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INTRODUCTION

AS WE WATCH A FORMATION of swans or geese fly across the late afternoon sky, our focus may shift back and forth between the individual and the group. When we consider the individual bird, we may wonder about its motion: How fast does it fly? How quickly can it adjust its speed? When our perspective moves to the formation, we have questions about the smooth curve that we imagine connecting the birds: How does the curve change shape? Does the same curve occur in other natural phenomena? What geometric properties does this curve have? In this chapter, we begin to study the *derivative*, a principal tool of calculus designed to help answer some of these questions.

We begin the chapter by considering two applied problems in Section 2.1. The first is to find the slope of the tangent line at a point on the graph of a function, and the second is to define the velocity of an object moving along a line. Remarkably, these seemingly diverse applications lead to the same concept: the derivative.

Our discussion provides insight into the power and generality of mathematics. Specifically, we eliminate the geometric and physical aspects of the two problems and define the derivative in Section 2.2 as the limit of an expression involving a function f. This allows us to apply the derivative concept to any quantity that can be represented by a function. Since quantities of this type occur in nearly every field of knowledge, applications of the derivative are numerous and varied, but each concerns a *rate of change*. Thus, returning to the two problems that started it all, we see that the slope of the tangent line may be used to describe the rate at which a graph rises (or falls) and velocity is the rate at which distance changes with respect to time.

Our main objective in Section 2.2 is to define derivatives and develop rules to find them without using limits. Section 2.3 presents the basic techniques for differentiation. We examine ways to determine derivatives for polynomials and trigonometric functions (Section 2.4) and for more complicated functions that can be built up from them by addition, subtraction, multiplication, division, and composition. We consider the chain rule, which is fundamental for the differentiation of composite functions, in Section 2.5. We then turn to derivatives where functions are described either explicitly or implicitly in Section 2.6, which presents implicit differentiation techniques.

In the final sections of the chapter, we consider two important applications of the derivative that involve estimation and approximation: linear approximations and differentials in Section 2.8 and Newton's method in Section 2.9. We shall discuss many more applications in subsequent chapters.

The flight of a flock of birds suggests questions about motion and curves that the derivative helps answer.

CHAPTER 2

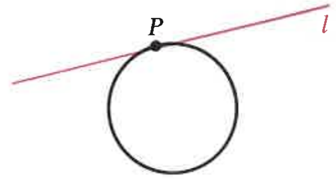
The Derivative

2.1 TANGENT LINES AND RATES OF CHANGE

In this section, we examine two general problems whose solutions use limits of the same form. First, we consider how to define the tangent line to the graph of a function. Then, we turn to the problem of measuring rates of change, with particular emphasis on velocity as the rate of change of position of a moving object.

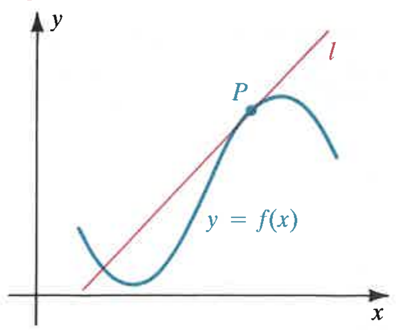
TANGENT LINES

Figure 2.1



Tangent lines to graphs are useful in many applications of calculus. In geometry, the tangent line l at a point P on a circle may be interpreted as the line that intersects the circle only at P, as illustrated in Figure 2.1. We cannot extend this interpretation to the graph of a function f, since a line may "touch" the graph of f at some isolated point P and then intersect it again at another point, as illustrated in Figure 2.2. Our plan is to define the *slope* of the tangent line at P, for if the slope is known, we can find an equation for l by using the point–slope form of the equation for lines (p. 14).

Figure 2.2



To define the slope of the tangent line l at $P(a, f(a))$ on the graph of f, we first choose another point $Q(x, f(x))$ (see Figure 2.3a) and consider the line through P and Q. This line is called a **secant line** for the graph.

We shall use the following notation:

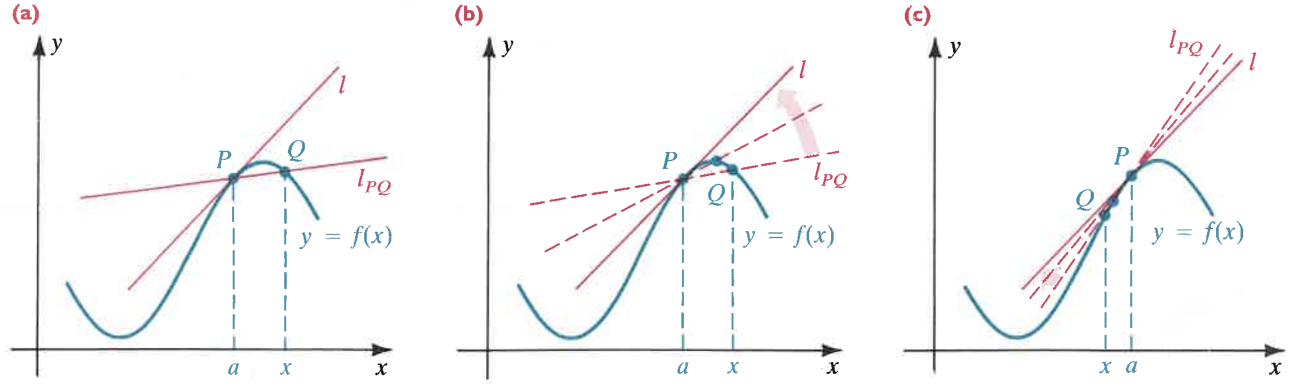
l_{PQ}: the secant line through P and Q

m_{PQ}: the slope of l_{PQ}

m_a: the slope of the tangent line l at $P(a, f(a))$

If Q is close to P, it appears that m_{PQ} is an approximation to m_a. Moreover, we would expect this approximation to improve if we take Q closer to P. With this in mind, we let *Q approach P*— that is, we (intuitively) let Q get closer to P— but $Q \neq P$. If Q approaches P from the right, we have the situation illustrated in Figure 2.3(b), where dashed lines indicate possible positions for l_{PQ}. In Figure 2.3(c), Q approaches P from the left. We could also let Q approach P in other ways, such as by taking points on the graph that are alternately to the left and to the right of P. If m_{PQ} has a limiting value— that is, if m_{PQ} gets closer to some number as Q approaches P— then that number is the slope m_a of the tangent line l.

Figure 2.3



Let us rephrase this discussion in terms of the function f. Referring to Figure 2.3 and using the coordinates of $P(a, f(a))$ and $Q(x, f(x))$, we see that the slope of the secant line l_{PQ} is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}.$$

Note that in order to have a secant line, P and Q must be distinct points, so we must have $x \neq a$. If f is continuous at a, we can make $Q(x, f(x))$ approach $P(a, f(a))$ by letting x approach a. This leads to the following definition for the slope m_a of l at $P(a, f(a))$:

$$m_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists.

