $2 \neq -2$. In order to define the *inverse of a function*, it is essential that different numbers in the domain *always* give different values of f. Such functions are called *one-to-one functions*.

Definition 6.1

> A function f with domain D and range R is a **one-to-one function** if whenever $a \neq b$ in D, then $f(a) \neq f(b)$ in R.

Figure 6.1



The diagram in Figure 6.1 illustrates a one-to-one function, because each function value in the range R corresponds to *exactly one* element in the domain D. The function whose graph is illustrated in Figure 6.2 is *not* one-to-one, because $a \neq b$ but $f(a) = f(b)$. Note that the horizontal line $y = f(a)$ (or $y = f(b)$) intersects the graph in more than one point. Thus, *if any horizontal line intersects the graph of a function f in more than one point, then f is not one-to-one*. Every increasing function is one-to-one, because if $a < b$, then $f(a) < f(b)$, and if $b < a$, then $f(b) < f(a)$. Thus, if $a \neq b$, then $f(a) \neq f(b)$. Similarly, every decreasing function is one-to-one.

If f is a one-to-one function with domain D and range R, then for each number y in R, there is *exactly one* number x in D such that $y = f(x)$, as illustrated by the arrow in Figure 6.3(a). Since x is *unique*, we may define a function g from R to D by means of the rule $x = g(y)$. As in Figure 6.3(b), *g reverses the correspondence given by f*. We call g the *inverse function* of f, as in the following definition.

Figure 6.2

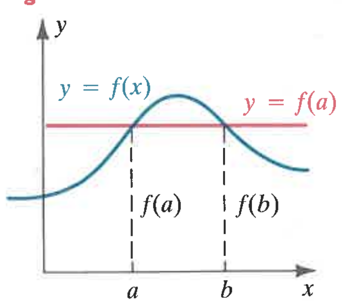
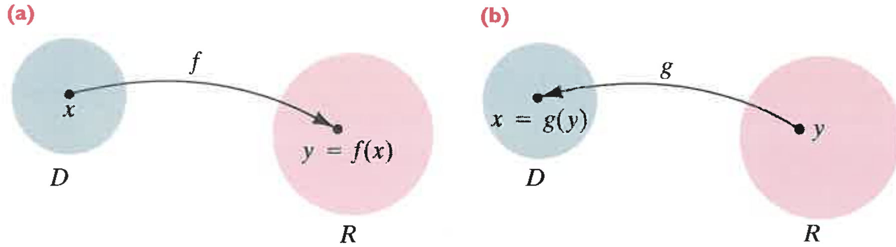


Figure 6.3
(a) (b)



Definition 6.2

> Let f be a one-to-one function with domain D and range R. A function g with domain R and range D is the **inverse function** of f, provided the following condition is true for every x in D and every y in R:
>
> $$y = f(x) \quad \text{if and only if} \quad x = g(y)$$

The following theorem can be used to verify that a function g is the inverse of f.

Theorem 6.3

> Let f be a one-to-one function with domain D and range R. If g is a function with domain R and range D, then g is the inverse function of f if and only if both of the following conditions are true:
>
> **(i)** $g(f(x)) = x$ for every x in D
>
> **(ii)** $f(g(y)) = y$ for every y in R

PROOF Let us first prove that if g is the inverse function of f, then conditions (i) and (ii) are true. By the definition of an inverse function,

$$y = f(x) \quad \text{if and only if} \quad x = g(y)$$

for every x in D and every y in R. If we substitute $f(x)$ for y in the equation $x = g(y)$, we obtain condition (i): $x = g(f(x))$. Similarly, if we substitute $g(y)$ for x in the equation $y = f(x)$, we obtain condition (ii): $y = f(g(y))$. Thus, if g is the inverse function of f, then conditions (i) and (ii) are true.

Conversely, let g be a function with domain R and range D, and suppose that conditions (i) and (ii) are true. To show that g is the inverse function of f, we must prove that

$$y = f(x) \quad \text{if and only if} \quad x = g(y)$$

for every x in D and every y in R.

First suppose that $y = f(x)$. Since (i) is true, $g(f(x)) = x$—that is, $g(y) = x$. Thus, if $y = f(x)$, then $x = g(y)$.

Next suppose that $x = g(y)$. Since (ii) is true, $f(g(y)) = y$—that is, $f(x) = y$. Thus, if $x = g(y)$, then $y = f(x)$, which completes the proof. ■

A one-to-one function f can have only one inverse function. Conditions (i) and (ii) of Theorem (6.3) imply that if g is the inverse function of f, then f is the inverse function of g. We say that *f and g are inverse functions of each other.*

If a function f has an inverse function g, we often denote g by f^{-1}. The -1 used in this notation should not be mistaken for an exponent—that is, $f^{-1}(y)$ *does not mean* $1/[f(y)]$. The reciprocal $1/[f(y)]$ may be denoted by $[f(y)]^{-1}$. It is important to remember the following relationships.

Domains and Ranges of f and f^{-1} 6.4

> domain of f^{-1} = range of f
>
> range of f^{-1} = domain of f

When we discuss functions, we often let x denote an arbitrary number in the domain. Thus, for the inverse function f^{-1}, we may consider

$f^{-1}(x)$, *where* x *is in the domain of* f^{-1}. In this case, the two conditions in Theorem (6.3) are written as follows:

(i) $f^{-1}(f(x)) = x$ for every x in the domain of f

(ii) $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

In some cases, we can find the inverse of a one-to-one function by solving the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = g(y)$. If the two conditions $g(f(x)) = x$ and $f(g(x)) = x$ are true for every x in the domains of f and g, respectively, then g is the required inverse function f^{-1}. The following guidelines summarize this procedure. In guideline (2), in anticipation of finding f^{-1}, we shall write $x = f^{-1}(y)$ instead of $x = g(y)$.

Guidelines for Finding f^{-1} in Simple Cases 6.5

1. Verify that f is a one-to-one function (or that f is increasing or is decreasing) throughout its domain.
2. Solve the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = f^{-1}(y)$.
3. Verify the two conditions

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

for every x in the domains of f and f^{-1}, respectively.

The success of this method depends on the nature of the equation $y = f(x)$, since we must be able to solve for x in terms of y. For this reason, we include *simple cases* in the title of the guidelines.

EXAMPLE 1 Let $f(x) = 3x - 5$. Find the inverse function of f.

SOLUTION We shall follow the three guidelines. First, we note that the graph of the linear function f is a line of slope 3. Since f is increasing throughout $\mathbb{R}$, f is one-to-one, and hence the inverse function f^{-1} exists. Moreover, since the domain and the range of f are $\mathbb{R}$, the same is true for f^{-1}.

As in guideline (2), we consider the equation

$$y = 3x - 5$$

and then solve for x in terms of y, obtaining

$$x = \frac{y+5}{3}.$$

We now let

$$f^{-1}(y) = \frac{y+5}{3}.$$

Since the symbol used for the variable is immaterial, we may also write

$$f^{-1}(x) = \frac{x+5}{3}.$$

We next verify the conditions (i) $f^{-1}(f(x)) = x$ and (ii) $f(f^{-1}(x)) = x$:

(i)
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(3x-5) && \text{definition of } f \\ &= \frac{(3x-5)+5}{3} && \text{definition of } f^{-1} \\ &= x && \text{simplifying} \end{aligned}$$

(ii)
$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{x+5}{3}\right) && \text{definition of } f^{-1} \\ &= 3\left(\frac{x+5}{3}\right) - 5 && \text{definition of } f \\ &= x && \text{simplifying} \end{aligned}$$

Thus, by Theorem (6.3), the inverse function of f is given by $f^{-1}(x) = (x+5)/3$.

EXAMPLE 2 Let $f(x) = x^2 - 3$ for $x \geq 0$. Find the inverse function of f.

SOLUTION The graph of f is sketched in Figure 6.4. The domain of f is $[0, \infty)$, and the range is $[-3, \infty)$. Since f is increasing, it is one-to-one and hence has an inverse function f^{-1} that has domain $[-3, \infty)$ and range $[0, \infty)$.

As in guideline (2), we consider the equation

$$y = x^2 - 3$$

and solve for x, obtaining

$$x = \pm\sqrt{y+3}.$$

Since x is nonnegative, we reject $x = -\sqrt{y+3}$ and let

$$f^{-1}(y) = \sqrt{y+3}, \quad \text{or, equivalently,} \quad f^{-1}(x) = \sqrt{x+3}.$$

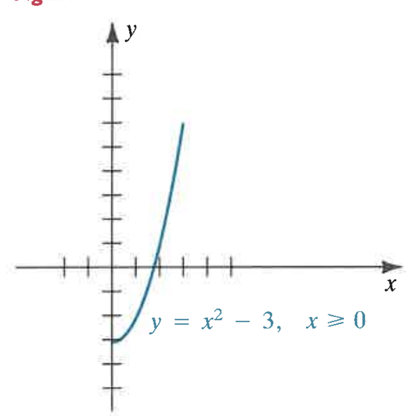
Finally, we verify that (i) $f^{-1}(f(x)) = x$ for x in $[0, \infty)$ and that (ii) $f(f^{-1}(x)) = x$ for x in $[-3, \infty)$:

(i)
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(x^2-3) \\ &= \sqrt{(x^2-3)+3} = \sqrt{x^2} = |x| = x \quad \text{if } x \geq 0 \end{aligned}$$

(ii)
$$\begin{aligned} f(f^{-1}(x)) &= f(\sqrt{x+3}) \\ &= (\sqrt{x+3})^2 - 3 = (x+3) - 3 = x \quad \text{if } x \geq -3 \end{aligned}$$

Thus, the inverse function is given by $f^{-1}(x) = \sqrt{x+3}$ for $x \geq -3$.

Figure 6.4



There is an interesting relationship between the graph of a function f and the graph of its inverse function f^{-1}. We first note that $b = f(a)$ is equivalent to $a = f^{-1}(b)$. These equations imply that *the point* (a, b) *is on the graph of* f *if and only if the point* (b, a) *is on the graph of* f^{-1}.

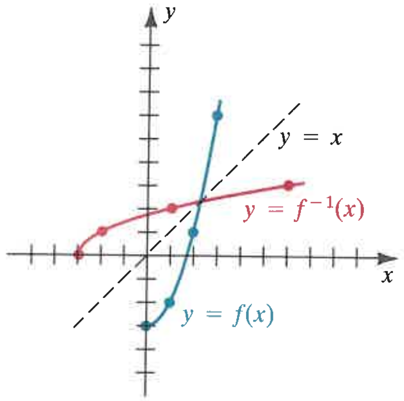


Figure 6.5

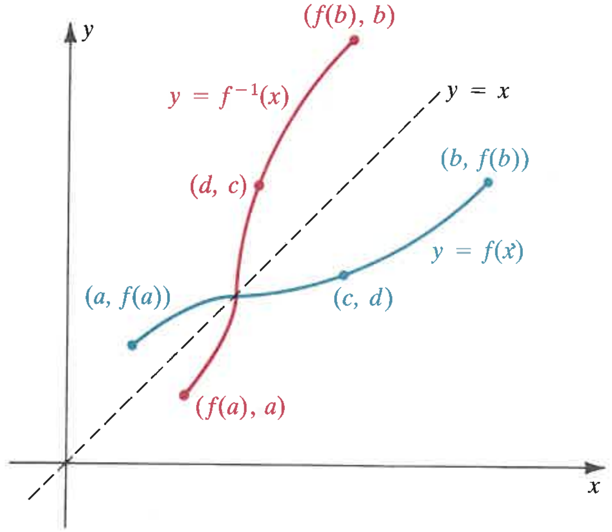
As an illustration, in Example 2 we found that the functions f and f^{-1} given by

$$f(x) = x^2 - 3 \quad \text{and} \quad f^{-1}(x) = \sqrt{x+3}$$

are inverse functions of each other, provided that x is suitably restricted. Some points on the graph of f are $(0, -3)$, $(1, -2)$, $(2, 1)$, and $(3, 6)$. Corresponding points on the graph of f^{-1} are $(-3, 0)$, $(-2, 1)$, $(1, 2)$, and $(6, 3)$. The graphs of f and f^{-1} are sketched on the same coordinate plane in Figure 6.5. If the page is folded along the line $y = x$ that bisects quadrants I and III (as indicated by the dashed line in the figure), then the graphs of f and f^{-1} coincide. The two graphs are *reflections* of each other through the line $y = x$. This reflective property is typical of the graph of every function f that has an inverse function f^{-1}(see Exercise 14).

Figure 6.6 illustrates the graphs of an arbitrary one-to-one function f and its inverse function f^{-1}. As indicated in the figure, (c, d) is on the graph of f if and only if (d, c) is on the graph of f^{-1}. Thus, if we restrict the domain of f to the interval $[a, b]$, then the domain of f^{-1} is restricted to $[f(a), f(b)]$. If f is continuous, then the graph of f has no breaks or holes, and hence the same is true for the (reflected) graph of f^{-1}. Thus, we see intuitively that if f is continuous on $[a, b]$, then f^{-1} is continuous on $[f(a), f(b)]$. We can also show that if f is increasing, then so is f^{-1}. The next theorem states these facts, and Appendix I contains a proof.

Figure 6.6



Theorem 6.6

> If f is continuous and increasing on $[a, b]$, then f has an inverse function f^{-1} that is continuous and increasing on $[f(a), f(b)]$.

We can also prove the analogous result obtained by replacing the word *increasing* in Theorem (6.6) with the word *decreasing*.

The next theorem provides a method for finding the derivative of an inverse function.

Theorem 6.7

> If a differentiable function f has an inverse function $g = f^{-1}$ and if $f'(g(c)) \neq 0$, then g is differentiable at c and
>
> $$g'(c) = \frac{1}{f'(g(c))}.$$

PROOF By Definition (2.6),

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

Let $y = g(x)$ and $a = g(c)$. Since f and g are inverse functions of each other,

$$g(x) = y \quad \text{if and only if} \quad f(y) = x$$

and

$$g(c) = a \quad \text{if and only if} \quad f(a) = c.$$

Because f is differentiable, it is continuous and hence, by Theorem (6.6), so is the inverse function $g = f^{-1}$. Thus, if $x \to c$, then $g(x) \to g(c)$; that is, $y \to a$. If $y \to a$, then $f(y) \to f(a)$. Thus, we may write

$$\begin{aligned} g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{y \to a} \frac{y - a}{f(y) - f(a)} \\ &= \lim_{y \to a} \frac{1}{\dfrac{f(y) - f(a)}{y - a}} \\ &= \frac{1}{\displaystyle\lim_{y \to a} \frac{f(y) - f(a)}{y - a}} \\ &= \frac{1}{f'(a)} = \frac{1}{f'(g(c))}. \end{aligned}$$ ■

It is convenient to restate Theorem (6.7) as follows.

Corollary 6.8

> If g is the inverse function of a differentiable function f and if $f'(g(x)) \neq 0$, then
>
> $$g'(x) = \frac{1}{f'(g(x))}.$$

Theorem (6.7) and Corollary (6.8) are useful because they enable us to compute the derivative of the inverse of a function without having an explicit formula for the inverse function. In the next example, we need the derivative of an inverse function so that we can find the slope of the tangent line to a point on its graph.

EXAMPLE ■ 3 If $f(x) = x^3 + 2x - 1$, prove that f has an inverse function g, and find the slope of the tangent line to the graph of g at the point $P(2, 1)$.

SOLUTION Since $f'(x) = 3x^2 + 2 > 0$ for every x, f is increasing and hence is one-to-one. Thus, f has an inverse function g. Since $f(1) = 2$, it follows that $g(2) = 1$, and consequently the point $P(2, 1)$ is on the graph of g. It would be difficult to find g using Guidelines (6.5), because we would have to solve the equation $y = x^3 + 2x - 1$ for x in terms of y. However, even if we cannot find g explicitly, we *can* find the slope $g'(2)$ of the tangent line to the graph of g at $P(2, 1)$. Thus, by Theorem (6.7),

$$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

An easy way to remember Corollary (6.8) is to let $y = f(x)$. If g is the inverse function of f, then $g(y) = g(f(x)) = x$. From (6.8),

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{f'(x)}.$$

This shows that, in a sense, the derivative of the inverse function g is the reciprocal of the derivative of f. A disadvantage of this formula is that it is not stated in terms of the independent variable for the inverse function. To illustrate, in Example 3, let $y = x^3 + 2x - 1$ and $x = g(y)$. Then

$$g'(y) = \frac{1}{3x^2 + 2} = \frac{1}{3(g(y))^2 + 2}.$$

We may also write this in the form

$$g'(x) = \frac{1}{3(g(x))^2 + 2}.$$

Consequently, to find $g'(x)$, it is necessary to know $g(x)$, just as in Corollary (6.8).