It is often desirable to use an alternative form for m_a, which can be obtained by changing from the variable x to a variable h as follows.

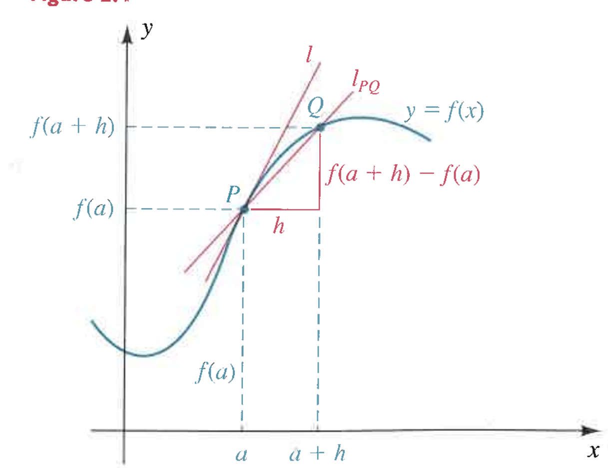
$$\text{Let} \quad h = x - a, \quad \text{or, equivalently,} \quad x = a + h.$$

If h is small, then x is close to a and the secant line through P and Q will be close to the tangent line at P. Referring to Figure 2.4 and using the coordinates $P(a, f(a))$ and $Q(a + h, f(a + h))$, we see that the slope m_{PQ} of the secant line is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}.$$

Since $x \to a$ is equivalent to $h \to 0$, our definition of the slope m_a of the tangent line l may be stated as follows.

Figure 2.4



Definition 2.1

> The **slope m_a of the tangent line** to the graph of a function f at $P(a, f(a))$ is
>
> $$m_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$
>
> provided the limit exists.

If the limit in Definition (2.1) does not exist, then the slope of the tangent line at $P(a, f(a))$ is undefined.

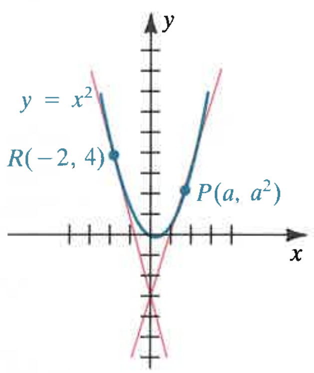
EXAMPLE ■ 1 Let $f(x) = x^2$, and let a be any real number.

(a) Find the slope of the tangent line to the graph of f at $P(a, a^2)$.

(b) Find an equation for the tangent line at $R(-2, 4)$.

SOLUTION

Figure 2.5



(a) The graph of $y = x^2$ and a typical point $P(a, a^2)$ are shown in Figure 2.5. Applying Definition (2.1), we see that the slope of the tangent line at P is

$$\begin{aligned} m_a &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2}{h} \\ &= \lim_{h \to 0} (2a + h) = 2a. \end{aligned}$$

(b) The slope of the tangent line at the point $R(-2, 4)$ is the special case of the formula $m_a = 2a$ with $a = -2$; that is,

$$m_{-2} = 2(-2) = -4.$$

Using the point–slope form, we can express an equation for the tangent line as

$$y - 4 = -4(x + 2), \quad \text{or} \quad y = -4x - 4.$$

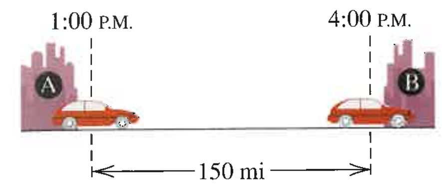
RATES OF CHANGE

Limits of the form given in Definition (2.1) occur in many applied problems where we wish to measure the rate of change of one variable with respect to another. Let us begin with the familiar problem of determining the velocity of a moving object. We consider **rectilinear motion,** in which an object travels along a line. Here the *average velocity* during a time interval is the ratio of the net distance traveled to the time elapsed.

Definition 2.2

> The **average velocity** v_{av} of an object that travels a net distance d in time t is
>
> $$v_{av} = \frac{d}{t}.$$

Figure 2.6



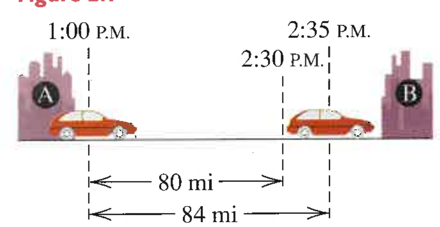
To illustrate, if an automobile leaves city A at 1:00 P.M. and travels along a straight highway, arriving at city B, 150 mi from A, at 4:00 P.M. (see Figure 2.6), then using Definition (2.2) with $d = 150$ and $t = 3$ (hours) yields the average velocity during the time interval $[1, 4]$:

$$v_{av} = \tfrac{150}{3} = 50 \text{ mi/hr}$$

This result is the velocity that, if maintained for 3 hr, would enable the automobile to travel the 150 mi from A to B.

The average velocity gives no information whatsoever about the velocity at any instant. At 2:30 P.M., for example, the automobile may be standing still or its speedometer may register 40 or 60 mi/hr. We can estimate the velocity at 2:30 P.M. if we know the position *near* this time. For example, suppose that at 2:30 P.M. the automobile is 80 mi from A and 5 min later, at 2:35 P.M., it is 84 mi from A, as Figure 2.7 illustrates. The net distance traveled in this 5 min, or $\frac{1}{12}$ hr, is 4 mi, and the average velocity during this time interval is

Figure 2.7

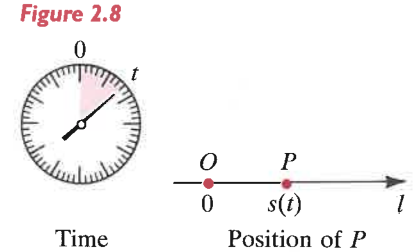


$$v_{av} = \frac{4}{\frac{1}{12}} = 48 \text{ mi/hr}.$$

Note that this result is not necessarily an accurate indication of the velocity at 2:30 P.M., since, for example, the automobile may have been traveling very slowly at 2:30 P.M. and then increased speed considerably to arrive at the point 84 mi from A at 2:35 P.M. Evidently, we obtain a better approximation by using the average velocity during a smaller time interval, say from 2:30 P.M. to 2:31 P.M. The best procedure seems to require taking smaller and smaller time intervals near 2:30 P.M. and studying the average velocity in each time interval. The approach leads us into a limiting process similar to that discussed for tangent lines.

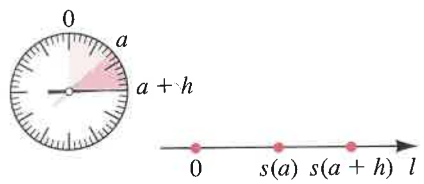
To make our discussion more precise, let us represent the position of an object moving rectilinearly by a point P on a coordinate line l. We sometimes refer to the motion of the *point* P on l, or the motion of an object whose position is specified by P. We shall assume that we know the position of P at every instant in a given interval of time. If $s(t)$ denotes the coordinate of P at time t, then the function s is called the **position function** for P. If we keep track of time by means of a clock, then, as illustrated in Figure 2.8, for each t the point P is $s(t)$ units from the origin.

Figure 2.8



To define the velocity of P at time a, we first determine the average velocity in a (small) time interval near a. Thus, we consider times a and

Figure 2.9



Change in time

Change in position of P

$a+h$, where h is a (small) nonzero real number. The corresponding positions of P are $s(a)$ and $s(a+h)$, as illustrated in Figure 2.9. The amount of change in the position of P is $s(a+h)-s(a)$. This number may be positive, negative, or zero. Note that $s(a+h)-s(a)$ is not necessarily the distance traveled by P between times a and $a+h$ since, for example, P may have moved beyond the point corresponding to $s(a+h)$ and then returned to that point at time a.

By Definition (2.2), the average velocity of P between times a and $a+h$ is

$$v_{\text{av}} = \frac{\text{change in distance}}{\text{change in time}} = \frac{s(a+h)-s(a)}{h}.$$

As in our previous discussion, we assume that the closer h is to 0, the closer v_{av} is to the velocity of P at time a. Thus, we *define* the velocity as the limit, as h approaches 0, of v_{av}, as in the following definition.

Definition 2.3

Suppose a point P moves on a coordinate line l such that its coordinate at time t is $s(t)$. The **velocity** v_a of P at time a is

$$v_a = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h},$$

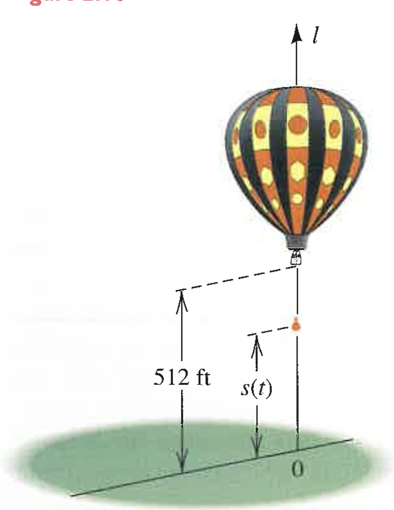
provided the limit exists.

The limit in Definition (2.3) is also called the **instantaneous velocity** of P at time a.

If $s(t)$ is measured in centimeters and t in seconds, then the unit of velocity is centimeters per second (cm/sec). If $s(t)$ is in miles and t in hours, then the unit of velocity is miles per hour. Other units of measurement may, of course, be used.

We shall return to the velocity concept in Chapter 3, where we will show that if the velocity is positive in a given time interval, then the point is moving in the positive direction on l. If the velocity is negative, the point is moving in the negative direction. Although these facts have not been proved, we shall use them in the following example.

Figure 2.10



EXAMPLE ■ 2 A sandbag is dropped from a hot-air balloon that is hovering at a height of 512 ft above the ground. If air resistance is disregarded, then the distance $s(t)$ from the ground to the sandbag after t seconds is given by

$$s(t) = -16t^2 + 512.$$

Find the velocity of the sandbag at

(a) $t = a$ sec **(b)** $t = 2$ sec **(c)** the instant it strikes the ground

SOLUTION

(a) As shown in Figure 2.10, we consider the sandbag to be moving along a vertical coordinate line l with origin at ground level. Note that at the instant it is dropped, $t = 0$ and

$$s(0) = -16(0) + 512 = 512 \text{ ft}.$$

To find the velocity of the sandbag at $t = a$, we use Definition (2.3), obtaining

$$\begin{aligned} v_a &= \lim_{h\to 0} \frac{s(a+h)-s(a)}{h} \\ &= \lim_{h\to 0} \frac{[-16(a+h)^2+512]-(-16a^2+512)}{h} \\ &= \lim_{h\to 0} \frac{-16(a^2+2ah+h^2)+512+16a^2-512}{h} \\ &= \lim_{h\to 0} \frac{-32ah-16h^2}{h} \\ &= \lim_{h\to 0}(-32a-16h) = -32a \text{ ft/sec}. \end{aligned}$$

The negative sign indicates that the motion of the sandbag is in the negative direction (downward) on l.

(b) To find the velocity at $t = 2$, we substitute 2 for a in the formula $v_a = -32a$, obtaining

$$v_2 = -32(2) = -64 \text{ ft/sec}.$$

(c) The sandbag strikes the ground when the distance above the ground is zero — that is, when

$$s(t) = -16t^2 + 512 = 0, \quad \text{or} \quad t^2 = \tfrac{512}{16} = 32.$$

This result gives us $t = \sqrt{32} = 4\sqrt{2} \approx 5.7$ sec. If we use the formula $v_a = -32a$ from part (a) with $a = 4\sqrt{2}$, we obtain the following impact velocity:

$$-32(4\sqrt{2}) = -128\sqrt{2} \approx -181 \text{ ft/sec}$$

There are many other applications that require limits similar to those in (2.1) and (2.3). In some, the independent variable is time t, as in the definition of velocity. For example, over a period of time, a chemist may be interested in the rate at which a certain substance dissolves in water; an electrical engineer may wish to know the rate of change of current in part of an electrical circuit; a biologist may be concerned with the rate at which the bacteria in a culture increase or decrease. In the social sciences, an economist may wish to determine the rate at which the gross national product is growing; a demographer or geographer may wish to analyze the rate of urbanization of a population; a sociologist may study the rate at which measures of alienation fluctuate; a political scientist may be concerned with the rate at which the public's approval of a national leader changes.

We can also consider rates of change with respect to quantities other than time. To illustrate, Boyle's law for a confined gas states that if the temperature remains constant, then the volume v and pressure p are related by the formula $v = c/p$ for some constant c. If the pressure is changing, a typical problem is to find the rate at which the volume is changing per unit change in pressure. This rate is known as *the instantaneous rate of change of v with respect to p*. To develop general methods that can be applied to different problems of this type, let us use x and y for variables and suppose that $y = f(x)$ for some function f. (In the preceding illustration, $y = v$, $x = p$, and $f(x) = c/x$.) We define rates of change of a variable y with respect to a variable x as follows.

Definition 2.4

Let $y = f(x)$, where f is defined on an open interval containing a.

(i) The **average rate of change** of $y = f(x)$ with respect to x on the interval $[a, a+h]$ is

$$y_{\text{av}} = \frac{f(a+h) - f(a)}{h}.$$

(ii) The **instantaneous rate of change** of y with respect to x at a is

$$y_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

We shall use the phrase *rate of change* interchangeably with *instantaneous rate of change*.