We may use a graphing utility to obtain the graphs of a function and its inverse simultaneously by exploiting the result that the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. Once the utility has plotted a point (a, b), we ask it to plot the point (b, a) as well. The formal mechanism for achieving this result utilizes *parametric equations*, which we will study in more depth in Chapter 9. For now, we represent the graph of $y = f(x)$ for some domain $a \le x \le b$ by the points $(t, f(t))$ for $a \le t \le b$. The variable t is called a *parameter*. We actually have a pair of parametric equations:

$$x = t, \qquad y = f(t) \quad \text{for} \quad a \le t \le b$$

The graph of the inverse function (if it exists) is obtained by reversing the roles of x and y. This reversal is easily accomplished by plotting a second pair of parametric equations:

$$x = f(t), \qquad y = t, \quad \text{for} \quad a \le t \le b$$

If f is not one-to-one on the interval $[a, b]$, then the plot of this second pair of equations is not the graph of a function. To see that the plot of the first pair of equations is a reflection of the second pair about the line $y = x$, we also plot the line by using a third pair of parametric equations:

$$x = t, \qquad y = t, \quad \text{for} \quad a \le t \le b$$

In using a graphing utility, we must indicate by special notation or command that what is being requested is the plotting of parametric equations.

EXAMPLE ■ 4

(a) Use a graphing utility and parametric equations to view the graphs of the function f given by $f(x) = x^3 + 0.3x - 2$, the inverse of f, and the line $y = x$ for $-1 \le x \le 2$.

(b) Verify that the function f is one-to-one on this interval.

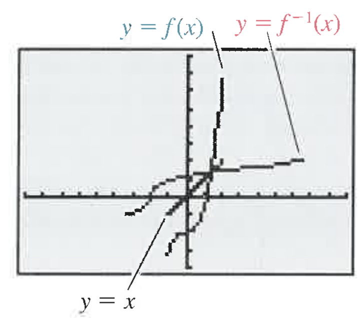
SOLUTION

(a) We set the graphing utility to plot the following parametric equations:

$X_{1T} = T$,	$Y_{1T} = T \wedge 3 + 0.3T - 2$	equations for f
$X_{2T} = Y_{1T}$,	$Y_{2T} = T$	equations for its inverse
$X_{3T} = T$,	$Y_{3T} = T$	equations for the line $y = x$

Note that we use an uppercase T instead of the lower-case t since most graphing calculators use uppercase letters. The graphing utility plots each of the points $(t, t^3 + 0.3t - 2)$, $(t^3 + 0.3t - 2, t)$, and (t, t) for each t in the interval $-1 \le t \le 2$ to produce the graphs shown in Figure 6.7.

Figure 6.7
$-9 \le x \le 9, -4 \le y \le 8$



(b) Both a visual inspection of the graph and observation of the fact that $f'(x) = 3x^2 + 0.3 > 0$ confirm that f is strictly increasing on $[-1, 2]$, and hence f is one-to-one on this interval.

EXAMPLE ■ 5

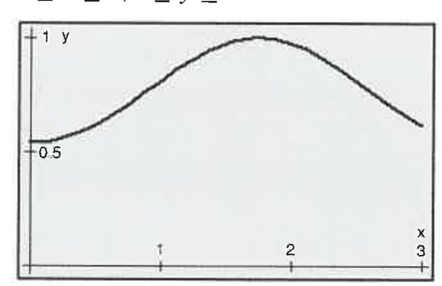
(a) Graph $f(x) = \cos[\cos(0.9x)]$ on the interval $[0, 3]$.

(b) Estimate the largest interval $[a, b]$ with $0 \le a < b \le 3$ on which f is one-to-one.

(c) If g is the function with domain interval $[a, b]$ such that $g(x) = f(x)$ for $a \le x \le b$, estimate the domain and the range of g^{-1}.

(d) Make use of parametric equations to view the graphs of g and g^{-1} on the same coordinate axes.

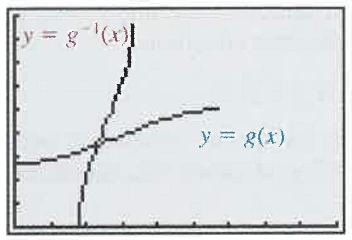
Figure 6.8
$0 \le x \le 3, 0 \le y \le 1$



SOLUTION

(a) Using a graphing utility gives the graph shown in Figure 6.8.

Figure 6.9
$0 \le x \le 2.7, 0 \le y \le 1.8$



(b) From the graph in part (a), we see that as x increases from 0, the function f initially increases from a value of $\cos(\cos 0) \approx 0.5403023$ until it reaches a maximum value and then begins to decrease. Using the trace feature on the graphing utility, we find that the maximum occurs at approximately $x = 1.7453293$ where the value of the function is $\cos[\cos(0.9)(1.7453293)] \approx 1$. Thus, f is one-to-one on the interval $[0, 1.7453293]$ and also on the interval $[1.7453293, 3]$. Since the first interval is longer, we select $[a, b] = [0, 1.7453293]$.

(c) From the analysis in part (b), we have $g(x) = \cos[\cos(0.9x)]$ with domain $[0, 1.7453293]$ and range $[0.5403023, 1]$. Hence, the inverse g^{-1} has domain $[0.5403023, 1]$ and range $[0, 1.7453293]$.

(d) We use parametric equations to generate the points $(t, \cos[\cos(0.9t)])$ and $(\cos[\cos(0.9t)], t)$ for $0 \le t \le 1.7453293$ to obtain the graphs of g and g^{-1} shown in Figure 6.9.

EXERCISES 6.1

Exer. 1–12: Find $f^{-1}(x)$.

1 $f(x) = 3x + 5$

2 $f(x) = 7 - 2x$

3 $f(x) = \dfrac{1}{3x - 2}$

4 $f(x) = \dfrac{1}{x + 3}$

5 $f(x) = \dfrac{3x + 2}{2x - 5}$

6 $f(x) = \dfrac{4x}{x - 2}$

7 $f(x) = 2 - 3x^2, \quad x \le 0$

8 $f(x) = 5x^2 + 2, \quad x \ge 0$

9 $f(x) = \sqrt{3 - x}$

10 $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

11 $f(x) = \sqrt[3]{x} + 1$

12 $f(x) = (x^3 + 1)^5$

13 **(a)** Prove that the linear function defined by $f(x) = ax + b$ with $a \ne 0$ has an inverse function, and find $f^{-1}(x)$.

(b) Does a constant function have an inverse? Explain.

14 Show that the graph of f^{-1} is the reflection of the graph of f through the line $y = x$ by verifying the following conditions:

(i) If $P(a, b)$ is on the graph of f, then $Q(b, a)$ is on the graph of f^{-1}.

(ii) The midpoint of line segment PQ is on the line $y = x$.

(iii) The line PQ is perpendicular to the line $y = x$.

Exer. 15–18: The graph of a one-to-one function f is shown in the figure. (a) Use a reflection to sketch the graph of f^{-1}. (b) Find the domain and the range of f. (c) Find the domain and the range of f^{-1}.

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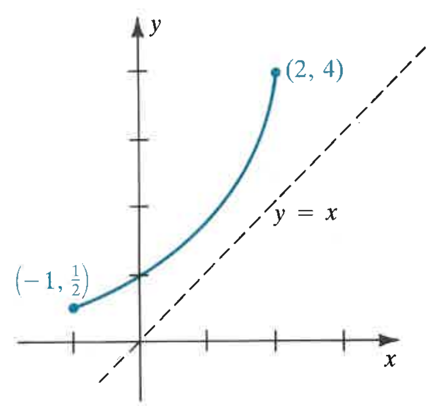

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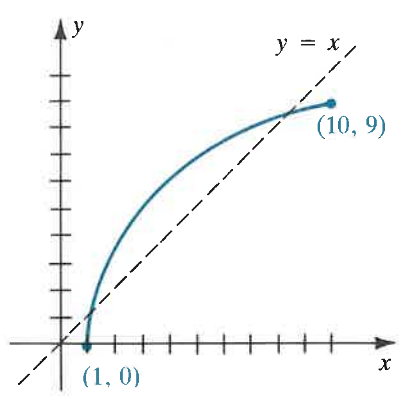

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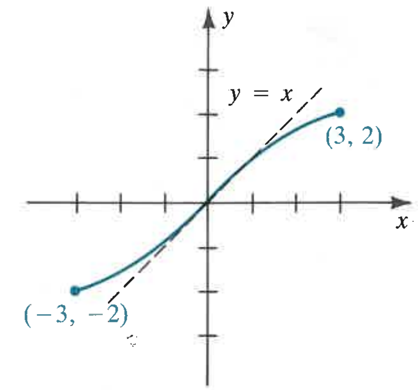

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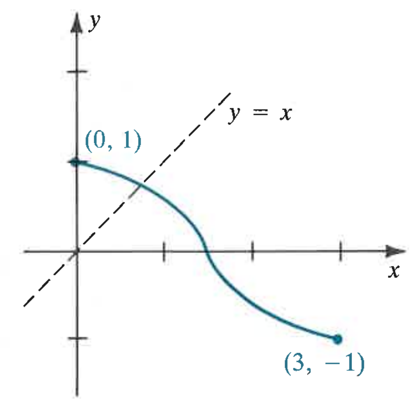


c **Exer. 19–24: Graph f on the given interval. (a) Estimate the largest interval $[a, b]$ with $a < b$ on which f is one-to-one. (b) If g is the function with domain $[a, b]$ such that $g(x) = f(x)$ for $a \le x \le b$, estimate the domain and the range of g^{-1}. (c) Use parametric equations to view the graphs of g and g^{-1} on the same coordinate axes.**

19 $f(x) = 2.1x^3 - 2.98x^2 - 2.11x + 3; \quad [-1, 2]$

20 $f(x) = 16x^5 + 8x^4 - 20x^3 - 8x^2 + 5x + 1; \quad [-1, 1]$

21 $f(x) = \sin[\sin(1.1x)]; \quad [-2, 2]$

22 $f(x) = \sin(x^3 + 2x^2 - 0.3); \quad [-1, 1]$

23 $f(x) = 2^{\cos(x-1)}; \quad [-3, 2]$

24 $f(x) = 3^{(2x^2-3x-1)}; \quad [-1, 2]$

Exer. 25–30: (a) Prove that f has an inverse function g. (b) State the domain of g. (c) Use Corollary (6.8) to find $g'(x)$.

25 $f(x) = \sqrt{2x + 3}$

26 $f(x) = \sqrt[3]{5x + 2}$

27 $f(x) = 4 - x^2, \quad x \ge 0$

28 $f(x) = x^2 - 4x + 5, \quad x \ge 2$

29 $f(x) = 1/x, \quad x \ne 0$

30 $f(x) = \sqrt{9 - x^2}, \quad 0 \le x \le 3$

Exer. 31–36: (a) Use f' to prove that f has an inverse function. (b) Find the slope of the tangent line at the point P on the graph of f^{-1}.

31 $f(x) = x^5 + 3x^3 + 2x - 1; \quad P(5, 1)$

32 $f(x) = 2 - x - x^3; \quad P(-8, 2)$

33 $f(x) = -2x + (8/x^3), \quad x > 0; \quad P(-3, 2)$

34 $f(x) = 4x^5 - (1/x^3), \quad x > 0; \quad P(3, 1)$

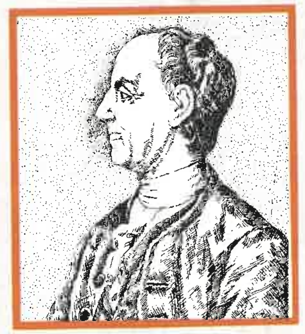
35 $f(x) = x^3 + 4x - 1; \quad P(15, 2)$

36 $f(x) = x^5 + x; \quad P(2, 1)$

Mathematicians and Their Times

LEONHARD EULER

IN THE EIGHTEENTH CENTURY, Europe witnessed a growing conflict between religion and science and the first of the political revolutions that overturned monarchies and founded new democratic forms of government. In his own life, Leonhard Euler (1707–1783), the leading mathematician and theoretical physicist of his time, balanced traditional religious beliefs with the demanding rationalist logic of deductive thought.



Euler was the most prolific mathematician in history, perhaps the most prolific author in any field. His writings fill 100 large books and contain contributions to mechanics, optics, acoustics, hydrodynamics, astronomy, chemistry, and medicine, as well as especially profound work in every branch of pure and applied mathematics. Born in Switzerland, Euler first studied to become a Calvinist minister as his father was, but mathematics led him down another path.

As a young man, Euler lost the sight in one eye, from an illness brought on by prolonged scientific work. Later in life, cataracts took the vision in his other eye. Although he was completely blind for 17 years, Euler's research never slackened. Blessed with a phenomenal memory and the ability to concentrate on difficult problems while surrounded by playing children (he had 13!), Euler could accurately complete complex problems mentally. As the physicist Arago noted, "He calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind."

In Euler's time, the major centers of scientific research were often academies funded by royalty. As a young student of John Bernoulli, Euler became friends with Bernoulli's sons Daniel and Nicolaus, who helped him secure a position at the Russian Academy in St. Petersburg. Euler remained there from 1727 until 1741 when Frederick the Great invited him to join the Berlin Academy. Euler did most of his best work during the quarter century he spent in Berlin. Since relations with Frederick were never very cordial (the emperor derided Euler as a "mathematical cyclops"), Euler gladly accepted the invitation of Catherine the Great to return to St. Petersburg in 1766.

Besides hundreds of research monographs, Euler also wrote influential mathematical textbooks at all levels. He introduced or established the use of many common mathematical notations: $f(x)$ for a function of x, π, e, $\sum$, $\log x$, $\sin x$, $\cos x$, and i. He also discovered the relationship

$$e^{\pi i} + 1 = 0,$$

the "mystical formula" linking the five most significant numbers in mathematics. George F. Simmons accurately describes Euler as "the Shakespeare of mathematics—universal, richly detailed, and inexhaustible."

6.2 THE NATURAL LOGARITHM FUNCTION

In this section, we define the natural logarithm function as a definite integral. At first you may think it strange to do so; later, however, you will see that the function we obtain obeys the familiar laws of logarithms considered in precalculus courses.

Let f be a function that is continuous on a closed interval $[a, b]$. As in the proof of Part I of the fundamental theorem of calculus (4.30), we can define a function F by

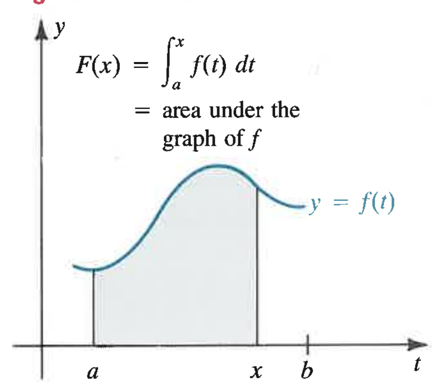
$$F(x) = \int_a^x f(t)\,dt$$

for x in $[a, b]$. If $f(t) \geq 0$ throughout $[a, b]$, then $F(x)$ is the area under the graph of f from a to x, as illustrated in Figure 6.10. For the special case $f(t) = t^n$, where n is a rational number and $n \neq -1$, we can find an explicit form for F. Thus, by the power rule for integrals,

$$F(x) = \int_a^x t^n\,dt = \left[\frac{t^{n+1}}{n+1}\right]_a^x$$

$$= \frac{1}{n+1}\left(x^{n+1} - a^{n+1}\right) \quad \text{if} \quad n \neq -1.$$

Figure 6.10



As indicated, we cannot use $t^{-1} = 1/t$ for the integrand, since $1/(n+1)$ is undefined if $n = -1$. Up to this point in our work, we have been unable to determine an antiderivative of $1/x$—that is, a function F such that $F'(x) = 1/x$. The next definition will remedy this situation.

Definition 6.9

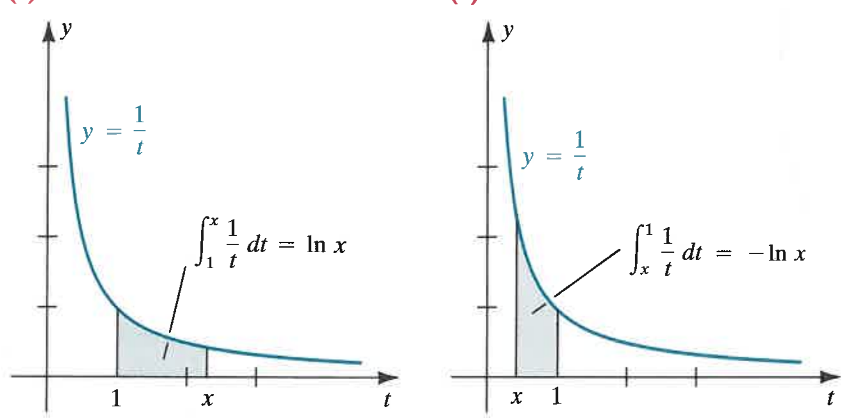
> The **natural logarithm function**, denoted by **ln**, is defined by
> $$\ln x = \int_1^x \frac{1}{t}\,dt$$
> for every $x > 0$.

The expression $\ln x$ (read *ell-en of* x) is called the **natural logarithm of** $\boldsymbol{x}$. We use this terminology because, as we shall see, ln has the same properties as the logarithmic functions studied in precalculus courses. The restriction $x > 0$ is necessary because if $x \le 0$, the integrand $1/t$ has an infinite discontinuity between x and 1 and hence $\int_1^x (1/t)\,dt$ does not exist.