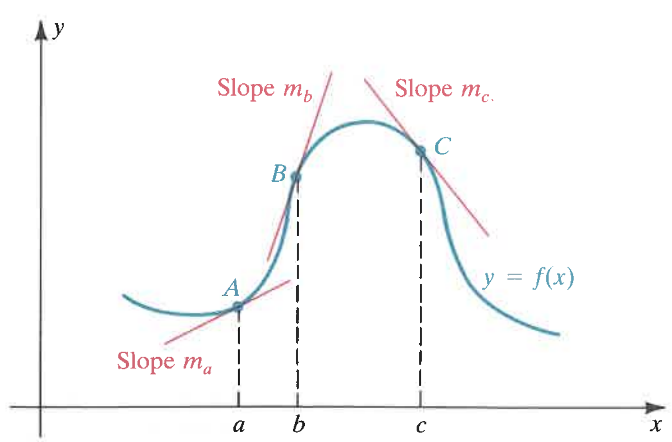
If, in Definition (2.4), we consider the special case $x = t$ (time) and $y = s(t)$ (position on a coordinate line), we obtain the following interpretations for rectilinear motion:

average velocity (v_{av}): the average rate of change of s with respect to t in some interval of time

velocity v_a: the instantaneous rate of change of s with respect to t at time a.

To interpret Definition (2.4)(ii) geometrically, imagine a point P traveling from left to right along the graph of $y = f(x)$ in Figure 2.11. The

Figure 2.11



instantaneous rate of change of y with respect to x gives us information about the rate at which the graph rises or falls per unit change in x. In Figure 2.11, m_a (the slope of the tangent line at A) is less than m_b (the slope of the tangent line at B), and the rate y_a at which y changes with respect to x at a is less than the rate y_b at which y changes with respect to x at b. Also note that since $m_c < 0$, the slope of the tangent line at C is negative, and y *decreases* as x increases.

The next two examples are physical and social science applications of Definition (2.4).

EXAMPLE 3 The voltage in a certain electrical circuit is 100 volts. If the current (in amperes) is I and the resistance (in ohms) is R, then by Ohm's law, $I = 100/R$. If R is increasing, find the instantaneous rate of change of I with respect to R at

(a) any resistance R **(b)** a resistance of 20 ohms

SOLUTION

(a) Using Definition (2.4)(ii) with $y = I$, $x = R$, and $f(R) = 100/R$ yields the instantaneous rate of change of I with respect to R at a resistance of R ohms:

$$\begin{aligned}
I_R &= \lim_{h \to 0} \frac{f(R+h) - f(R)}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{100}{R+h} - \dfrac{100}{R}}{h} \\
&= \lim_{h \to 0} \frac{100R - 100(R+h)}{h(R+h)R} \\
&= \lim_{h \to 0} \frac{-100h}{h(R+h)R} \\
&= \lim_{h \to 0} \frac{-100}{(R+h)R} \\
&= -\frac{100}{R^2}
\end{aligned}$$

The negative sign indicates that the current is decreasing as the resistance is increasing.

(b) Using the formula $I_R = -100/R^2$ from part (a), we find the instantaneous rate of change of I with respect to R at $R = 20$ to be

$$I_{20} = -\frac{100}{20^2} = -\frac{1}{4}.$$

Thus, when $R = 20$, the current is *decreasing* at a rate of $\frac{1}{4}$ ampere per ohm.

EXAMPLE ■ 4 The expression $P = \sqrt{at+b}$, with $a = 920$ and $b = 151.3^2$, gives a good approximation for the population P (in millions) of the United States during the period 1950–1990, where $t = 0$ corresponds to the year 1950. Find the instantaneous rate of change of P with respect to t at

(a) any time t **(b)** $t = 39$ (the year 1989)

SOLUTION

(a) Using Definition (2.4)(ii) with $y = P$, $x = t$, and $f(t) = \sqrt{at+b}$ gives the instantaneous rate of change of P with respect to t at time t years:

$$P_t = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{\sqrt{a(t+h)+b}-\sqrt{at+b}}{h}$$

Evaluating this limit by rationalizing the numerator and then simplifying yields

$$\begin{aligned}P_t &= \lim_{h\to 0}\frac{\sqrt{a(t+h)+b}-\sqrt{at+b}}{h}\cdot\frac{\sqrt{a(t+h)+b}+\sqrt{at+b}}{\sqrt{a(t+h)+b}+\sqrt{at+b}}\\ &= \lim_{h\to 0}\frac{[a(t+h)+b]-(at+b)}{h(\sqrt{a(t+h)+b}+\sqrt{at+b})}\\ &= \lim_{h\to 0}\frac{at+ah+b-at-b}{h(\sqrt{a(t+h)+b}+\sqrt{at+b})}\\ &= \lim_{h\to 0}\frac{ah}{h(\sqrt{a(t+h)+b}+\sqrt{at+b})}\\ &= \lim_{h\to 0}\frac{a}{\sqrt{a(t+h)+b}+\sqrt{at+b}} \qquad (h \neq 0)\\ &= \frac{a}{\sqrt{at+b}+\sqrt{at+b}}\\ &= \frac{a}{2\sqrt{at+b}}.\end{aligned}$$

(b) Using the formula for P_t from part (a), with $a = 920$ and $b = 151.3^2$, we find the instantaneous rate of change of P with respect to t at $t = 39$ to be

$$P_{39} = \frac{920}{2\sqrt{920(39)+151.3^2}} = \frac{460}{\sqrt{58{,}771.69}} \approx 1.897.$$

Thus, in 1989, the United States population was growing at an approximate rate of 1.9 million people per year.

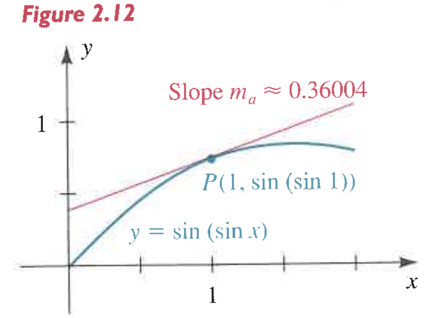
To find slopes of tangent lines and instantaneous rates of change, we need to determine limits of the form given in Definition (2.1). In Examples 1–4, we have determined the limits algebraically. In Section 2.3, we will discuss other algebraic techniques for evaluating similar limits. Calculators and computers can help us *approximate* values for such limits by evaluating the difference quotient $[f(a+h)-f(a)]/h$ for values of h close to 0, but there are some difficulties to be considered. For example, difficulty in evaluating calculations can occur for values of h extremely close to 0. Most computing devices perform arithmetic with a finite number of significant digits, usually between 7 and 14 digits. When h is close to 0, the numerator $f(a+h)-f(a)$ may be difficult to evaluate because the terms $f(a+h)$ and $f(a)$ differ by an amount too small for the calculator to distinguish from zero. Hence, the approximations of the difference quotient may get worse rather than better, as illustrated in the next example.

EXAMPLE ■ 5 Use Definition (2.1) to approximate the slope m_a of the tangent line to the graph of $f(x) = \sin(\sin x)$ at $P(a, f(a))$ when $a = 1$ by letting $h = 100^{-n}$ for $n = 1, 2, \ldots, 7$.

SOLUTION We use a calculator to generate the results in the following table:

n	h	$f(a+h)$	$f(a)$	$f(a+h)-f(a)$	$\frac{f(a+h)-f(a)}{h}$
1	0.01	0.749185709107	0.745624141665	0.003561567441	0.356156744109
2	0.0001	0.745660141723	0.745624141665	3.600005706E-5	0.3600005706
3	0.000001	0.745624501705	0.745624141665	3.6003911E-7	0.36003911
4	0.00000001	0.745624145266	0.745624141665	3.60045E-9	0.360045
5	0.0000000001	0.745624141702	0.745624141665	3.605E-11	0.3605
6	0.000000000001	0.745624141666	0.745624141665	4E-13	0.4
7	0.00000000000001	0.745624141665	0.745624141665	0	0

Figure 2.12



It appears from the table's first four lines that the limit is approximately 0.36004. Examining a graph of the function indicates that this approximate value appears to be correct for the slope of the tangent line. Figure 2.12 shows a graph of f and a graph of the tangent line through $(1, \sin(\sin 1))$ with slope 0.36004. In Section 2.5, we will see that this approximate value is correct to five significant digits.

Note that below the fourth line of the table, where the values of h are extremely small, the estimates of the difference quotient are not getting closer to 0.36004, but rather, farther away. Because of finite precision arithmetic, the numbers in the third and fourth columns become so close that there are few significant digits in the fifth column. When h gets so small that both $f(a+h)$ and $f(a)$ round to exactly the same value in the calculator, the value 0 is obtained in the last two columns.

EXERCISES 2.1

Exer. 1–6: (a) Use Definition (2.1) to find the slope of the tangent line to the graph of f at $P(a, f(a))$. (b) Find an equation of the tangent line at $P(2, f(2))$.

1 $f(x) = 5x^2 - 4x$ 2 $f(x) = 3 - 2x^2$

3 $f(x) = x^3$ 4 $f(x) = x^4$

5 $f(x) = 3x + 2$ 6 $f(x) = 4 - 2x$

Exer. 7–10: (a) Use Definition (2.1) to find the slope of the tangent line to the graph of the equation at the point with x-coordinate a. (b) Find an equation of the tangent line at P. (c) Sketch the graph and the tangent line at P.

7 $y = \sqrt{x}$; $P(4, 2)$

8 $y = \sqrt[3]{x}$; $P(-8, -2)$

9 $y = 1/x$; $P(2, \frac{1}{2})$

10 $y = 1/x^2$; $P(2, \frac{1}{4})$

Exer. 11–12: (a) Sketch the graph of the equation and the tangent lines at the points with x-coordinates $-2, -1, 1$, and 2. (b) Find the point on the graph at which the slope of the tangent line is the given number m.

11 $y = x^2$; $m = 6$ 12 $y = x^3$; $m = 9$

Exer. 13–14: The position function s of a point P moving on a coordinate line l is given, with t in seconds and $s(t)$ in centimeters. (a) Find the average velocity of P in the following time intervals: [1, 1.2], [1, 1.1], and [1, 1.01]. (b) Find the velocity of P at $t = 1$.

13 $s(t) = 4t^2 + 3t$ 14 $s(t) = 2t - 3t^2$

15 A rescue helicopter drops a crate of supplies from a height of 160 ft. After t seconds, the crate is $160 - 16t^2$ feet above the ground.

(a) Find the velocity of the crate at $t = 1$.

(b) With what velocity does the crate strike the ground?

16 A projectile is fired directly upward from the ground with an initial velocity of 112 ft/sec. Its distance above the ground after t seconds is $112t - 16t^2$ feet.

(a) Find the velocity of the projectile at $t = 2$, $t = 3$, and $t = 4$.

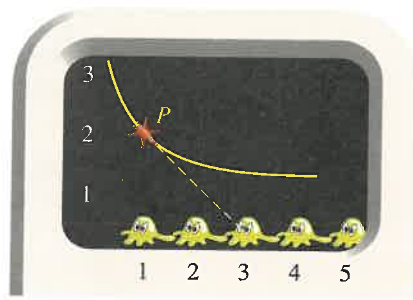
(b) When does the projectile strike the ground?

(c) Find the velocity at the instant it strikes the ground.

17 In the video game shown in the figure, airplanes fly from left to right along the path $y = 1 + (1/x)$ and can shoot their bullets in the tangent direction at creatures placed along the x-axis at $x = 1, 2, 3, 4$, and 5. Determine whether a creature will be hit if the player shoots when the plane is at

(a) $P(1, 2)$ (b) $Q(\frac{3}{2}, \frac{5}{3})$

Exercise 17

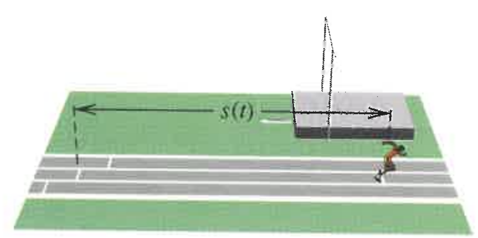


18 An athlete runs the hundred-meter dash in such a way that the distance $s(t)$ run after t seconds is given by $s(t) = \frac{1}{5}t^2 + 8t$ meters (see figure). Find the athlete's velocity at

(a) the start of the dash (b) $t = 5$ sec

(c) the finish line

Exercise 18



Exer. 19–20: (a) Find the average rate of change of y with respect to x on the given interval. (b) Find the instantaneous rate of change of y with respect to x at the left endpoint of the interval.

19 $y = x^2 + 2$; $[3, 3.5]$ 20 $y = 3 - 2x^2$; $[2, 2.4]$

21 Boyle's law states that if the temperature remains constant, the pressure p and volume v of a confined gas are related by $p = c/v$ for some constant c. If, for a certain gas, $c = 200$ and v is increasing, find the instantaneous rate of change of p with respect to v at

(a) any volume v (b) a volume of 10

22 Using the Lorentz contraction formula,

$$L = L_0\sqrt{1 - v^2/c^2},$$

find the instantaneous rate of change of the length L of an object with respect to the velocity v at

(a) any velocity v

(b) $v = 0.9c$

c 23 Graph $f(x) = \sin(\pi x)$ on the interval [0, 2].