If $x > 1$, the definite integral $\int_1^x (1/t)\,dt$ may be interpreted as the area of the region under the graph of $y = 1/t$ from $t = 1$ to $t = x$ (see Figure 6.11a).

Figure 6.11

(a) (b)



If $0 < x < 1$, then, since
$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt,$$
the integral is the *negative* of the area of the region under the graph of $y = 1/t$ from $t = x$ to $t = 1$ (see Figure 6.11b). Thus, $\ln x$ *is negative for* $0 < x < 1$ *and positive for* $x > 1$. Also note that, by Definition (4.18),
$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

Applying Theorem (4.35) yields
$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$
for every $x > 0$. Substituting $\ln x$ for $\int_1^x (1/t)\,dt$ gives us the following theorem.

Theorem 6.10

> $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By Theorem (6.10), $\ln x$ *is an antiderivative of* $1/x$. Since $\ln x$ is differentiable and its derivative $1/x$ is positive for every $x > 0$, it follows from Theorems (2.12) and (3.15) that *the natural logarithmic function is continuous and increasing throughout its domain.* Also note that
$$\frac{d^2}{dx^2}(\ln x) = \frac{d}{dx}\left(\frac{d}{dx}(\ln x)\right) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2},$$
which is negative for every $x > 0$. Hence, by (3.18), the graph of the natural logarithmic function is concave downward on $(0, \infty)$.

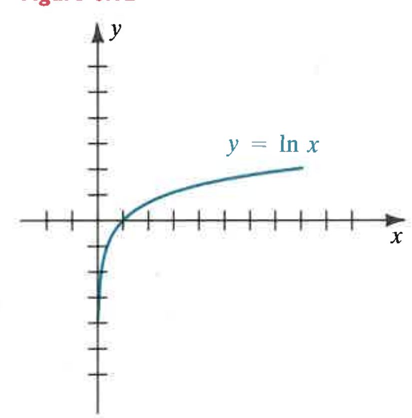
Let us sketch the graph of $y = \ln x$. If $0 < x < 1$, then $\ln x < 0$ and the graph is below the x-axis. If $x > 1$, the graph is above the x-axis. Since $\ln 1 = 0$, the x-intercept is 1. We may approximate y-coordinates of points on the graph by applying the trapezoidal rule or Simpson's rule. If $x = 2$, then, by Example 3 in Section 4.7,
$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx 0.693.$$

We will show in Theorem (6.12) that if $a > 0$, then $\ln a^r = r \ln a$ for every rational number r. Using this result yields the following:
$$\begin{aligned}
\ln 4 &= \ln 2^2 = 2\ln 2 \approx 2(0.693) \approx 1.386\\
\ln 8 &= \ln 2^3 = 3\ln 2 \approx 2.079\\
\ln \tfrac{1}{2} &= \ln 2^{-1} = -\ln 2 \approx -0.693\\
\ln \tfrac{1}{4} &= \ln 2^{-2} = -2\ln 2 \approx -1.386\\
\ln \tfrac{1}{8} &= \ln 2^{-3} = -3\ln 2 \approx -2.079
\end{aligned}$$

Plotting the points that correspond to the y-coordinates we have calculated and using the fact that ln is continuous and increasing gives us the sketch in Figure 6.12.

At the end of this section, we prove that
$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty.$$

Figure 6.12



The first of these results tells us that $y = \ln x$ increases without bound as $x \to \infty$. Note, however, that the *rate of change* of y with respect to x is very small if x is large. For example, if $x = 10^6$, then
$$\left.\frac{dy}{dx}\right]_{10^6} = \left.\frac{1}{x}\right]_{10^6} = \frac{1}{10^6} = 0.000001.$$

Thus, the tangent line is *almost* horizontal at the point on the graph with x-coordinate 10^6, and hence the graph is very flat near that point. The fact that $\lim_{x\to 0^+} \ln x = -\infty$ tells us that the line $x = 0$ (the y-axis) is a vertical asymptote for the graph (see Figure 6.12).

The next result generalizes Theorem (6.10).

Theorem 6.11

If $u = g(x)$ and g is differentiable, then

(i) $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u}\dfrac{du}{dx}$ if $u > 0$

(ii) $\dfrac{d}{dx}(\ln |u|) = \dfrac{1}{u}\dfrac{du}{dx}$ if $u \neq 0$

PROOF

(i) If we let $y = \ln u$ and $u = g(x)$, then, by the chain rule and Theorem (6.10),

$$\frac{d}{dx}(\ln u) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

(ii) If $u > 0$, then $|u| = u$ and, by part (i),

$$\frac{d}{dx}(\ln |u|) = \frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}.$$

If $u < 0$, then $|u| = -u > 0$ and, by part (i),

$$\frac{d}{dx}(\ln |u|) = \frac{d}{dx}(\ln(-u)) = \frac{1}{-u}\frac{d}{dx}(-u)$$
$$= -\frac{1}{u}(-1)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

In examples and exercises, if a function is defined in terms of the natural logarithm function, its domain will not usually be stated explicitly. Instead *we shall tacitly assume that x is restricted to values for which the logarithmic expression has meaning*. Thus, in Example 1, we assume $x^2 - 6 > 0$; that is, $|x| > \sqrt{6}$. In Example 2, we assume $x + 1 > 0$.

EXAMPLE 1 If $f(x) = \ln(x^2 - 6)$, find $f'(x)$.

SOLUTION Letting $u = x^2 - 6$ in Theorem (6.11)(i) yields

$$f'(x) = \frac{d}{dx}\left(\ln(x^2 - 6)\right) = \frac{1}{x^2 - 6}\frac{d}{dx}(x^2 - 6) = \frac{2x}{x^2 - 6}.$$

EXAMPLE 2 If $y = \ln\sqrt{x + 1}$, find dy/dx.

SOLUTION Letting $u = \sqrt{x + 1}$ in Theorem (6.11)(i) gives us

$$\frac{dy}{dx} = \frac{d}{dx}(\ln\sqrt{x+1})$$
$$= \frac{1}{\sqrt{x+1}}\frac{d}{dx}(\sqrt{x+1}) = \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2}(x+1)^{-1/2}$$
$$= \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2}\frac{1}{\sqrt{x+1}} = \frac{1}{2(x+1)}.$$

EXAMPLE 3 If $f(x) = \ln|4 + 5x - 2x^3|$, find $f'(x)$.

SOLUTION Using Theorem (6.11)(ii) with $u = 4 + 5x - 2x^3$ yields

$$f'(x) = \frac{d}{dx}\left(\ln|4 + 5x - 2x^3|\right)$$
$$= \frac{1}{4 + 5x - 2x^3}\frac{d}{dx}(4 + 5x - 2x^3) = \frac{5 - 6x^2}{4 + 5x - 2x^3}.$$

The next result states that natural logarithms satisfy the laws of logarithms studied in precalculus mathematics courses.

Laws of Natural Logarithms 6.12

If $p > 0$ and $q > 0$, then

(i) $\ln pq = \ln p + \ln q$

(ii) $\ln \dfrac{p}{q} = \ln p - \ln q$

(iii) $\ln p^r = r \ln p$ for every rational number r

PROOF

(i) If $p > 0$, then using Theorem (6.11) with $u = px$ gives us

$$\frac{d}{dx}(\ln px) = \frac{1}{px}\frac{d}{dx}(px) = \frac{1}{px}p = \frac{1}{x}.$$

Thus, $\ln px$ and $\ln x$ are both antiderivatives of $1/x$, and hence, by Theorem (4.2),

$$\ln px = \ln x + C$$

for some constant C. Letting $x = 1$, we obtain

$$\ln p = \ln 1 + C.$$

Since $\ln 1 = 0$, we see that $C = \ln p$, and therefore

$$\ln px = \ln x + \ln p.$$

Substituting q for x in the last equation gives us

$$\ln pq = \ln q + \ln p,$$

which is what we wished to prove.

(ii) Using the formula $\ln p + \ln q = \ln pq$ with $p = 1/q$, we see that

$$\ln\frac{1}{q} + \ln q = \ln\left(\frac{1}{q} \cdot q\right) = \ln 1 = 0$$

and hence $\qquad \ln\dfrac{1}{q} = -\ln q.$

Consequently,

$$\ln\frac{p}{q} = \ln\left(p\cdot\frac{1}{q}\right) = \ln p + \ln\frac{1}{q} = \ln p - \ln q.$$

(iii) If r is a rational number and $x > 0$, then, by Theorem (6.11) with $u = x^r$,

$$\frac{d}{dx}(\ln x^r) = \frac{1}{x^r}\frac{d}{dx}(x^r) = \frac{1}{x^r}rx^{r-1} = r\left(\frac{1}{x}\right) = \frac{r}{x}.$$

By Theorems (2.18)(iv) and (6.7), we may also write

$$\frac{d}{dx}(r\ln x) = r\frac{d}{dx}(\ln x) = r\left(\frac{1}{x}\right) = \frac{r}{x}.$$

Since $\ln x^r$ and $r\ln x$ are both antiderivatives of r/x, it follows from Theorem (4.2) that

$$\ln x^r = r\ln x + C$$

for some constant C. If we let $x = 1$ in the last formula, we obtain

$$\ln 1 = r\ln 1 + C.$$

Since $\ln 1 = 0$, this implies that $C = 0$ and, therefore,

$$\ln x^r = r\ln x.$$

In Section 6.5, we shall extend this law to irrational exponents. ■

As shown in the following illustration, sometimes it is convenient to use laws of natural logarithms *before* differentiating.

ILLUSTRATION

$f(x)$	$f(x)$ after using laws of logarithms	$f'(x)$
■ $\ln[(x+2)(3x-5)]$	$\ln(x+2)+\ln(3x-5)$	$\dfrac{1}{x+2}+\dfrac{1}{3x-5}\cdot 3 = \dfrac{6x+1}{(x+2)(3x-5)}$
■ $\ln\dfrac{x+2}{3x-5}$	$\ln(x+2)-\ln(3x-5)$	$\dfrac{1}{x+2}-\dfrac{1}{3x-5}\cdot 3 = \dfrac{-11}{(x+2)(3x-5)}$
■ $\ln(x^2+1)^5$	$5\ln(x^2+1)$	$5\cdot\dfrac{1}{x^2+1}\cdot 2x = \dfrac{10x}{x^2+1}$
■ $\ln\sqrt{x+1}$	$\dfrac{1}{2}\ln(x+1)$	$\dfrac{1}{2}\cdot\dfrac{1}{x+1} = \dfrac{1}{2(x+1)}$

An advantage of using laws of logarithms before differentiating may be seen by comparing the method of finding $(d/dx)\ln\sqrt{x+1}$ in the preceding illustration with the solution of Example 2.

In the next two examples, we apply laws of logarithms to complicated expressions before differentiating.

EXAMPLE ■ 4 If $f(x) = \ln[\sqrt{6x-1}(4x+5)^3]$, find $f'(x)$.

SOLUTION We first write $\sqrt{6x-1} = (6x-1)^{1/2}$ and then use laws of logarithms (i) and (iii):

$$\begin{aligned} f(x) &= \ln[(6x-1)^{1/2}(4x+5)^3] \\ &= \ln(6x-1)^{1/2} + \ln(4x+5)^3 \\ &= \tfrac{1}{2}\ln(6x-1) + 3\ln(4x+5) \end{aligned}$$

By Theorem (6.11),

$$\begin{aligned} f'(x) &= \left(\frac{1}{2}\cdot\frac{1}{6x-1}\cdot 6\right) + \left(3\cdot\frac{1}{4x+5}\cdot 4\right) \\ &= \frac{3}{6x-1} + \frac{12}{4x+5} \\ &= \frac{84x+3}{(6x-1)(4x+5)}. \end{aligned}$$

EXAMPLE ■ 5 If $y = \ln\sqrt[3]{\dfrac{x^2-1}{x^2+1}}$, find $\dfrac{dy}{dx}$.

SOLUTION We first use laws of logarithms to change the form of y as follows:

$$\begin{aligned} y &= \ln\left(\frac{x^2-1}{x^2+1}\right)^{1/3} = \frac{1}{3}\ln\left(\frac{x^2-1}{x^2+1}\right) \\ &= \tfrac{1}{3}[\ln(x^2-1) - \ln(x^2+1)] \end{aligned}$$

Next we use Theorem (6.11) to obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{3}\left(\frac{1}{x^2-1}\cdot 2x - \frac{1}{x^2+1}\cdot 2x\right) \\ &= \frac{2x}{3}\left(\frac{1}{x^2-1} - \frac{1}{x^2+1}\right) \\ &= \frac{2x}{3}\left[\frac{2}{(x^2-1)(x^2+1)}\right] = \frac{4x}{3(x^2-1)(x^2+1)}. \end{aligned}$$

Given $y = f(x)$, we may sometimes find dy/dx by **logarithmic differentiation**. This method is especially useful if $f(x)$ involves complicated products, quotients, or powers. In the following guidelines, it is assumed that $f(x) > 0$; however, we shall show that the same steps can be used if $f(x) < 0$.

Guidelines for Logarithmic Differentiation 6.13

1	$y = f(x)$	given
2	$\ln y = \ln f(x)$	take natural logarithms and simplify
3	$\dfrac{d}{dx}(\ln y) = \dfrac{d}{dx}(\ln f(x))$	differentiate implicitly
4	$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln f(x))$	by Theorem (6.11)
5	$\dfrac{dy}{dx} = f(x)\dfrac{d}{dx}(\ln f(x))$	multiply by $y = f(x)$

Of course, to complete the solution we must differentiate $\ln f(x)$ at some stage after guideline (3). If $f(x) < 0$ for some x, then guideline (2) is invalid, since $\ln f(x)$ is undefined. In this event, we can replace guideline (1) with $|y| = |f(x)|$ and take natural logarithms, obtaining $\ln |y| = \ln |f(x)|$. If we now differentiate implicitly and use Theorem (6.11)(ii), we again arrive at guideline (4). Thus, negative values of $f(x)$ do not change the outcome, and we are not concerned whether $f(x)$ is positive or negative. The method should not be used to find $f'(a)$ if $f(a) = 0$, since $\ln 0$ is undefined.

EXAMPLE 6 If

$$y = \frac{(5x-4)^3}{\sqrt{2x+1}},$$

use logarithmic differentiation to find dy/dx.

SOLUTION As in guideline (2), we begin by taking the natural logarithm of each side, obtaining

$$\begin{aligned}\ln y &= \ln(5x-4)^3 - \ln\sqrt{2x+1}\\ &= 3\ln(5x-4) - \tfrac{1}{2}\ln(2x+1).\end{aligned}$$

Applying guidelines (3) and (4), we differentiate implicitly with respect to x and then use Theorem (6.8) to obtain

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \left(3\cdot\frac{1}{5x-4}\cdot 5\right) - \left(\frac{1}{2}\cdot\frac{1}{2x+1}\cdot 2\right)\\ &= \frac{25x+19}{(5x-4)(2x+1)}.\end{aligned}$$

Finally, as in guideline (5), we multiply both sides of the last equation by y (that is, by $(5x-4)^3/\sqrt{2x+1}$) to get

$$\begin{aligned}\frac{dy}{dx} &= \frac{25x+19}{(5x-4)(2x+1)}\cdot\frac{(5x-4)^3}{\sqrt{2x+1}}\\ &= \frac{(25x+19)(5x-4)^2}{(2x+1)^{3/2}}.\end{aligned}$$

We could check this result by applying the quotient rule to y.

An application of natural logarithms to growth processes is given in the next example. Many additional applied problems involving ln appear in other examples and exercises of this chapter.

EXAMPLE 7 The *Count model* is an empirically based formula that can be used to predict the height of a preschooler. If $h(x)$ denotes the height (in centimeters) at age x (in years) for $\frac{1}{4} \le x \le 6$, then $h(x)$ can be approximated by

$$h(x) = 70.228 + 5.104x + 9.222\ln x.$$

(a) Predict the height and rate of growth when a child reaches age 2.

(b) When is the rate of growth largest?

SOLUTION

(a) The height at age 2 is approximately

$$h(2) = 70.228 + 5.104(2) + 9.222\ln 2 \approx 86.8 \text{ cm}.$$

The rate of change of h with respect to x is

$$h'(x) = 5.104 + 9.222\left(\frac{1}{x}\right).$$

Letting $x = 2$ gives us

$$h'(2) = 5.104 + 9.222(\tfrac{1}{2}) = 9.715.$$

Hence the rate of growth on the child's second birthday is about 9.7 cm/yr.

(b) To determine the maximum value of the rate of growth $h'(x)$, we first find the critical numbers of h'. Differentiating $h'(x)$, we obtain

$$h''(x) = 9.222\left(-\frac{1}{x^2}\right) = -\frac{9.222}{x^2}.$$

Since $h''(x)$ is negative for every x in $[\frac{1}{4}, 6]$, h' has no critical numbers in $(\frac{1}{4}, 6)$. It follows from Theorem (3.15) that h' is decreasing on $[\frac{1}{4}, 6]$. Consequently, the maximum value of $h'(x)$ occurs at $x = \frac{1}{4}$; that is, the rate of growth is largest at the age of 3 months.

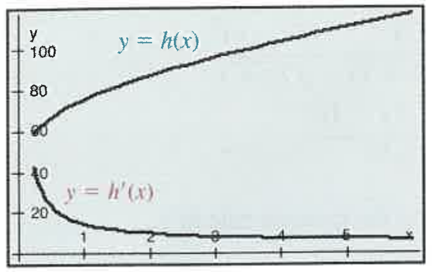


Figure 6.13
$0.25 \le x \le 6, 0 \le y \le 120$

COMPUTATIONAL METHOD The graphs of the functions h and h' on the interval $[0.25, 6]$ are shown in Figure 6.13. Using the trace operation, we find that $h(2) \approx 86.8$. We also see that h' decreases on the interval so that its maximum is at the left endpoint, $x = 0.25$.

We conclude this section by examining $\ln x$ as $x \to \infty$ and as $x \to 0^+$. If $x > 1$, we may interpret the integral $\int_1^x (1/t)\,dt = \ln x$ as the area of the region shown in Figure 6.14. The sum of the areas of the three rectangles shown in Figure 6.15 is

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{13}{12}.$$

Since the area under the graph of $y = 1/t$ from $t = 1$ to $t = 4$ is $\ln 4$, we see that

$$\ln 4 > \tfrac{13}{12} > 1.$$

It follows that if M is any positive rational number, then

$$M \ln 4 > M, \quad \text{or} \quad \ln 4^M > M.$$

If $x > 4^M$, then since ln is an increasing function,

$$\ln x > \ln 4^M > M.$$

This proves that $\ln x$ can be made as large as desired by choosing x sufficiently large—that is,

$$\lim_{x\to\infty} \ln x = \infty.$$

To investigate the case $x \to 0^+$, we first note that

$$\ln \frac{1}{x} = \ln 1 - \ln x = 0 - \ln x = -\ln x.$$

Hence,
$$\lim_{x\to 0^+} \ln x = \lim_{x\to 0^+}\left(-\ln \frac{1}{x}\right).$$

Figure 6.14

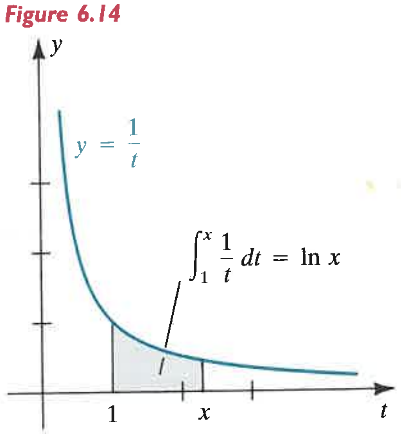
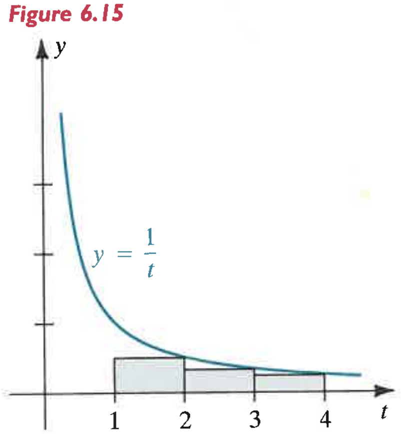


Figure 6.15



As x approaches zero through positive values, $1/x$ increases without bound and, therefore, so does $\ln(1/x)$. Consequently, $-\ln(1/x)$ *decreases* without bound and

$$\lim_{x\to 0^+} \ln x = -\infty.$$

EXERCISES 6.2

Exer. 1–34: Find $f'(x)$ if $f(x)$ is the given expression.

1 $\ln(9x + 4)$
2 $\ln(x^4 + 1)$
3 $\ln(3x^2 - 2x + 1)$
4 $\ln(4x^3 - x^2 + 2)$
5 $\ln|3 - 2x|$
6 $\ln|4 - 3x|$
7 $\ln|2 - 3x|^5$
8 $\ln|5x^2 - 1|^3$
9 $\ln\sqrt{7 - 2x^3}$
10 $\ln\sqrt[3]{6x + 7}$
11 $x \ln x$
12 $\ln(\ln x)$
13 $\ln\sqrt{x} + \sqrt{\ln x}$
14 $\ln x^3 + (\ln x)^3$
15 $\dfrac{1}{\ln x} + \ln\dfrac{1}{x}$
16 $\dfrac{x^2}{\ln x}$
17 $\ln[(5x - 7)^4(2x + 3)^3]$
18 $\ln[\sqrt[3]{4x - 5}(3x + 8)^2]$
19 $\ln\dfrac{\sqrt{x^2 + 1}}{(9x - 4)^2}$
20 $\ln\dfrac{x^2(2x - 1)^3}{(x + 5)^2}$
21 $\ln\sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$
22 $\ln\sqrt{\dfrac{4 + x^2}{4 - x^2}}$
23 $\ln(x + \sqrt{x^2 - 1})$
24 $\ln(x + \sqrt{x^2 + 1})$
25 $\ln\cos 2x$
26 $\cos(\ln 2x)$
27 $\ln\tan^3 3x$
28 $\ln\cot(x^2)$
29 $\ln\ln\sec 2x$
30 $\ln\csc^2 4x$
31 $\ln|\sec x|$
32 $\ln|\sin x|$
33 $\ln|\sec x + \tan x|$
34 $\ln|\csc x - \cot x|$

Exer. 35–38: Use implicit differentiation to find y'.

35 $3y - x^2 + \ln xy = 2$
36 $y^2 + \ln(x/y) - 4x = -3$
37 $x\ln y - y\ln x = 1$
38 $y^3 + x^2\ln y = 5x + 3$

Exer. 39–44: Use logarithmic differentiation to find dy/dx.

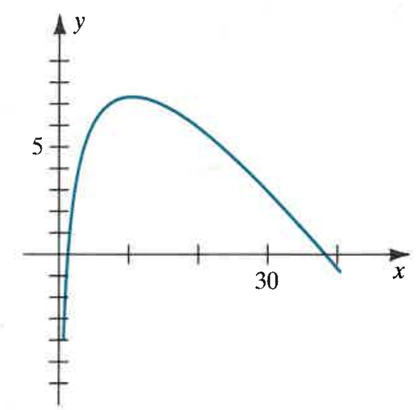
39 $y = (5x + 2)^3(6x + 1)^2$
40 $y = (x + 1)^2(x + 2)^3(x + 3)^4$
41 $y = \sqrt{4x + 7}(x - 5)^3$
42 $y = \sqrt{(3x^2 + 2)\sqrt{6x - 7}}$
43 $y = \dfrac{(x^2 + 3)^5}{\sqrt{x + 1}}$
44 $y = \dfrac{(x^2 + 3)^{2/3}(3x - 4)^4}{\sqrt{x}}$

45 Find an equation of the tangent line to the graph of $y = x^2 + \ln(2x - 5)$ at the point $P(3, 9)$.

46 Find an equation of the tangent line to the graph of $y = x + \ln x$ that is perpendicular to the line whose equation is $2x + 6y = 5$.

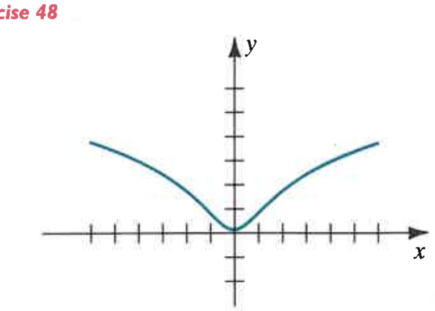
47 Shown in the figure is a graph of $y = 5\ln x - \frac{1}{2}x$. Find the coordinates of the highest point, and show that the graph is concave downward for $x > 0$.

Exercise 47



48 Shown in the figure is a graph of $y = \ln(x^2 + 1)$. Find the points of inflection.

Exercise 48



49 An approximation to the age T (in years) of a female blue whale can be obtained from a length measurement L (in feet) using $T = -2.57\ln[(87 - L)/63]$. A blue whale has been spotted by a research vessel, and her length is estimated to be 80 ft. If the maximum error in estimating L is ± 2 ft, use differentials to approximate the maximum error in T.

50 The *Ehrenberg relation*, $\ln W = \ln 2.4 + 0.0184h$, is an empirically based formula relating the height h (in centimeters) to the weight W (in kilograms) for children aged 5–13. The formula, with minor changes in constants, has been verified in many different countries. Find the relationship between the rates of change dW/dt and dh/dt, for time t (in years).

51 A rocket of mass m_1 is filled with fuel of mass m_2, which will be burned at a constant rate of b kg/sec. If the fuel is expelled from the rocket at a constant rate, the distance $s(t)$ (in meters) that the rocket has traveled after t seconds is

$$s(t) = ct + \frac{c}{b}(m_1 + m_2 - bt)\ln\left(\frac{m_1 + m_2 - bt}{m_1 + m_2}\right)$$

for some constant $c > 0$.

(a) Find the initial velocity and the initial acceleration of the rocket.

(b) Burnout occurs when $t = m_2/b$. Find the velocity and the acceleration at burnout.

52 One method of estimating the thickness of the ozone layer is to use the formula $\ln(I/I_0) = -\beta T$, where I_0 is the intensity of a particular wavelength of light from the sun before it reaches the atmosphere, I is the intensity of the same wavelength after passing through a layer of ozone T centimeters thick, and β is the absorption coefficient for that wavelength. Suppose that for a wavelength of 3055×10^{-8} cm with $\beta \approx 2.7$, I_0/I is measured as 2.3.

(a) Approximate the thickness of the ozone layer to the nearest 0.01 cm.

(b) If the maximum error in the measured value of I_0/I is ± 0.1, use differentials to approximate the maximum error in the approximation obtained in part (a).

53 Describe the difference between the graphs of $y = \ln(x^2)$ and $y = 2\ln x$.

54 Sketch the graphs of

(a) $y = \ln|x|$ **(b)** $y = |\ln x|$

c **Exer. 55–60: Graph f on the given interval. (a) Approximate the range of the function defined on this interval so that the graph just fits in the viewing window. (b) Identify x-intercepts, y-intercepts, and relative extrema in this viewing window, if any. Estimate these features to two decimal places.**

55 $f(x) = \ln(\sin x)$; $[0.1, 3.1]$

56 $f(x) = \sin(\ln x)$; $[0.1, 800]$

57 $f(x) = \dfrac{\ln[4(x+1)]}{\ln[2(x+1)]}$; $[-0.1, 3]$

58 $f(x) = x^2 \ln x$; $[0, 2]$

59 $f(x) = \ln[x(1.3 + \sin x)]$; $[0.1, 20]$

60 $f(x) = \ln(x^4 - 4x^2 - 0.8x + 5.4)$; $[-3, 3]$

c **Exer. 61–64: Use Newton's method or a solving routine to approximate the real root(s) of the equation to four decimal places. Use a graphing utility to ensure that all roots are found.**

61 $\ln x + x = 0$

62 $\ln(x^2 - 1.8x + 1) = 3$

63 $2 - 0.3x - 0.2x^2 - \ln[\ln(x^2 + 1.5)] = 0$

64 $\ln[x(1 - 0.9\cos x)] = 0$

c **Exer. 65–68: Use a numerical integration method or routine to approximate the definite integral to four decimal places.**

65 $\displaystyle\int_1^4 \ln(\sin x - x\cos x)\,dx$

66 $\displaystyle\int_1^{50} \ln(1 + \ln x)\,dx$

67 $\displaystyle\int_{0.5}^2 \frac{\ln 4x}{\ln 3x}\,dx$

68 $\displaystyle\int_2^5 \frac{x^2}{\ln x}\,dx$

c **69** Approximate the area bounded by the graphs of $y = \ln x$, $y = 1/x$, and $x = 10$.

c **70** Approximate the volume of the solid generated by revolving the graph of $y = (\ln x)^2/x$, $1 \le x \le 5$, about the x-axis.

c **71** Approximate the arc length of the part of the curve $y = \ln x$ that lies inside the circle $x^2 + y^2 = 25$.

6.3 THE EXPONENTIAL FUNCTION

In Section 6.2, we saw that

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty.$$

These facts are used in the proof of the following result.

Theorem 6.14

> To every real number x there corresponds exactly one positive real number y such that $\ln y = x$.

PROOF First note that if $x = 0$, then $y = 1$. Moreover, since ln is an increasing function, 1 is the only value of y such that $\ln y = 0$.

If x is positive, then we may choose a number b such that

$$\ln 1 < x < \ln b.$$

Since ln is continuous, it takes on every value between ln 1 and ln b (see the intermediate value theorem (1.26)). Thus, there is a number y between 1 and b such that $\ln y = x$. Since ln is an increasing function, there is only one such number.

Finally, if x is negative, then there is a number $b > 0$ such that

$$\ln b < x < \ln 1,$$

and, as before, there is exactly one number y between b and 1 such that $\ln y = x$. ■

It follows from Theorem (6.14) that the range of the natural logarithms is $\mathbb{R}$. Since ln is an increasing function, it is one-to-one and therefore has an inverse function, to which we give the following special name.

Definition 6.15

> The **natural exponential function**, denoted by **exp**, is the inverse of the natural logarithm function.

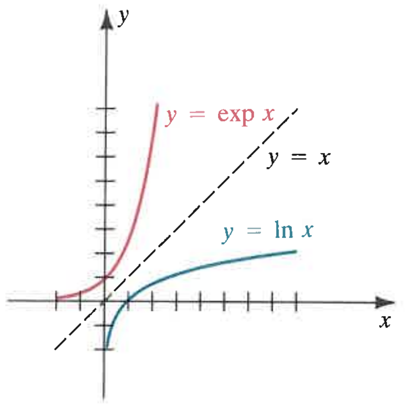
The reason for the term *exponential* in this definition will become clear shortly. Since exp is the inverse of ln, its domain is $\mathbb{R}$ and its range is $(0, \infty)$. Moreover, as in (6.2),

$$y = \exp x \quad \text{if and only if} \quad x = \ln y,$$

where x is any real number and $y > 0$. By Theorem (6.3), we may also write

$$\ln(\exp x) = x \quad \text{and} \quad \exp(\ln y) = y.$$

Figure 6.16



As we observed in Section 6.1, if two functions are inverses of each other, then their graphs are reflections through the line $y = x$. Hence the graph of $y = \exp x$ can be obtained by reflecting the graph of $y = \ln x$ through this line, as illustrated in Figure 6.16. Note that

$$\lim_{x\to\infty} \exp x = \infty \quad \text{and} \quad \lim_{x\to -\infty} \exp x = 0.$$

By Theorem (6.14), there exists exactly one positive real number whose natural logarithm is 1. This number is denoted by e. The great Swiss mathematician Leonhard Euler was among the first to study its properties extensively. (See *Mathematicians and Their Times*.)

Definition of e 6.16

> The letter e denotes the positive real number such that $\ln e = 1$.

Several values of ln were calculated in Section 6.2. We can show, by means of the trapezoidal rule, that

$$\int_1^{2.7} \frac{1}{t}\,dt < 1 < \int_1^{2.8} \frac{1}{t}\,dt.$$

Applying Definitions (6.9) and (6.16) yields

$$\ln 2.7 < \ln e < \ln 2.8$$

and hence

$$2.7 < e < 2.8.$$

Later, in Theorem (6.32), we show that

$$e = \lim_{h\to 0}(1+h)^{1/h}.$$

This limit formula can be used to approximate e to any degree of accuracy. In Section 6.5, the first five decimal places in the following 32-decimal-place approximation for e will be justified in this way.

Approximation to e 6.17

> $e \approx 2.71828182845904523536028747135266$

It can be shown that e is an irrational number.

If r is any *rational* number, then

$$\ln e^r = r \ln e = r(1) = r.$$

The formula $\ln e^r = r$ may be used to motivate a definition of e^x for every *real* number x. Specifically, we shall *define* e^x as the real number y such that $\ln y = x$. The following statement is a convenient way to remember this definition.

Definition of e^x 6.18

> If x is any real number, then
>
> $$e^x = y \quad \text{if and only if} \quad \ln y = x.$$

Since exp is the inverse function of ln,

$$\exp x = y \quad \text{if and only if} \quad \ln y = x.$$

Comparing this relationship with Definition (6.18), we see that

$$e^x = \exp x \quad \text{for every } x.$$

This result shows the reason for calling exp an *exponential* function and referring to it as the **exponential function with base e**. The graph of $y = e^x$ is the same as that of $y = \exp x$, illustrated in Figure 6.16. Hereafter *we shall use e^x instead of* $\exp x$ *to denote values of the natural exponential function*. The most commonly used exponential function in mathematics and its applications is the natural exponential function exp. For this reason, the function exp is often called "the exponential function."

The fact that $\ln(\exp x) = x$ for every x and $\exp(\ln x) = x$ for every $x > 0$ may now be written as follows:

Theorem 6.19

> **(i)** $\ln e^x = x$ for every x
>
> **(ii)** $e^{\ln x} = x$ for every $x > 0$

Some special cases of this theorem are given in the following illustration.

ILLUSTRATION

$\ln e^5 = 5$	$\ln e^{\sqrt{x+1}} = \sqrt{x+1}$
$e^{\ln 5} = 5$	$e^{\ln\sqrt{x+1}} = \sqrt{x+1}$
$e^{3\ln x} = e^{\ln(x^3)} = x^3$	$e^{k\ln x} = e^{\ln(x^k)} = x^k$

NOTE To obtain numerical approximations for e^x, use the natural exponential function provided by your calculator or computer. It gives a better approximation than entering an approximation for the number e and then raising this approximation to a power.