(a) Use the graph to estimate the slope of the tangent line at $P(1.4, f(1.4))$.

(b) Use Definition (2.1) with $h = \pm 0.0001$ to approximate the slope in part (a).

c 24 Graph $f(x) = \dfrac{10\cos x}{x^2 + 4}$ on the interval $[-2, 2]$.

(a) Use the graph to estimate the slope of the tangent line at $P(-0.5, f(-0.5))$.

(b) Use Definition (2.1) with $h = \pm 0.0001$ to approximate the slope in part (a).

(c) Find an (approximate) equation of the tangent line to the graph at P.

c 25 An object's position on a coordinate line is given by

$$s(t) = \frac{\cos^2 t + t^2 \sin t}{t^2 + 1},$$

where $s(t)$ is in feet and t is in seconds. Approximate its velocity at $t = 2$ by using Definition (2.3) with $h = 0.01, 0.001$, and 0.0001.

c 26 The position function s of an object moving on a coordinate line is given by

$$s(t) = \frac{t - t^2 \sin t}{t^2 + 1},$$

where $s(t)$ is in meters and t is in minutes.

(a) Graph s for $0 \le t \le 10$.

(b) Approximate the time intervals in which its velocity is positive.

c **Exer. 27–30: (a) Use Definition (2.1) with the given value of h to approximate the slope of the tangent line at the indicated points. (b) Graph the function and the three approximated tangent lines over the given interval.**

27 $f(x) = \sin(\pi x)$ on [0, 2]; $P(0.7, f(0.7))$, $P(1.1, f(1.1))$, $P(1.4, f(1.4))$; $h = 0.001$

28 $f(x) = 3^{-x^2}$ on $[-2, 2]$; $P(-0.67, f(-0.67))$, $P(0.3, f(0.3))$, $P(1.14, f(1.14))$; $h = 0.0002$

29 $f(x) = 0.625\sqrt{64 - x^2}$ on $[-8, 8]$; $P(-7, f(-7))$, $P(3, f(3))$, $P(5, f(5))$; $h = 0.0005$

30 $f(x) = 0.4\sqrt{25 + x^2}$ on $[-10, 10]$; $P(-5, f(-5))$, $P(-1, f(-1))$, $P(9, f(9))$; $h = 0.0001$

c 31 An object's position on a coordinate line is given by

$$s(t) = \sin(\sin t),$$

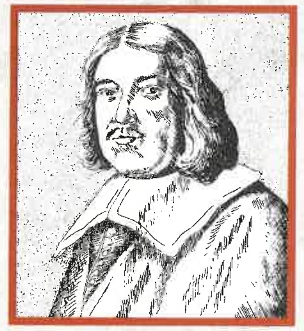
where $s(t)$ is in miles and t is in hours. Use Definition (2.3), with $h = -0.1, -0.05$, and -0.001 to approximate its velocity at

(a) $t = -1$ (b) $t = 2$

Mathematicians and Their Times

PIERRE DE FERMAT

PIERRE DE FERMAT WAS PERHAPS the greatest mathematician of the seventeenth century. He made fundamental contributions to analytic geometry, calculus, probability, and number theory. Most astounding to us in this age of specialized knowledge and major research centers is that Fermat was not a professional mathematician. He did not even have a degree in mathematics.



To others, it appeared that Fermat's life was quiet and uneventful. Born in Beaumont-de-Lomagne, France, in August 1601, Fermat was a shy and retiring person. His father was a leather merchant. His mother's family boasted a number of public service lawyers; Fermat followed this occupation. He rose to the rank of King's Councilor in the Parlement of Toulouse and discharged this position with great skill and integrity for 17 years until his death on January 12, 1665.

Fermat's vocation may have been the law and public service, but his passionate avocation was mathematics. Although Fermat and Descartes were independent inventors of analytic geometry, Fermat went considerably further than Descartes, introducing perpendicular axes and finding equations for straight lines, circles, ellipses, parabolas, and hyperbolas.

While Newton and Leibniz share credit for the invention of calculus, Fermat had made critically important discoveries in this field more than a decade before either of them was born. He found the equations of tangent lines, located the maximum and minimum points, and computed the area beneath many different curves. He was also able to solve these three principal problems of calculus for a wide variety of functions.

Fermat's favorite branch of mathematics was number theory: the study of integers and relations between them. For 350 years, the most famous unsolved problem in mathematics was *Fermat's Last Theorem*. Next to a passage on integer solutions to the equation $x^2 + y^2 = z^2$ (examples: $3^2 + 4^2 = 5^2$, and $5^2 + 12^2 = 13^2$), Fermat penned the most memorable margin note in the history of science: "I have discovered a truly wonderful proof which this margin is too narrow to contain." He meant that if $n > 2$, then there are no integer solutions to $x^n + y^n = z^n$. In June 1993, Andrew Wiles, a Princeton University mathematician, announced that he had proved the truth of Fermat's claim.

Although primarily interested in "pure" mathematics, Fermat also made profound discoveries in applications as well. He formulated the idea that the path along which a light ray travels as it moves from one medium to another — say, from air to water — is the path that minimizes the total travel time. *Fermat's principle of least time*, as it is called today, led to the calculus of variations and provided the basis for Hamilton's principle of least action, a powerful unifying idea in physics.

2.2 DEFINITION OF DERIVATIVE

In the preceding section, we examined several different problems whose solutions all involved similar limits. Whether in determining the slope of the tangent to a curve, or finding the velocity of an object moving along a line, or discovering the instantaneous rate of change of an electrical current with respect to voltage, we ultimately use limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

or, equivalently,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This limit is the basis for one of the fundamental concepts of calculus, the *derivative*. The derivative occurs throughout calculus in problems concerned with rates of change and thus has applications in many fields of study.

In this section, we begin with equivalent definitions of the derivative, given in terms of limits, that can be applied to any function. We then look at some simple rules that allow us to find derivatives without directly evaluating limits. We also consider some basic properties of the derivative and its notation.

DEFINING THE DERIVATIVE

Definition 2.5

The **derivative** of a function f is the function f' whose value at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The symbol f' in Definition (2.5) is read *f prime*. It is important to note that in determining $f'(x)$, we regard x as an arbitrary real number and consider the limit as h approaches 0. Once we have obtained $f'(x)$, we can find $f'(a)$ for a specific real number a by substituting a for x.

The statement $f'(x)$ *exists* means that the limit in Definition (2.5) exists. If $f'(x)$ exists, we say that **f is differentiable at x**, or that **f has a derivative at x**. If the limit does not exist, then f is not differentiable at x. The terminology *differentiate* $f(x)$ or *find the derivative of* $f(x)$ means to find $f'(x)$.

Occasionally we will find it convenient to use the following alternative form of Definition (2.5) to find $f'(a)$.

Alternative Definition of Derivative 2.6

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

This formula was first used to define m_a (see p. 145).

The following applications are restatements of Definitions (2.1) and (2.4)(ii) using $f'(x)$. These interpretations of the derivative are very important and will be used in many examples and exercises throughout the text.

Applications of the Derivative 2.7

(i) Tangent line: The slope of the tangent line to the graph of the function $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$.

(ii) Rate of change: If $y = f(x)$, the instantaneous rate of change of y with respect to x at a is $f'(a)$.

As a special case of (2.7)(ii), recall from Definition (2.3) that if $x = t$ denotes time and $y = s(t)$ is the position of a point P on a coordinate line, then $s'(a)$ is the velocity of P at time a.

EXAMPLE ▪ 1 If $f(x) = 3x^2 - 12x + 8$, find

(a) $f'(x)$ **(b)** $f'(4)$, $f'(-2)$, and $f'(a)$

SOLUTION

(a) By Definition (2.5),

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[3(x+h)^2 - 12(x+h) + 8] - (3x^2 - 12x + 8)}{h} \\ &= \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 - 12x - 12h + 8) - (3x^2 - 12x + 8)}{h} \\ &= \lim_{h \to 0} \frac{6xh + 3h^2 - 12h}{h} = \lim_{h \to 0} (6x + 3h - 12) = 6x - 12. \end{aligned}$$

(b) Substituting for x in $f'(x) = 6x - 12$, we obtain

$$f'(4) = 6(4) - 12 = 12,$$
$$f'(-2) = 6(-2) - 12 = -24,$$

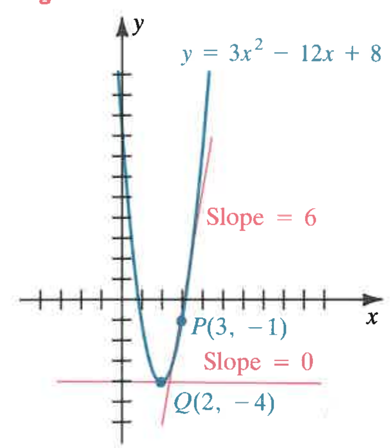
and $$f'(a) = 6a - 12.$$

EXAMPLE ▪ 2 If $y = 3x^2 - 12x + 8$, use the results of Example 1 to find

(a) the slope of the tangent line to the graph of this equation at the point $P(3, -1)$

(b) the point on the graph at which the tangent line is horizontal

Figure 2.13



SOLUTION

(a) If we let $f(x) = 3x^2 - 12x + 8$, then, by (2.7)(i) and Example 1, the slope of the tangent line at $(x, f(x))$ is $f'(x) = 6x - 12$. In particular, the slope at $P(3, -1)$ is

$$f'(3) = 6(3) - 12 = 6.$$

(b) Since the tangent line is horizontal if the slope $f'(x)$ is 0, we solve $6x - 12 = 0$, obtaining $x = 2$. The corresponding value of y is -4. Hence the tangent line is horizontal at the point $Q(2, -4)$.

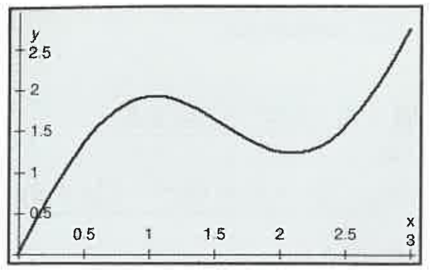
The graph of f (a parabola) and the tangent lines at P and Q are sketched in Figure 2.13. Note that the vertex of the parabola is the point $Q(2, -4)$.

EXAMPLE ▪ 3 As a swan begins its flight, its height above the ground (in meters) after x seconds is observed to be given by the function

$$f(x) = x + \sin 2x.$$

Use a graphing utility to obtain a graph of f on the interval $[0, 3]$, estimate the times when the tangent line is horizontal, and interpret the results in terms of rates of change of the swan's distance from the ground.

Figure 2.14
$0 \le x \le 3, 0 \le y \le 3$



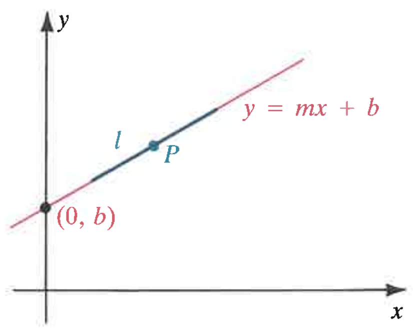
SOLUTION Using a graphing utility and the viewing window shown, we obtain the graph of $f(x) = x + \sin 2x$ in Figure 2.14. If we sketch or visualize tangent lines at various points on the graph, we can determine that they have positive slopes on the open interval extending from 0 to approximately 1.05 and again on the open interval from approximately 2.10 to 3. On these intervals, the rate of change of the swan's height above the ground is positive, and the distance between the swan and the ground is increasing. On the open interval extending from 1.05 to 2.10, the tangent lines have negative slopes. During this time interval, the swan's height above the ground is decreasing. At approximate times of 1.05 and 2.10, the tangent lines are horizontal. At these two times, the swan's distance above the ground is neither decreasing nor increasing.

BASIC RULES OF DIFFERENTIATION

The process of finding a derivative by means of Definition (2.5) can be very tedious if $f(x)$ is a complicated expression. Fortunately, we can establish general formulas and rules that enable us to find $f'(x)$ without using limits.

If f is a linear function, then $f(x) = mx + b$ for real numbers m and b. The graph of f is the line with slope m and y-intercept b (see Figure 2.15).

Figure 2.15



As indicated in the figure, the tangent line l at any point P coincides with the graph of f and hence has slope m. From Definition (2.7)(i), we conclude that $f'(x) = m$ for every x. Thus, we obtain the following rule, which we can also prove directly from Definition (2.5).

Derivative of a Linear Function 2.8

If $f(x) = mx + b$, then $f'(x) = m$.

The following result is the special case of (2.8) with $m = 0$.

Derivative of a Constant Function 2.9

If $f(x) = b$, then $f'(x) = 0$.

The preceding result is also graphically evident, because the graph of a constant function is a horizontal line and hence has slope 0.

Some special cases of (2.8) and (2.9) are given in the following illustration.

ILLUSTRATION

$f(x)$	$3x - 7$	$-4x + 2$	$7x$	x	13	π^2	$\sqrt[3]{10}$
$f'(x)$	3	-4	7	1	0	0	0

Many algebraic expressions contain a variable x raised to some power n. The next result, appropriately called the *power rule*, provides a simple formula for finding the derivative if n is an integer.

Power Rule 2.10

Let n be an integer.

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$,

provided $x \neq 0$ when $n \le 0$.