The next theorem states that the laws of exponents are true for powers of e.

Theorem 6.20

If p and q are real numbers and r is a rational number, then

(i) $e^p e^q = e^{p+q}$

(ii) $\dfrac{e^p}{e^q} = e^{p-q}$

(iii) $(e^p)^r = e^{pr}$

PROOF Using Theorems (6.12) and (6.19), we obtain

$$\ln e^p e^q = \ln e^p + \ln e^q = p + q = \ln e^{p+q}.$$

Since the natural logarithm function is one-to-one,

$$e^p e^q = e^{p+q}.$$

Thus, we have proved (i). The proofs for (ii) and (iii) are similar. We show in Section 6.5 that (iii) is also true if r is irrational. ■

By Theorem (6.7), the inverse function of a differentiable function is differentiable, and hence $(d/dx)(e^x)$ exists. The next theorem states that *e^x is its own derivative.*

Theorem 6.21

$$\frac{d}{dx}(e^x) = e^x$$

PROOF By (i) of Theorem (6.19),

$$\ln e^x = x.$$

Differentiating each side of this equation and using Theorem (6.11)(i) with $u = e^x$ gives us the following:

$$\frac{d}{dx}(\ln e^x) = \frac{d}{dx}(x)$$

$$\frac{1}{e^x}\frac{d}{dx}(e^x) = 1$$

$$\frac{d}{dx}(e^x) = e^x \quad ■$$

EXAMPLE ■ 1 If $f(x) = x^2 e^x$, find $f'(x)$.

SOLUTION By the product rule and Theorem (6.21),

$$\begin{aligned} f'(x) &= x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2) \\ &= x^2e^x + e^x(2x) = xe^x(x+2). \end{aligned}$$

The next result is a generalization of Theorem (6.21).

Theorem 6.22

If $u = g(x)$ and g is differentiable, then

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}.$$

PROOF Letting $y = e^u$ with $u = g(x)$, and using the chain rule and Theorem (6.21), we have

$$\frac{d}{dx}(e^u) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{du}{dx}. \quad ■$$

If $u = x$, then Theorem (6.22) reduces to (6.21).

EXAMPLE ■ 2 If $y = e^{\sqrt{x^2+1}}$, find dy/dx.

SOLUTION By Theorem (6.22),

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}\left(e^{\sqrt{x^2+1}}\right) &= e^{\sqrt{x^2+1}}\frac{d}{dx}\left(\sqrt{x^2+1}\right) \\ &= e^{\sqrt{x^2+1}}\frac{d}{dx}((x^2+1)^{1/2}) \\ &= e^{\sqrt{x^2+1}}(\tfrac{1}{2})(x^2+1)^{-1/2}(2x) \\ &= e^{\sqrt{x^2+1}}\cdot\frac{x}{\sqrt{x^2+1}} \\ &= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}. \end{aligned}$$

EXAMPLE ■ 3 The function f defined by $f(x) = e^{-x^2/2}$ occurs in the branch of mathematics called *probability*. Find the local extrema of f, discuss concavity, find the points of inflection, and sketch the graph of f.

SOLUTION By Theorem (6.22),

$$f'(x) = e^{-x^2/2}\frac{d}{dx}\left(-\frac{x^2}{2}\right) = e^{-x^2/2}\left(-\frac{2x}{2}\right) = -xe^{-x^2/2}.$$

Since $e^{-x^2/2}$ is always positive, the only critical number of f is 0. If $x < 0$, then $f'(x) > 0$, and if $x > 0$, then $f'(x) < 0$. It follows from the first derivative test that f has a local maximum at 0. The maximum value is $f(0) = e^{-0} = 1$.

Applying the product rule to $f'(x)$ yields

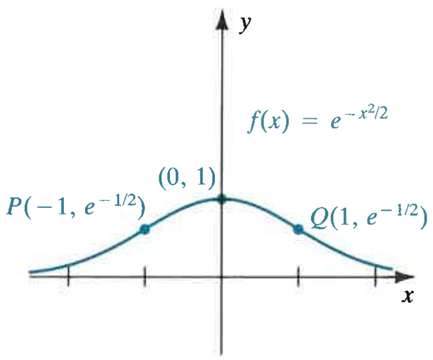
$$\begin{aligned} f''(x) &= -x\frac{d}{dx}\left(e^{-x^2/2}\right) + e^{-x^2/2}\frac{d}{dx}(-x) \\ &= -xe^{-x^2/2}(-2x/2) - e^{-x^2/2} \\ &= e^{-x^2/2}(x^2 - 1), \end{aligned}$$

and hence the second derivative is zero at -1 and 1. If $-1 < x < 1$, then $f''(x) < 0$ and, by (3.18), the graph of f is concave downward in the open interval $(-1, 1)$. If $x < -1$ or $x > 1$, then $f''(x) > 0$ and, therefore, the graph is concave upward throughout the infinite intervals $(-\infty, -1)$ and $(1, \infty)$. Consequently, $P(-1, e^{-1/2})$ and $Q(1, e^{-1/2})$ are points of inflection. From the expression

$$f(x) = \frac{1}{e^{x^2/2}}$$

it is evident that as x increases numerically, $f(x)$ approaches 0. We can prove that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$—that is, the x-axis is a horizontal asymptote. The graph of f is sketched in Figure 6.17.

Figure 6.17



Exponential functions play an important role in the field of *radiotherapy*, the treatment of tumors by radiation. The fraction of cells in a tumor that survive a treatment, called the *surviving fraction*, depends not only on the energy and nature of the radiation, but also on the depth, size, and characteristics of the tumor itself. The exposure to radiation may be thought of as a number of potentially damaging events, where only one *hit* is required to kill a tumor cell. Suppose that each cell has exactly one *target* that must be hit. If k denotes the average target size of a tumor cell and if x is the number of damaging events (the *dose*), then the surviving fraction $f(x)$ is given by

$$f(x) = e^{-kx}$$

and is called the *one-target–one-hit surviving fraction*.

Suppose next that each cell has n targets and that hitting each target once results in the death of a cell. In this case, the *n-target–one-hit surviving fraction* is given by

$$f(x) = 1 - (1 - e^{-kx})^n.$$

In the next example, we examine the case where $n = 2$.

EXAMPLE 4 If each cell of a tumor has two targets, then the two-target–one-hit surviving fraction is given by

$$f(x) = 1 - (1 - e^{-kx})^2,$$

where k is the average size of a cell. Analyze the graph of f to determine what effect increasing the dosage x has on decreasing the surviving fraction of tumor cells.

SOLUTION First note that if $x = 0$, then $f(0) = 1$; that is, if there is no dose, then all cells survive. Differentiating, we obtain

$$\begin{aligned} f'(x) &= 0 - 2(1 - e^{-kx})\frac{d}{dx}(1 - e^{-kx}) \\ &= -2(1 - e^{-kx})(ke^{-kx}) \\ &= -2ke^{-kx}(1 - e^{-kx}). \end{aligned}$$

Since $f'(x) < 0$ for every $x > 0$ and $f'(0) = 0$, the function f is decreasing and the graph has a horizontal tangent line at the point (0, 1). We may verify that the second derivative is

$$f''(x) = 2k^2e^{-kx}(1 - 2e^{-kx}).$$

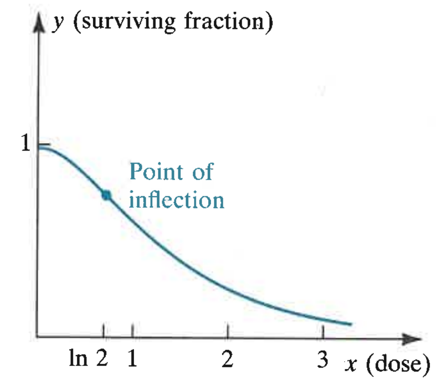
We see that $f''(x) = 0$ if $1 - 2e^{-kx} = 0$—that is, if $e^{-kx} = \frac{1}{2}$, or, equivalently, $-kx = \ln\frac{1}{2} = -\ln 2$. We thus obtain

$$x = \frac{1}{k}\ln 2.$$

It can be verified that if $0 \le x < (1/k)\ln 2$, then $f''(x) < 0$, and hence the graph is concave downward. If $x > (1/k)\ln 2$, then $f''(x) > 0$, and the graph is concave upward. The implication is that a point of inflection exists at x-coordinate $(1/k)\ln 2$. The y-coordinate of this point is

$$\begin{aligned} f\left(\frac{1}{k}\ln 2\right) &= 1 - (1 - e^{-\ln 2})^2 \\ &= 1 - (1 - \tfrac{1}{2})^2 = \tfrac{3}{4}. \end{aligned}$$

Figure 6.18
Surviving fraction of tumor cells after a radiation treatment



The graph is sketched in Figure 6.18 for the case $k = 1$. The *shoulder* on the curve near the point (0, 1) represents the threshold nature of the treatment—that is, a small dose results in very little tumor elimination. Note that if x is large, then an increase in dosage has little effect on the surviving fraction. To determine the ideal dose that should be administered to a given patient, specialists in radiation therapy must also take into account the number of healthy cells that are killed during a treatment.

EXERCISES 6.3

Exer. 1–30: Find $f'(x)$ if $f(x)$ equals the given expression.

1 e^{-5x}	2 e^{3x}
3 e^{3x^2}	4 e^{1-x^3}
5 $\sqrt{1+e^{2x}}$	6 $1/(e^x+1)$
7 $e^{\sqrt{x+1}}$	8 xe^{-x}
9 x^2e^{-2x}	10 $\sqrt{e^{2x}+2x}$
11 $e^x/(x^2+1)$	12 $x/e^{(x^2)}$
13 $(e^{4x}-5)^3$	14 $(e^{3x}-e^{-3x})^4$
15 $e^{1/x}+(1/e^x)$	16 $e^{\sqrt{x}}+\sqrt{e^x}$
17 $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$	18 $e^{x\ln x}$
19 $e^{-2x}\ln x$	20 $\ln e^x$
21 $\sin e^{5x}$	22 $e^{\sin 5x}$
23 $\ln\cos e^{-x}$	24 $e^{-3x}\cos 3x$
25 $e^{3x}\tan\sqrt{x}$	26 $\sec e^{-2x}$
27 $\sec^2(e^{-4x})$	28 $e^{-x}\tan^2 x$
29 $xe^{\cot x}$	30 $\ln(\csc e^{3x})$

Exer. 31–34: Use implicit differentiation to find y'.

31 $e^{xy}-x^3+3y^2=11$

32 $xe^y+2x-\ln(y+1)=3$

33 $e^x\cot y = xe^{2y}$

34 $e^x\cos y = xe^y$

35 Find an equation of the tangent line to the graph of $y=(x-1)e^x+3\ln x+2$ at the point $P(1,2)$.

36 Find an equation of the tangent line to the graph of $y=x-e^{-x}$ that is parallel to the line $6x-2y=7$.

Exer. 37–42: Find the local extrema of f. Determine where f is increasing or is decreasing, discuss concavity, find the points of inflection, and sketch the graph of f.

37 $f(x)=xe^x$	38 $f(x)=x^2e^{-2x}$
39 $f(x)=e^{1/x}$	40 $f(x)=xe^{-x}$
41 $f(x)=x\ln x$	42 $f(x)=(1-\ln x)^2$

43 A radioactive substance decays according to the formula $q(t)=q_0e^{-ct}$, where q_0 is the initial amount of the substance, c is a positive constant, and $q(t)$ is the amount remaining after time t. Show that the rate at which the substance decays is proportional to $q(t)$.

44 The current $I(t)$ at time t in an electrical circuit is given by $I(t)=I_0e^{-Rt/L}$, where R is the resistance, L is the inductance, and I_0 is the current at time $t=0$. Show that the rate of change of the current at any time t is proportional to $I(t)$.

45 If a drug is injected into the bloodstream, then its concentration t minutes later is given by

$$C(t)=\frac{k}{a-b}(e^{-bt}-e^{-at})$$

for positive constants a, b, and k.

(a) At what time does the maximum concentration occur?

(b) What can be said about the concentration after a long period of time?

46 If a beam of light that has intensity k is projected vertically downward into water, then its intensity $I(x)$ at a depth of x meters is $I(x)=ke^{-1.4x}$.

(a) At what rate is the intensity changing with respect to depth at 1 m? 5 m? 10 m?

(b) At what depth is the intensity one-half its value at the surface? one-tenth its value?

47 The *Jenss model* is generally regarded as the most accurate formula for predicting the height of a preschooler. If $h(x)$ denotes the height (in centimeters) at age x (in years) for $\frac{1}{4}\le x\le 6$, then $h(x)$ can be approximated by

$$h(x)=79.041+6.39x-e^{3.261-0.993x}.$$

(Compare with Example 7 of Section 6.2.)

(a) Predict the height and the rate of growth when a child reaches the age of 1.

(b) When is the rate of growth largest, and when is it smallest?

48 For a population of female African elephants, the weight $W(t)$ (in kilograms) at age t (in years) may be approximated by a *von Bertanlanffy growth function* W such that

$$W(t)=2600(1-0.51e^{-0.075t})^3.$$

(a) Approximate the weight and the rate of growth of a newborn.

(b) Assuming that an adult female weighs 1800 kg, estimate her age and her rate of growth at present.

(c) Find and interpret $\lim_{t\to\infty} W(t)$.

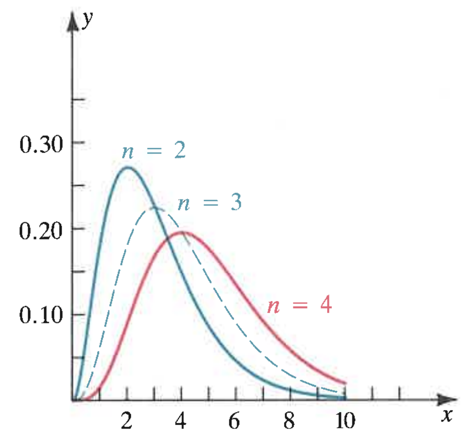
(d) Show that the rate of growth is largest between the ages of 5 and 6.

49 Gamma distributions, which are important in traffic control studies and probability theory, are determined by $f(x)=cx^ne^{-ax}$ for $x>0$, a positive integer n, a positive constant a, and $c=a^{n+1}/n!$. Shown in the figure are graphs corresponding to $a=1$ for $n=2, 3,$ and 4.

(a) Show that f has exactly one local maximum.

(b) If $n=4$, determine where $f(x)$ is increasing most rapidly.

Exercise 49



50 The relative number of gas molecules in a container that travel at a velocity of v cm/sec can be computed by means of the *Maxwell–Boltzmann speed distribution*, $F(v)=cv^2e^{-mv^2/(2kT)}$, where T is the temperature (in °K), m is the mass of a molecule, and c and k are positive constants. Show that the maximum value of F occurs when $v=\sqrt{2kT/m}$.

51 An *urban density model* is a formula that relates the population density (in number per square mile) to the distance r (in miles) from the center of the city. The formula $D=ae^{-br+cr^2}$, where a, b, and c are positive constants, has been found to be appropriate for certain cities. Determine the shape of the graph for $r\ge 0$.

52 The effect of light on the rate of photosynthesis can be described by

$$f(x)=x^ae^{(a/b)(1-x^b)}$$

for $x>0$ and positive constants a and b.

(a) Show that f has a maximum at $x=1$.

(b) Conclude that if $x_0>0$ and $y_0>0$, then $g(x)=y_0f(x/x_0)$ has a maximum $g(x_0)=y_0$.

53 The rate R at which a tumor grows is related to its size x by the equation $R=rx\ln(K/x)$, where r and K are positive constants. Show that the tumor is growing most rapidly when $x=e^{-1}K$.

54 If p denotes the selling price (in dollars) of a commodity and x is the corresponding demand (in number sold per day), then the relationship between p and x may be given by $p=p_0e^{-ax}$ for positive constants p_0 and a. Suppose $p=300e^{-0.02x}$. Find the selling price that will maximize daily revenues (see page 332).

55 In statistics, the probability density function for the normal distribution is defined by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-z^2/2}\quad\text{with}\quad z=\frac{x-\mu}{\sigma}$$

for real numbers μ and $\sigma>0$ (μ is the *mean* and σ^2 is the *variance* of the distribution). Find the local extrema of f, and determine where f is increasing or is decreasing. Discuss concavity, find points of inflection, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and sketch the graph of f (see Example 3).

c 56 The integral $\int_a^b e^{-x^2}\,dx$ has applications in statistics. Use the trapezoidal rule, with $n=10$, to approximate this integral if $a=0$ and $b=1$.

c **Exer. 57–60: Graph f on the given interval. (a) Approximate the range of f on this interval. (b) Identify x-intercepts, y-intercepts, and relative extrema in this viewing window, if any. Estimate these features to two decimal places.**

57 $f(x)=1-(1-e^{-x})^3$; [0, 4]
(This function is the *three-target–one-hit surviving fraction*, with $k=1$; see Example 4.)

58 $f(x)=\dfrac{150}{1+5e^{-0.4x}}$; [−5, 25]

59 $f(x)=6e^{-0.39x}\cos(1.87x)$; [−1, 8]

60 $f(x)=\ln(8+e^x)$; [−4, 15]

c **Exer. 61–64: Use Newton's method or a solving routine to approximate the real root(s) of the equation to four decimal places. Use a graphing utility to ensure that all roots are found.**

61 $e^{-x}=x$

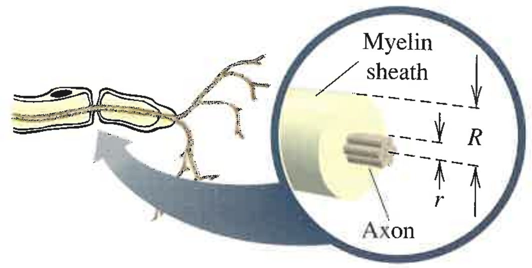
62 $e^{3x}-5e^{2x}+7e^x=2$

63 $xe^x=4$

64 $0.2e^{0.6x}+1.3e^{-0.1x}=4-2.8x^2$

65 Nerve impulses in the human body travel along nerve fibers that consist of an *axon*, which transports the impulse, and an insulating coating surrounding the axon, called the *myelin sheath* (see figure). The nerve fiber is similar to an insulated cylindrical cable, for which the velocity v of an impulse is given by $v = -k(r/R)^2 \ln(r/R)$, where r is the radius of the cable and R is the insulation radius. Find the value of r/R that maximizes v. (In most nerve fibers, $r/R \approx 0.6$.)

Exercise 65



6.4 INTEGRATION USING NATURAL LOGARITHM AND EXPONENTIAL FUNCTIONS

We may use differentiation formulas for ln to obtain formulas for integration. In particular, by Theorem (6.11),

$$\frac{d}{dx}\left(\ln|g(x)|\right) = \frac{1}{g(x)}g'(x),$$

which gives us the integration formula

$$\int \frac{1}{g(x)}g'(x)\,dx = \ln|g(x)| + C.$$

This result is restated in the next theorem in terms of the variable u.

Theorem 6.23

> If $u = g(x) \neq 0$ and g is differentiable, then
>
> $$\int \frac{1}{u}\,du = \ln|u| + C.$$

Of course, if $u > 0$, then the absolute value sign may be deleted. A special case of Theorem (6.23) is

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

EXAMPLE 1 Evaluate $\displaystyle\int \frac{x}{3x^2-5}\,dx$.

SOLUTION Rewriting the integral as

$$\int \frac{x}{3x^2-5}\,dx = \int \frac{1}{3x^2-5}\,x\,dx$$

suggests that we use Theorem (6.23) with $u = 3x^2 - 5$. Thus, we make the substitution

$$u = 3x^2 - 5, \qquad du = 6x\,dx.$$

Introducing a factor 6 in the integrand and using Theorem (6.23) yields

$$\int \frac{x}{3x^2-5}\,dx = \frac{1}{6}\int \frac{1}{3x^2-5}6x\,dx = \frac{1}{6}\int \frac{1}{u}\,du$$
$$= \tfrac{1}{6}\ln|u| + C = \tfrac{1}{6}\ln|3x^2-5| + C.$$

Another technique is to replace the expression $x\,dx$ in the integral by $\frac{1}{6}\,du$ and then integrate.

EXAMPLE 2 Evaluate $\displaystyle\int_2^4 \frac{1}{9-2x}\,dx$.

SOLUTION Since $1/(9-2x)$ is continuous on $[2, 4]$, the definite integral exists. One method of evaluation consists of using an indefinite integral to find an antiderivative of $1/(9-2x)$. We let

$$u = 9 - 2x, \qquad du = -2\,dx$$

and proceed as follows:

$$\int \frac{1}{9-2x}\,dx = -\frac{1}{2}\int \frac{1}{9-2x}(-2)\,dx$$
$$= -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|u| + C$$
$$= -\tfrac{1}{2}\ln|9-2x| + C$$

Applying the fundamental theorem of calculus yields

$$\int_2^4 \frac{1}{9-2x}\,dx = -\tfrac{1}{2}\left[\ln|9-2x|\right]_2^4$$
$$= -\tfrac{1}{2}(\ln 1 - \ln 5) = \tfrac{1}{2}\ln 5.$$

Another method is to use the same substitution in the *definite* integral and change the limits of integration. Since $u = 9 - 2x$, we obtain the following:

(i) If $x = 2$, then $u = 5$.
(ii) If $x = 4$, then $u = 1$.

Thus,

$$\int_2^4 \frac{1}{9-2x}\,dx = -\frac{1}{2}\int_2^4 \frac{1}{9-2x}(-2)\,dx$$
$$= -\frac{1}{2}\int_5^1 \frac{1}{u}\,du = -\frac{1}{2}\left[\ln|u|\right]_5^1$$
$$= -\tfrac{1}{2}(\ln 1 - \ln 5) = \tfrac{1}{2}\ln 5.$$

EXAMPLE ■ 3 Evaluate $\int \frac{\sqrt{\ln x}}{x}\,dx$.

SOLUTION Two possible substitutions are $u = \sqrt{\ln x}$ and $u = \ln x$. If we use

$$u = \ln x, \qquad du = \frac{1}{x}\,dx,$$

then

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{\ln x}\cdot\frac{1}{x}\,dx = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C$$
$$= \tfrac{2}{3}(\ln x)^{3/2} + C.$$

The substitution $u = \sqrt{\ln x}$ could also be used; however, the algebraic manipulations would be somewhat more involved.

The derivative formula $(d/dx)(e^{g(x)}) = e^{g(x)}g'(x)$ gives us the following integration formula for the natural exponential function:

$$\int e^{g(x)}g'(x)\;dx = e^{g(x)} + C$$

This result is restated in the next theorem in terms of the variable u.

Theorem 6.24

If $u = g(x)$ and g is differentiable, then

$$\int e^u\,du = e^u + C.$$

As a special case of Theorem (6.24), if $u = x$, then

$$\int e^x\,dx = e^x + C.$$

EXAMPLE ■ 4 Evaluate:

(a) $\int \frac{e^{3/x}}{x^2}\,dx$ **(b)** $\int_1^2 \frac{e^{3/x}}{x^2}\,dx$

SOLUTION

(a) Rewriting the integral as

$$\int \frac{e^{3/x}}{x^2}\,dx = \int e^{3/x}\frac{1}{x^2}\,dx$$

suggests that we use Theorem (6.24) with $u = 3/x$. Thus, we make the substitution

$$u = \frac{3}{x}, \qquad du = -\frac{3}{x^2}\,dx.$$

The integrand may be written in the form of Theorem (6.24) by introducing the factor -3. Doing this and compensating by multiplying the integral by $-\frac{1}{3}$, we obtain

$$\int \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int e^{3/x}\left(-\frac{3}{x^2}\right)dx$$
$$= -\tfrac{1}{3}\int e^u\,du$$
$$= -\tfrac{1}{3}e^u + C$$
$$= -\tfrac{1}{3}e^{3/x} + C.$$

(b) Using the antiderivative found in part (a) and applying the fundamental theorem of calculus yields

$$\int_1^2 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\left[e^{3/x}\right]_1^2$$
$$= -\tfrac{1}{3}(e^{3/2} - e^3) \approx 5.2.$$

We can also evaluate the integral by using the method of substitution. As in part (a), we let $u = 3/x$, $du = (-3/x^2)\,dx$, and we note that if $x = 1$, then $u = 3$, and if $x = 2$, then $u = \frac{3}{2}$. Consequently,

$$\int_1^2 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_1^2 e^{3/x}\left(-\frac{3}{x^2}\right)dx$$
$$= -\tfrac{1}{3}\int_3^{3/2} e^u\;du$$
$$= -\tfrac{1}{3}\left[e^u\right]_3^{3/2} = -\tfrac{1}{3}(e^{3/2} - e^3) \approx 5.2.$$

The integral $\int e^{ax}\,dx$, with $a \neq 0$, occurs frequently. We can show that

$$\int e^{ax}\;dx = \frac{1}{a}\,e^{ax} + C$$

either by using Theorem (6.24) or by showing that $(1/a)e^{ax}$ is an antiderivative of e^{ax}.

ILLUSTRATION

- $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$
- $\int e^{-5x}\,dx = -\frac{1}{5}e^{-5x} + C$
- $\int e^{-x}\,dx = -e^{-x} + C$

In the next example, we solve a differential equation that contains exponential expressions.

EXAMPLE ■ 5 Solve the differential equation

$$\frac{dy}{dx} = 3e^{2x} + 6e^{-3x}$$

subject to the initial condition $y = 4$ if $x = 0$.

SOLUTION As in Example 6 of Section 4.1, we may multiply both sides of the equation by dx and then integrate as follows:

$$dy = (3e^{2x} + 6e^{-3x})\,dx$$

$$\int dy = \int (3e^{2x} + 6e^{-3x})\,dx = 3\int e^{2x}\,dx + 6\int e^{-3x}\,dx$$

$$y = 3(\tfrac{1}{2})e^{2x} + 6(-\tfrac{1}{3})e^{-3x} + C$$

$$= \tfrac{3}{2}e^{2x} - 2e^{-3x} + C$$

Using the initial condition $y = 4$ if $x = 0$ gives us

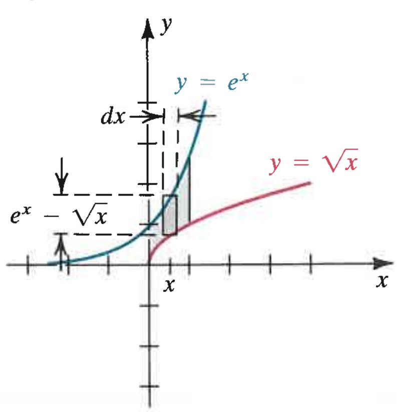
$$4 = \tfrac{3}{2}e^{0} - 2e^{0} + C = \tfrac{3}{2} - 2 + C.$$

Hence, $C = 4 - \frac{3}{2} + 2 = \frac{9}{2}$, and the solution of the differential equation is

$$y = \tfrac{3}{2}e^{2x} - 2e^{-3x} + \tfrac{9}{2}.$$

EXAMPLE ■ 6 Find the area of the region bounded by the graphs of the equations $y = e^x$, $y = \sqrt{x}$, $x = 0$, and $x = 1$.

Figure 6.19



SOLUTION The region and a typical rectangle of the type considered in Chapter 5 are shown in Figure 6.19. As usual, we list the following:

width of rectangle: dx

length of rectangle: $e^x - \sqrt{x}$

area of rectangle: $(e^x - \sqrt{x})\,dx$

We next take a limit of sums of these rectangular areas by applying the operator $\int_0^1$:

$$\int_0^1 (e^x - \sqrt{x})\,dx = \int_0^1 (e^x - x^{1/2})\,dx$$

$$= \left[e^x - \tfrac{2}{3}x^{3/2}\right]_0^1 = e - \tfrac{5}{3} \approx 1.05$$

In Chapter 4, we obtained integration formulas for the sine and cosine functions. We were unable to consider the remaining four trigonometric functions at that time because, as indicated in the next theorem, their integrals are logarithmic functions. In the theorem, we assume that $u = g(x)$, with g differentiable whenever the function is defined.

Theorem 6.25

(i) $\int \tan u\,du = -\ln|\cos u| + C$

(ii) $\int \cot u\,du = \ln|\sin u| + C$

(iii) $\int \sec u\,du = \ln|\sec u + \tan u| + C$

(iv) $\int \csc u\,du = \ln|\csc u - \cot u| + C$

PROOF It is sufficient to consider the case $u = x$, since the formulas for $u = g(x)$ then follow from the Chain Rule, Theorem (4.7).

To find $\int \tan x\,dx$, we first use a trigonometric identity to express $\tan x$ in terms of $\sin x$ and $\cos x$ as follows:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{1}{\cos x}\sin x\,dx$$

The form of the integrand on the right suggests that we make the substitution

$$v = \cos x, \qquad dv = -\sin x\,dx.$$

This gives us

$$\int \tan x\,dx = -\int \frac{1}{v}\,dv.$$

If $\cos x \neq 0$, then by Theorem (6.11)(ii),

$$\int \tan x\,dx = -\ln|v| + C = -\ln|\cos x| + C.$$

A formula for $\int \cot x\,dx$ may be obtained in similar fashion by first writing $\cot x = (\cos x)/(\sin x)$.

To find a formula for $\int \sec x\,dx$, we begin as follows:

$$\int \sec x\,dx = \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)\,dx$$

Using the substitution

$$v = \sec x + \tan x, \qquad dv = (\sec x \tan x + \sec^2 x)\,dx$$

gives us

$$\begin{aligned}\int \sec x\,dx &= \int \frac{1}{v}\,dv\\ &= \ln|v| + C\\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

A similar proof can be given for (iv).

If we use $\cos u = 1/\sec u$, $\sin u = 1/\csc u$, and $\ln(1/v) = -\ln v$, then formulas (i) and (ii) of Theorem (6.25) can be written as follows:

$$\int \tan u\,du = \ln|\sec u| + C$$

$$\int \cot u\,du = -\ln|\csc u| + C$$

EXAMPLE 7 Evaluate $\int x\cot x^2\,dx$.

SOLUTION To obtain the form $\int \cot u\;du$, we make the substitution

$$u = x^2, \qquad du = 2x\,dx.$$

We next introduce the factor 2 in the integrand as follows:

$$\int x\cot x^2\,dx = \tfrac{1}{2}\int (\cot x^2)2x\,dx$$

Since $u = x^2$ and $du = 2x\,dx$,

$$\begin{aligned}\int x\cot x^2\,dx = \tfrac{1}{2}\int \cot u\,du &= \tfrac{1}{2}\ln|\sin u| + C\\ &= \tfrac{1}{2}\ln|\sin x^2| + C.\end{aligned}$$

EXAMPLE 8 Evaluate $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$.

SOLUTION We make the substitution

$$u = \frac{x}{2}, \qquad du = \frac{1}{2}\,dx$$

and note that $u = 0$ if $x = 0$, and $u = \pi/4$ if $x = \pi/2$. Thus,

$$\begin{aligned}\int_0^{\pi/2} \tan\frac{x}{2}\,dx &= 2\int_0^{\pi/2} \tan\frac{x}{2}\cdot\frac{1}{2}\,dx\\ &= 2\int_0^{\pi/4} \tan u\,du = 2\,[\ln \sec u]_0^{\pi/4}.\end{aligned}$$

In this case, we may drop the absolute value sign given in Theorem (6.25)(iii), because $\sec u$ is positive if u is between 0 and $\pi/4$. Since $\ln\sec(\pi/4) = \ln\sqrt{2} = \frac{1}{2}\ln 2$ and $\ln\sec 0 = \ln 1 = 0$, it follows that

$$\int_0^{\pi/2} \tan\frac{x}{2}\,dx = 2\cdot\frac{1}{2}\ln 2 = \ln 2 \approx 0.69.$$

EXAMPLE 9 Evaluate $\int e^{2x}\sec e^{2x}\,dx$.

SOLUTION We let

$$u = e^{2x}, \qquad du = 2e^{2x}\,dx$$

and proceed as follows:

$$\begin{aligned}\int e^{2x}\sec e^{2x}\,dx &= \tfrac{1}{2}\int (\sec e^{2x})2e^{2x}\,dx\\ &= \tfrac{1}{2}\int \sec u\;du\\ &= \tfrac{1}{2}\ln|\sec u + \tan u| + C\\ &= \tfrac{1}{2}\ln|\sec e^{2x} + \tan e^{2x}| + C\end{aligned}$$

EXAMPLE 10 Evaluate $\int (\csc x - 1)^2\,dx$.

SOLUTION

$$\begin{aligned}\int (\csc x - 1)^2\,dx &= \int (\csc^2 x - 2\csc x + 1)\,dx\\ &= \int \csc^2 x\,dx - 2\int \csc x\;dx + \int dx\\ &= -\cot x - 2\ln|\csc x - \cot x| + x + C.\end{aligned}$$

We shall discuss additional methods for integrating trigonometric expressions in Chapter 7.

EXERCISES 6.4

Exer. 1–36: Evaluate the integral.

1 (a) $\int \frac{1}{2x+7}\,dx$ **(b)** $\int_{-2}^{1} \frac{1}{2x+7}\,dx$

2 (a) $\int \frac{1}{4-5x}\,dx$ **(b)** $\int_{-1}^{0} \frac{1}{4-5x}\,dx$

3 (a) $\int \frac{4x}{x^2-9}\,dx$ **(b)** $\int_1^2 \frac{4x}{x^2-9}\,dx$

4 (a) $\int \frac{3x}{x^2+4}\,dx$ **(b)** $\int_1^2 \frac{3x}{x^2+4}\,dx$

5 (a) $\int e^{-4x}\,dx$ (b) $\int_1^3 e^{-4x}\,dx$

6 (a) $\int x^2 e^{3x^3}\,dx$ (b) $\int_1^2 x^2 e^{3x^3}\,dx$

7 (a) $\int \tan 2x\,dx$ (b) $\int_0^{\pi/8} \tan 2x\,dx$

8 (a) $\int \cot \frac{1}{3}x\,dx$ (b) $\int_{3\pi/2}^{9\pi/4} \cot \frac{1}{3}x\,dx$

9 (a) $\int \csc \frac{1}{2}x\,dx$ (b) $\int_{\pi}^{5\pi/3} \csc \frac{1}{2}x\,dx$

10 (a) $\int \sec 3x\,dx$ (b) $\int_0^{\pi/12} \sec 3x\,dx$

11 $\int \frac{x-2}{x^2-4x+9}\,dx$ 12 $\int \frac{x^3}{x^4-5}\,dx$

13 $\int \frac{(x+2)^2}{x}\,dx$ 14 $\int \frac{(2+\ln x)^{10}}{x}\,dx$

15 $\int \frac{\ln x}{x}\,dx$ 16 $\int \frac{1}{x(\ln x)^2}\,dx$

17 $\int (x+e^{5x})\,dx$ 18 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

19 $\int \frac{3\sin x}{1+2\cos x}\,dx$ 20 $\int \frac{\sec^2 x}{1+\tan x}\,dx$

21 $\int \frac{(e^x+1)^2}{e^x}\,dx$ 22 $\int \frac{e^x}{(e^x+1)^2}\,dx$

23 $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$ 24 $\int \frac{e^x}{e^x+1}\,dx$

25 $\int \frac{\cot \sqrt[3]{x}}{\sqrt[3]{x^2}}\,dx$ 26 $\int e^x(1+\tan e^x)\,dx$

27 $\int \frac{1}{\cos 2x}\,dx$ 28 $\int (x+\csc 8x)\,dx$

29 $\int \frac{\tan e^{-3x}}{e^{3x}}\,dx$ 30 $\int e^{\cos x}\sin x\,dx$

31 $\int \frac{\cos^2 x}{\sin x}\,dx$ 32 $\int \frac{\tan^2 2x}{\sec 2x}\,dx$

33 $\int \frac{\cos x \sin x}{\cos^2 x - 1}\,dx$ 34 $\int (\tan 3x + \sec 3x)\,dx$

35 $\int (1+\sec x)^2\,dx$ 36 $\int \csc x(1-\csc x)\,dx$

Exer. 37–38: Find the area of the region bounded by the graphs of the given equations.

37 $y = e^{2x}, \quad y=0, \quad x=0, \quad x=\ln 3$

38 $y = 2\tan x, \quad y=0, \quad x=0, \quad x=\pi/4$

Exer. 39–40: Find the volume of the solid generated if the region bounded by the graphs of the equations is revolved about the indicated axis.

39 $y = e^{-x^2}, \quad x=0, \quad x=1, \quad y=0;$ y-axis

40 $y = \sec x, \quad x=-\pi/3, \quad x=\pi/3, \quad y=0;$ x-axis

Exer. 41–44: Solve the differential equation subject to the given conditions.

41 $y' = 4e^{2x} + 3e^{-2x};$ $y=4$ if $x=0$

42 $y' = 3e^{4x} - 8e^{-2x};$ $y=-2$ if $x=0$

43 $y'' = 3e^{-x};$ $y=-1$ and $y'=1$ if $x=0$

44 $y'' = 6e^{2x};$ $y=-3$ and $y'=2$ if $x=0$

Exer. 45–46: A nonnegative function f defined on a closed interval $[a,b]$ is called a *probability density function* if $\int_a^b f(x)\,dx = 1$. Determine c so that the resulting function is a probability density function.

45 $f(x) = \frac{cx}{x^2+4}$ for $0 \le x \le 3$

46 $f(x) = cxe^{-x^2}$ for $0 \le x \le 10$

47 A culture of bacteria is growing at a rate of $3e^{0.2t}$ per hour, with t in hours and $0 \le t \le 20$.

(a) How many new bacteria will be in the culture after the first five hours?

(b) How many new bacteria are introduced in the sixth through the fourteenth hours?

(c) For approximately what value of t will the culture contain 150 new bacteria?

48 If a savings bond is purchased for \$500 with interest compounded continuously at 7% per year, then after t years the bond will be worth $500e^{0.07t}$ dollars.

(a) Approximately when will the bond be worth \$1000?

(b) Approximately when will the value of the bond be growing at a rate of \$50 per year?

49 The specific heat c of a metal such as silver is constant at temperatures T above 200 °K. If the temperature of the metal increases from T_1 to T_2, the area under the curve $y = c/T$ from T_1 to T_2 is called the *change in entropy* ΔS, a measurement of the increased molecular disorder of the system. Express ΔS in terms of T_1 and T_2.

50 The 1952 earthquake in Assam had a magnitude of 8.7 on the Richter scale—the largest ever recorded. (The October 1989 San Francisco earthquake had a magnitude of 7.1.) Seismologists have determined that if the largest earthquake in a given year has magnitude R, then the energy E (in joules) released by all earthquakes

in that year can be estimated by using the formula

$$E = 9.13 \times 10^{12} \int_0^R e^{1.25x}\,dx.$$

Find E if $R = 8$.

51 In a circuit containing a 12-volt battery, a resistor, and a capacitor, the current $I(t)$ at time t is predicted to be $I(t) = 10e^{-4t}$ amperes. If $Q(t)$ is the charge (in coulombs) on the capacitor, then $I = dQ/dt$.

(a) If $Q(0) = 0$, find $Q(t)$.

(b) Find the charge on the capacitor after a long period of time.

52 A country that presently has coal reserves of 50 million tons used 6.5 million tons last year. On the basis of population projections, the rate of consumption R (in million tons/year) is expected to increase according to the formula $R = 6.5e^{0.02t}$, where t is the time in years. If the country uses only its own resources, estimate how many years the coal reserves will last.

53 A very small spherical particle (on the order of 5 microns in diameter) is projected into still air with an initial velocity of v_0 m/sec, but its velocity decreases because of drag forces. Its velocity after t seconds is given by $v(t) = v_0 e^{-t/k}$ for some positive constant k.

(a) Express the distance that the particle travels as a function of t.

(b) The *stopping distance* is the distance traveled by the particle before it comes to rest. Express the stopping distance in terms of v_0 and k.

54 If the temperature remains constant, the pressure p and the volume v of an expanding gas are related by the equation $pv = k$ for some constant k. Show that the work done if the gas expands from v_0 to v_1 is $k\ln(v_1/v_0)$. (*Hint:* See Example 5 of Section 5.6.)

c **Exer. 55–58: Use a numerical integration method or routine to approximate the definite integral to four decimal places.**

55 $\int_0^1 e^{-x^2}\,dx$ 56 $\int_{-4}^{8} e^{-x^2}\,dx$

57 $\int_{0.5}^{6.5} \frac{e^x}{x}\,dx$ 58 $\int_0^3 \sqrt{x+1}\,e^x\,dx$

c 59 Approximate the area bounded by the graphs of $y = e^x$ and $y = 4 - x^2$.

c 60 Approximate the volume of the solid generated by revolving the graph of $y = e^x$, $-10 \le x \le 1$, about the x-axis.

c 61 Approximate the arc length of the part of the curve $y = e^x$ that lies inside the circle $x^2 + y^2 = 25$.

6.5 GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Throughout this section, a will denote a positive real number. Let us begin by defining a^x for every real number x. If the exponent is a *rational* number r, then applying Theorems (6.19)(ii) and (6.12)(iii) yields

$$a^r = e^{\ln a^r} = e^{r \ln a}.$$

This formula is the motivation for the following definition of a^x.

Definition of a^x 6.26

$$a^x = e^{x \ln a}$$

for every $a > 0$ and every real number x.

ILLUSTRATION

- $2^{\pi} = e^{\pi \ln 2} \approx e^{2.1775860903} \approx 8.82497782708$
- $(\frac{1}{2})^{\sqrt{3}} = e^{\sqrt{3}\ln(1/2)} \approx e^{-1.20056613385} \approx 0.301023743931$

If $f(x) = a^x$, then f is the **exponential function with base a**. Since e^x is positive for every x, so is a^x. To approximate values of a^x, we may use a calculator or refer to standard tables of logarithmic and exponential functions.

It is now possible to prove that the law of logarithms stated in Theorem (6.12)(iii) is also true for irrational exponents. Thus, if u is any *real* number, then, by Definition (6.26) and Theorem (6.19)(i),

$$\ln a^u = \ln e^{u \ln a} = u \ln a.$$

The next theorem states that properties of rational exponents from elementary algebra are also true for real exponents.

***Laws of Exponents* 6.27**

Let $a > 0$ and $b > 0$. If u and v are any real numbers, then

$$a^u a^v = a^{u+v} \qquad (a^u)^v = a^{uv} \qquad (ab)^u = a^u b^u$$

$$\frac{a^u}{a^v} = a^{u-v} \qquad \left(\frac{a}{b}\right)^u = \frac{a^u}{b^u}$$

PROOF To show that $a^u a^v = a^{u+v}$, we use Definition (6.26) and Theorem (6.20)(i) as follows:

$$\begin{aligned} a^u a^v &= e^{u \ln a} e^{v \ln a} \\ &= e^{u \ln a + v \ln a} \\ &= e^{(u+v)\ln a} \\ &= a^{u+v} \end{aligned}$$

To prove that $(a^u)^v = a^{uv}$, we first use Definition (6.26) with a^u in place of a and $v = x$ to write

$$(a^u)^v = e^{v \ln a^u}.$$

Using the fact that $\ln a^u = u \ln a$ and then applying Definition (6.26), we obtain

$$(a^u)^v = e^{vu \ln a} = a^{vu} = a^{uv}.$$

The proofs of the remaining laws are similar. ■

As usual, in part (ii) of the next theorem, $u = g(x)$, where g is differentiable.

***Theorem* 6.28**

(i) $\dfrac{d}{dx}(a^x) = a^x \ln a$

(ii) $\dfrac{d}{dx}(a^u) = (a^u \ln a)\dfrac{du}{dx}$

PROOF Applying Definition (6.26) and Theorem (6.22), we obtain

$$\begin{aligned} \frac{d}{dx}(a^x) &= \frac{d}{dx}(e^{x \ln a}) \\ &= e^{x \ln a}\frac{d}{dx}(x \ln a) \\ &= e^{x \ln a}(\ln a). \end{aligned}$$

Since $e^{x \ln a} = a^x$, this gives us formula (i):

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Formula (ii) follows from the chain rule. ■

Note that if $a = e$, then Theorem (6.28)(i) reduces to Theorem (6.21), since $\ln e = 1$.

ILLUSTRATION

- $\dfrac{d}{dx}(3^x) = 3^x \ln 3$
- $\dfrac{d}{dx}(10^x) = 10^x \ln 10$
- $\dfrac{d}{dx}(3^{\sqrt{x}}) = (3^{\sqrt{x}} \ln 3)\dfrac{d}{dx}(\sqrt{x}) = (3^{\sqrt{x}} \ln 3)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{3^{\sqrt{x}} \ln 3}{2\sqrt{x}}$
- $\dfrac{d}{dx}(10^{\sin x}) = (10^{\sin x} \ln 10)\dfrac{d}{dx}(\sin x) = (10^{\sin x} \ln 10) \cos x$

If $a > 1$, then $\ln a > 0$ and, therefore, $(d/dx)(a^x) = a^x \ln a > 0$. Hence a^x is increasing on the interval $(-\infty, \infty)$ if $a > 1$.

If $0 < a < 1$, then $\ln a < 0$ and $(d/dx)(a^x) = a^x \ln a < 0$. Thus, a^x is decreasing for every x if $0 < a < 1$.

Figure 6.20

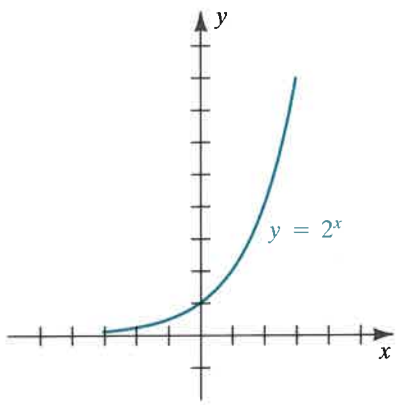
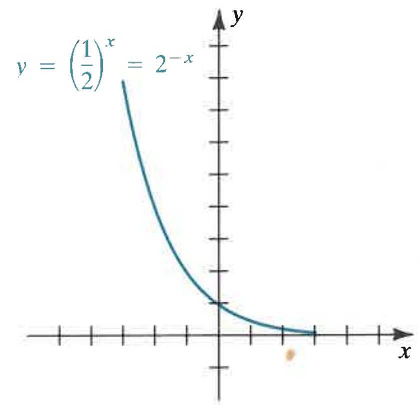


Figure 6.21



The graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x = 2^{-x}$ are sketched in Figures 6.20 and 6.21. The graph of $y = a^x$ has the general shape illustrated in Figure 6.20 or 6.21 if $a > 1$ or $0 < a < 1$, respectively.

If $u = g(x)$, *it is important to distinguish between expressions of the form* a^u *and* u^a. To differentiate a^u, we use Theorem (6.28); for u^a, the power rule must be employed, as illustrated in the next example.

EXAMPLE ■ 1 Find y' if $y = (x^2+1)^{10} + 10^{x^2+1}$.

SOLUTION Using the power rule for functions and Theorem (6.28), we obtain

$$\begin{aligned} y' &= 10(x^2+1)^9(2x) + (10^{x^2+1}\ln 10)(2x) \\ &= 20x[(x^2+1)^9 + 10^{x^2}\ln 10]. \end{aligned}$$

The integration formula in (i) of the next theorem may be verified by showing that the integrand is the derivative of the expression on the right side of the equation. Formula (ii) follows from Theorem (6.28)(ii), where $u = g(x)$.

Theorem 6.29

(i) $\displaystyle\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$

(ii) $\displaystyle\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$

EXAMPLE ■ 2 Evaluate:

(a) $\displaystyle\int 3^x\,dx$ **(b)** $\displaystyle\int x3^{(x^2)}\,dx$

SOLUTION

(a) Using (i) of Theorem (6.29) yields

$$\int 3^x\,dx = \left(\frac{1}{\ln 3}\right)3^x + C.$$

(b) To use (ii) of Theorem (6.29), we make the substitution

$$u = x^2, \qquad du = 2x\,dx$$

and proceed as follows:

$$\begin{aligned} \int x3^{(x^2)}\,dx &= \tfrac{1}{2}\int 3^{(x^2)}(2x)\,dx = \tfrac{1}{2}\int 3^u\,du \\ &= \frac{1}{2}\left(\frac{1}{\ln 3}\right)3^u + C = \left(\frac{1}{2\ln 3}\right)3^{(x^2)} + C \end{aligned}$$

EXAMPLE ■ 3 An important problem in oceanography is determining the light intensity at different ocean depths. The *Beer–Lambert law* states that at a depth x (in meters), the light intensity $I(x)$ (in calories/cm²/sec) is given by $I(x) = I_0a^x$, where I_0 and a are positive constants.

(a) What is the light intensity at the surface?

(b) Find the rate of change of the light intensity with respect to depth at a depth x.

(c) If $a = 0.4$ and $I_0 = 10$, find the average light intensity between the surface and a depth of x meters.

(d) Show that $I(x) = I_0e^{kx}$ for some constant k.

SOLUTION

(a) At the surface, $x = 0$ and

$$I(0) = I_0a^0 = I_0.$$

Hence the light intensity at the surface is I_0.

(b) The rate of change of $I(x)$ with respect to x is $I'(x)$. Thus,

$$I'(x) = I_0(a^x\ln a) = (\ln a)(I_0a^x) = (\ln a)I(x).$$

Hence the rate of change $I'(x)$ at depth x is directly proportional to $I(x)$, and the constant of proportionality is $\ln a$.

(c) If $I(x) = 10(0.4)^x$, then, by Definition (4.29), the average value of I on the interval [0, 5] is

$$\begin{aligned} I_{\text{av}} &= \frac{1}{5-0}\int_0^5 10(0.4)^x\,dx = 2\int_0^5 (0.4)^x\,dx \\ &= 2\left[\frac{1}{\ln(0.4)}(0.4)^x\right]_0^5 = \frac{2}{\ln(0.4)}[(0.4)^5 - (0.4)^0] \\ &= \frac{-1.97952}{\ln(0.4)} \approx 2.16. \end{aligned}$$

(d) Using Definition (6.26) yields

$$I(x) = I_0 a^x = I_0 e^{x \ln a} = I_0 e^{kx},$$

where $k = \ln a$.

If $a \neq 1$ and $f(x) = a^x$, then f is a one-to-one function. Its inverse function is denoted by $\log_a$ and is called the **logarithmic function with base a**. Another way of stating this relationship is as follows.

Definition of $\log_a x$ **6.30**

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

The expression $\log_a x$ is called **the logarithm of x with base a**. In this terminology, natural logarithms are logarithms with base e—that is,

$$\ln x = \log_e x.$$

Laws of logarithms similar to Theorem (6.12) are true for logarithms with base a.

To obtain the relationship between $\log_a$ and ln, consider $y = \log_a x$, or, equivalently, $x = a^y$. Taking the natural logarithm of both sides of the last equation gives us $\ln x = y \ln a$, or $y = (\ln x)/(\ln a)$ and thus proves that

$$\log_a x = \frac{\ln x}{\ln a}.$$

Differentiating both sides of the last equation leads to (i) of the next theorem. Using the chain rule and generalizing to absolute values as in Theorem (6.11) gives us (ii), where $u = g(x)$.

Theorem **6.31**

(i) $\dfrac{d}{dx}(\log_a x) = \dfrac{d}{dx}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln a} \cdot \dfrac{1}{x}$

(ii) $\dfrac{d}{dx}(\log_a |u|) = \dfrac{d}{dx}\left(\dfrac{\ln |u|}{\ln a}\right) = \dfrac{1}{\ln a} \cdot \dfrac{1}{u}\dfrac{du}{dx}$

ILLUSTRATION

- $\dfrac{d}{dx}(\log_2 x) = \dfrac{d}{dx}\left(\dfrac{\ln x}{\ln 2}\right) = \dfrac{1}{\ln 2} \cdot \dfrac{1}{x} = \dfrac{1}{(\ln 2)x}$
- $\dfrac{d}{dx}(\log_2 |x^2 - 9|) = \dfrac{d}{dx}\left(\dfrac{\ln |x^2 - 9|}{\ln 2}\right) = \dfrac{1}{\ln 2} \cdot \dfrac{1}{x^2 - 9} \cdot 2x = \dfrac{2x}{(\ln 2)(x^2 - 9)}$

Logarithms with base 10 are useful for certain applications (see Exercises 50–54). We refer to such logarithms as **common logarithms** and use

the symbol **$\log x$** as an abbreviation for $\log_{10} x$. This notation is used in the next example where we perform, in a new context, the familiar task of determining the tangent line to a graph.

EXAMPLE ■ 4 If $f(x) = \log \sqrt[3]{(2x+5)^2}$,

(a) find $f'(x)$

(b) graph both the function f and the line tangent to its graph at $x = -0.6$.

SOLUTION

(a) Although most graphing utilities can work with both common and natural logarithms, we express the function in terms of natural logarithms to make differentiating easier. We first write $f(x) = \log(2x+5)^{2/3}$. The law $\log u^r = r \log u$ is true only if $u > 0$; however, since $(2x+5)^{2/3} = |2x+5|^{2/3}$, we may proceed as follows:

$$\begin{aligned} f(x) &= \log(2x+5)^{2/3} \\ &= \log|2x+5|^{2/3} \\ &= \tfrac{2}{3}\log|2x+5| \\ &= \frac{2}{3}\frac{\ln|2x+5|}{\ln 10} \end{aligned}$$

Differentiating yields

$$f'(x) = \frac{2}{3} \cdot \frac{1}{\ln 10} \cdot \frac{1}{2x+5}(2) = \frac{4}{3(2x+5)\ln 10}.$$

(b) We begin by using a graphing utility to plot the function $f(x) = \log(2x+5)^{2/3}$. To graph the tangent line, we must first find an equation for it using the point–slope formula. Since

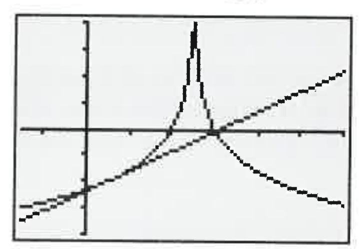
$$f(-0.6) = \tfrac{2}{3}\log 3.8 \approx 0.386522,$$

the point of tangency is $(-0.6, 0.386522)$. The slope of the tangent line is the value of the derivative f' at $x = -0.6$. From part (a), we have

$$f'(-0.6) = \frac{4}{11.4 \ln 10} \approx 0.152384.$$

Thus, an equation for the line tangent to the graph at $x = -0.6$ is approximately $y = 0.386522 + 0.152384(x + 0.6)$. We plot the tangent line with the graph of the function on the same coordinate axes, obtaining the results shown in Figure 6.22.

Figure 6.22
$-6 \le x \le 1.5,\ -0.8 \le y \le 0.8$



Now that we have defined irrational exponents, we may consider the **general power function** f given by $f(x) = x^c$ for any real number c. If c is irrational, then, by definition, the domain of f is the set of positive real numbers. Using Definition (6.26) and Theorems (6.22) and (6.11)(i), we have

$$\frac{d}{dx}(x^c) = \frac{d}{dx}(e^{c\ln x}) = e^{c\ln x}\frac{d}{dx}(c\ln x)$$
$$= e^{c\ln x}\left(\frac{c}{x}\right) = x^c\left(\frac{c}{x}\right) = cx^{c-1}.$$

This result proves that the power rule is true for irrational as well as rational exponents. The power rule for functions may also be extended to irrational exponents.

ILLUSTRATION

- $\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$
- $\frac{d}{dx}(1+e^{2x})^{\pi} = \pi(1+e^{2x})^{\pi-1}\frac{d}{dx}(1+e^{2x})$

$$= \pi(1+e^{2x})^{\pi-1}(2e^{2x}) = 2\pi e^{2x}(1+e^{2x})^{\pi-1}$$

EXAMPLE 5 If $y = x^x$ and $x > 0$,

(a) find dy/dx

(b) graph both the function and its derivative

SOLUTION

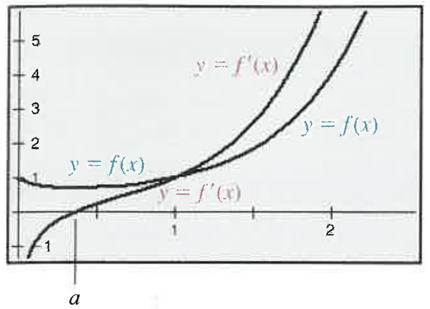
(a) Since the exponent in x^x is a variable, the power rule may not be used. Similarly, Theorem (6.28) is not applicable, since the base a is not a fixed real number. However, by Definition (6.26), $x^x = e^{x\ln x}$ for every $x > 0$, and hence

$$\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x\ln x})$$
$$= e^{x\ln x}\frac{d}{dx}(x\ln x)$$
$$= e^{x\ln x}\left[x\left(\frac{1}{x}\right) + (1)\ln x\right] = x^x(1+\ln x).$$

Another way of solving this problem is to use the method of logarithmic differentiation introduced in the preceding section. In this case, we take the natural logarithm of both sides of the equation $y = x^x$ and then differentiate implicitly as follows:

$$\ln y = \ln x^x = x\ln x$$
$$\frac{dy}{dx}(\ln y) = \frac{d}{dx}(x\ln x)$$
$$\frac{1}{y}\frac{dy}{dx}(y) = 1 + \ln x$$
$$\frac{dy}{dx}(y) = y(1+\ln x) = x^x(1+\ln x)$$

Figure 6.23
$f(x) = x^x$
$f'(x) = x^x(1+\ln x)$
$0 \le x \le 2.5,\ -1.5 \le y \le 6$



(b) In part (a), we saw that the expression x^x is unusual because both the base and the exponent are variable. The expression for the derivative f' is even more complicated, since it involves both x^x and $(1+\ln x)$. To gain a better understanding of the function $f(x) = x^x$, we use a graphing utility to plot its graph and the graph of its derivative, as shown in Figure 6.23. Several features of the function are evident from these graphs. First, when $x = 0$, the expression x^x becomes the undefined algebraic expression 0^0, but from the graph it appears that the function $f(x) = x^x$ approaches 1 as x approaches 0 from the right. We shall prove this result in Section 6.9. Second, if we examine the graph of the derivative f', we see that it is negative for $0 < x < a$ and positive for $x > a$. We can determine the value of a by examining the sign of $f'(x)$. Since $x^x > 0$, $f'(x) < 0$ if $1 + \ln x < 0$ or, equivalently, $x < e^{-1}$. Hence, $a = e^{-1} \approx 0.37$. Thus, the function $f(x) = x^x$ decreases to an absolute minimum at $x = a$ and then increases for $x > a$. It also appears from the graph of the derivative that it is unbounded in the negative direction as $x \to 0^+$, so the function $f(x) = x^x$ is not differentiable at $x = 0$.

We conclude this section by expressing the number e as a limit.

Theorem 6.32

(i) $\lim_{h\to 0}(1+h)^{1/h} = e$ **(ii)** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

PROOF Applying the definition of derivative (2.5) to $f(x) = \ln x$ and using laws of logarithms yields

$$f'(x) = \lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0}\frac{1}{h}\ln\frac{x+h}{x}$$
$$= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{1/h}.$$

Since $f'(x) = 1/x$, we have, for $x = 1$,

$$1 = \lim_{h\to 0}\ln(1+h)^{1/h}.$$

We next observe, from Theorem (6.19), that

$$(1+h)^{1/h} = e^{\ln(1+h)^{1/h}}.$$

Since the natural exponential function is continuous at 1, it follows from Theorem (1.25) that

$$\lim_{h\to 0}(1+h)^{1/h} = \lim_{h\to 0}[e^{\ln(1+h)^{1/h}}]$$
$$= e^{[\lim_{h\to 0}\ln(1+h)^{1/h}]} = e^1 = e.$$

This establishes part (i) of the theorem. The limit in part (ii) may be obtained by introducing the change of variable $n = 1/h$ with $h > 0$. ■

The formulas in Theorem (6.32) are sometimes used to *define* the number e. You may find it instructive to calculate $(1+h)^{1/h}$ for numerically small values of h. Some approximate values are given in the following table.

h	$(1+h)^{1/h}$	h	$(1+h)^{1/h}$
−0.01	2.73199902643	0.01	2.70481382942
−0.001	2.71964221644	0.001	2.71692393224
−0.0001	2.71841775501	0.0001	2.71814592683
−0.00001	2.71829541999	0.00001	2.71826823717
−0.000001	2.71828318760	0.000001	2.71828046932
−0.0000001	2.71828196437	0.0000001	2.71828169255

To five decimal places, $e \approx 2.71828$.

EXERCISES 6.5

Exer. 1–24: Find $f'(x)$ if $f(x)$ is the given expression.

1 7^x 2 5^{-x}

3 8^{x^2+1} 4 $9^{\sqrt{x}}$

5 $\log(x^4+3x^2+1)$ 6 $\log_3|6x-7|$

7 5^{3x-4} 8 3^{2-x^2}

9 $(x^2+1)10^{1/x}$ 10 $(10^x+10^{-x})^{10}$

11 $\log(3x^2+2)^5$ 12 $\log\sqrt{x^2+1}$

13 $\log_5\left|\dfrac{6x+4}{2x-3}\right|$ 14 $\log\left|\dfrac{1-x^2}{2-5x^3}\right|$

15 $\log\ln x$ 16 $\ln\log x$

17 x^e+e^x 18 $x^\pi\pi^x$

19 $(x+1)^x$ 20 x^{4+x^2}

21 $2^{\sin^2 x}$ 22 $4^{\sec 3x}$

23 (a) e^e (b) x^5 (c) $x^{\sqrt{5}}$ (d) $(\sqrt{5})^x$ (e) $x^{(x^2)}$

24 (a) π^π (b) x^4 (c) x^π (d) π^x (e) x^{2x}

c **Exer. 25–28: Plot the graph of the function and the line tangent to the graph at the point $(a, f(a))$.**

25 $f(x)=5^{3x-4}$; $a=1$

26 $f(x)=3^{2-x^2}$; $a=-1.5$

27 $f(x)=\log(3x^2+2)^5$; $a=5$

28 $f(x)=\log\sqrt{x^2+1}$; $a=10$

Exer. 29–44: Evaluate the integral.

29 (a) $\int 7^x\,dx$ (b) $\int_{-2}^{1} 7^x\,dx$

30 (a) $\int 3^x\,dx$ (b) $\int_{-1}^{0} 3^x\,dx$

31 (a) $\int 5^{-2x}\,dx$ (b) $\int_{1}^{2} 5^{-2x}\,dx$

32 (a) $\int 2^{3x-1}\,dx$ (b) $\int_{-1}^{1} 2^{3x-1}\,dx$

33 $\int 10^{3x}\,dx$ 34 $\int 5^{-5x}\,dx$

35 $\int x(3^{-x^2})\,dx$ 36 $\int \dfrac{(2^x+1)^2}{2^x}\,dx$

37 $\int \dfrac{2^x}{2^x+1}\,dx$ 38 $\int \dfrac{3^x}{\sqrt{3^x+4}}\,dx$

39 $\int \dfrac{1}{x\log x}\,dx$ 40 $\int \dfrac{10^{\sqrt{x}}}{\sqrt{x}}\,dx$

41 $\int 3^{\cos x}\sin x\,dx$ 42 $\int \dfrac{5^{\tan x}}{\cos^2 x}\,dx$

43 (a) $\int \pi^\pi\,dx$ (b) $\int x^4\,dx$

(c) $\int x^\pi\,dx$ (d) $\int \pi^x\,dx$

44 (a) $\int e^e\,dx$ (b) $\int x^5\,dx$

(c) $\int x^{\sqrt{5}}\,dx$ (d) $\int (\sqrt{5})^x\,dx$

45 Find the area of the region bounded by the graphs of $y=2^x$, $x+y=1$, and $x=1$.

46 The region under the graph of $y=3^{-x}$ from $x=1$ to $x=2$ is revolved about the x-axis. Find the volume of the resulting solid.

47 An economist predicts that the buying power $B(t)$ of a dollar t years from now will decrease according to the formula $B(t)=(0.95)^t$.

(a) At approximately what rate will the buying power be decreasing two years from now?

(b) Estimate the *average buying power* of the dollar over the next two years.

48 When a person takes a 100-mg tablet of an asthma drug orally, the rate R at which the drug enters the bloodstream is predicted to be $R=5(0.95)^t$ mg/min. If the bloodstream does not contain any trace of the drug when the tablet is taken, determine the number of minutes needed for 50 mg to enter the bloodstream.

49 One thousand trout, each one year old, are introduced into a large pond. The number still alive after t years is predicted to be $N(t)=1000(0.9)^t$.

(a) Approximate the death rate dN/dt at times $t=1$ and $t=5$. At what rate is the population decreasing when $N=500$?

(b) The weight $W(t)$ (in pounds) of an individual trout is expected to increase according to the formula $W(t)=0.2+1.5t$. After approximately how many years is the total number of pounds of trout in the pond a maximum?

50 The vapor pressure P (in psi), a measure of the volatility of a liquid, is related to its temperature T (in °F) by the *Antoine equation*: $\log P = a+[b/(c+T)]$, for constants a, b, and c. Vapor pressure increases rapidly with an increase in temperature. Find conditions on a, b, and c that guarantee that P is an increasing function of T.

51 Chemists use a number denoted by pH to describe quantitatively the acidity or basicity of solutions. By definition, $\mathrm{pH}=-\log[\mathrm{H}^+]$, where $[\mathrm{H}^+]$ is the hydrogen ion concentration in moles per liter. For a certain brand of vinegar, it is estimated (with a maximum percentage error of $\pm 0.5\%$) that $[\mathrm{H}^+]\approx 6.3\times 10^{-3}$. Calculate the pH and use differentials to estimate the maximum percentage error in the calculation.

52 The magnitude R (on the Richter scale) of an earthquake of intensity I may be found by means of the formula $R=\log(I/I_0)$, where I_0 is a certain minimum intensity. Suppose the intensity of an earthquake is estimated to be 100 times I_0. If the maximum percentage error in the estimate is $\pm 1\%$, use differentials to approximate the maximum percentage error in the calculated value of R.

53 Let $R(x)$ be the reaction of a subject to a stimulus of strength x. For example, if the stimulus x is *saltiness* (in grams of salt per liter), $R(x)$ may be the subject's estimate of how salty the solution tasted on a scale from 0 to 10. A function that has been proposed to relate R to x is given by the *Weber–Fechner formula*: $R = a\log(x/x_0)$, where a is a positive constant.

(a) Show that $R=0$ for the *threshold stimulus* $x=x_0$.

(b) The derivative $S=dR/dx$ is the *sensitivity* at stimulus level x and measures the ability to detect small changes in stimulus level. Show that S is inversely proportional to x, and compare $S(x)$ to $S(2x)$.

54 The loudness of sound, as experienced by the human ear, is based on intensity level. A formula used for finding the intensity level α that corresponds to a sound intensity I is $\alpha = 10\log(I/I_0)$ decibels, where I_0 is a special value of I agreed to be the weakest sound that can be detected by the ear under certain conditions. Find the rate of change of α with respect to I if

(a) I is 10 times as great as I_0

(b) I is 1000 times as great as I_0

(c) I is 10,000 times as great as I_0 (This is the intensity level of the average voice.)

55 If a principal of P dollars is invested in a savings account for t years and the yearly interest rate r (expressed as a decimal) is compounded n times per year, then the amount A in the account after t years is given by the *compound interest formula*:

$$A = P[1+(r/n)]^{nt}.$$

(a) Let $h=r/n$ and show that

$$\ln A = \ln P + rt\ln(1+h)^{1/h}.$$

(b) Let $n\to\infty$ and use the expression in part (a) to establish the formula $A=Pe^{rt}$ for interest *compounded continuously*.

56 Establish Theorem (6.32)(ii) by using the limit in part (i) and the change of variable $n=1/h$.