PROOF By Definition (2.5),

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

If n is a positive integer, then we can expand $(x+h)^n$ by using the binomial theorem, obtaining

$$f'(x) = \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-2)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right].$$

Since each term within the brackets, except the first, contains a power of h, we see that $f'(x) = nx^{n-1}$.

If n is negative and $x \neq 0$, then we can write $n = -k$ with k positive. Thus,

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^{-k} - x^{-k}}{h} = \lim_{h \to 0} \frac{x^k - (x+h)^k}{h(x+h)^k x^k}.$$

As before, if we use the binomial theorem to expand $(x+h)^k$ and then simplify and take the limit, we obtain

$$f'(x) = -kx^{-k-1} = nx^{n-1}.$$

If $n = 0$ and $x \neq 0$, then the power rule is also true, for in this case $f(x) = x^0 = 1$ and, by (2.9), $f'(x) = 0 = 0 \cdot x^{0-1}$. ■

Some special cases of the power rule are listed in the next illustration.

ILLUSTRATION

$f(x)$	x^2	x^3	x^4	x^{100}	$x^{-1} = \frac{1}{x}$	$x^{-2} = \frac{1}{x^2}$	$x^{-10} = \frac{1}{x^{10}}$
$f'(x)$	$2x$	$3x^2$	$4x^3$	$100x^{99}$	$(-1)x^{-2} = -\frac{1}{x^2}$	$-2x^{-3} = -\frac{2}{x^3}$	$-10x^{-11} = -\frac{10}{x^{11}}$

We can extend the power rule to rational exponents. In particular, in Appendix I we show that for every positive integer n,

$$\text{if} \quad f(x) = x^{1/n}, \quad \text{then} \quad f'(x) = \frac{1}{n}x^{(1/n)-1},$$

provided these expressions are defined. By using the power rule for functions (2.27), proved in Section 2.5, we can then show that for any rational number m/n,

$$\text{if} \quad f(x) = x^{m/n}, \quad \text{then} \quad f'(x) = \frac{m}{n}x^{(m/n)-1}.$$

In Chapter 6, we will prove that the power rule holds for every *real* number n. Some special cases of the power rule for rational exponents are given in the next illustration.

ILLUSTRATION

$f(x)$	$f'(x)$
$\sqrt{x} = x^{1/2}$	$\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$
$\sqrt[3]{x^2} = x^{2/3}$	$\frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$
$\sqrt[4]{x^5} = x^{5/4}$	$\frac{5}{4}x^{1/4} = \frac{5}{4}\sqrt[4]{x}$
$\frac{1}{x^{1/3}} = x^{-1/3}$	$-\frac{1}{3}x^{-4/3} = -\frac{1}{3x^{4/3}}$
$\frac{1}{x^{3/2}} = x^{-3/2}$	$-\frac{3}{2}x^{-5/2} = -\frac{3}{2x^{5/2}}$

By using the same type of proof that was used for the power rule (2.10), we can prove the following for any real number c.

Theorem 2.11

$$\text{If} \quad f(x) = cx^n, \quad \text{then} \quad f'(x) = (cn)x^{n-1}.$$

In words, *to differentiate* cx^n, *multiply the coefficient* c *by the exponent* n *and reduce the exponent by* 1.

EXAMPLE ■ 4 A spherical balloon is being inflated. Find the instantaneous rate of change of the surface area S of the balloon with respect to the radius x.

SOLUTION Using the formula for surface area S of a sphere, $S = f(x) = 4\pi x^2$, and Theorem (2.11), we can readily find the instantaneous rate of change of the surface area with respect to the radius:

$$f'(x) = (4\pi \cdot 2)x = 8\pi x$$

CONTINUITY AND DIFFERENTIABILITY

Not every function $f(x)$ is differentiable at every value of x in its domain. As we shall see from Theorem (2.12), if f is not continuous at a, then f is not differentiable at a. Moreover, many continuous functions can also fail to be differentiable.

To formally determine whether a function is differentiable or not, we must examine the limit in (2.5) or (2.6) and decide whether the limit exists or not. Informally, however, we can say that a function is continuous at a point if its graph has no breaks or jumps at the point, and if it is also differentiable, it passes through the point in a "smooth" fashion with no corners or vertical tangents. Our next example is perhaps the simplest familiar function that fails to be differentiable at a point where it is continuous.

EXAMPLE ■ 5 If $f(x) = |x|$, show that f is not differentiable at 0.

SOLUTION The graph of f is sketched in Figure 2.16. We can prove that $f'(0)$ does not exist by showing that the limit in Alternative Definition (2.6) does not exist. With $a = 0$ and $f(x) = |x|$, we have

$$\frac{f(x) - f(a)}{x - a} = \frac{|x| - |0|}{x - 0} = \frac{|x|}{x}.$$

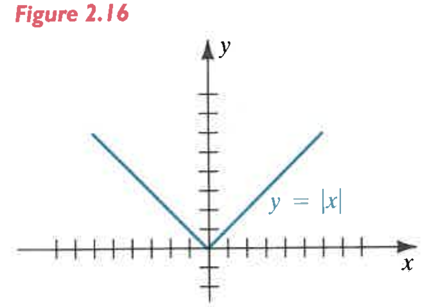
But, by the definition of the absolute value function,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

we have

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Figure 2.16



and this expression has no limit as x approaches 0 (see Example 7 of Section 1.1). Since $\lim_{x\to a}[f(x)-f(a)]/(x-a)$ does not exist at $a=0$, the function $f(x)=|x|$ is not differentiable at 0.

We observe from Example 5 that the absolute value function is continuous at 0 (see Example 1 of Section 1.5), but it is not differentiable at 0. The graph of $f(x)=|x|$ shown in Figure 2.16 displays a geometric distinction between continuous and differentiable functions. The graph of f is unbroken and has no jumps as it passes through $x=0$, so f is continuous. However, since it has a corner at (0, 0) and therefore no tangent line at that point, it is not differentiable. (A formal definition of *corner* is given later in this section.)

CAUTION Not every continuous function is differentiable. In contrast, as the next theorem states, *every differentiable function is continuous*.

Theorem 2.12

If a function f is differentiable at a, then f is continuous at a.

PROOF We shall use Alternative Definition (2.6):

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

We may write $f(x)$ in a form that contains $[f(x)-f(a)]/(x-a)$ as follows, provided $x\neq a$:

$$f(x)=\frac{f(x)-f(a)}{x-a}(x-a)+f(a).$$

Using limit theorems, we find that

$$\lim_{x\to a}f(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)+\lim_{x\to a}f(a)$$
$$=f'(a)\cdot 0+f(a)=f(a).$$

Thus, by Definition (1.20), f is continuous at a. ■

DIFFERENTIABILITY ON AN INTERVAL

Thus far, we have considered the differentiability of a function $f(x)$ at a particular single value of x. We now extend the concept to *differentiability on an interval*.

A function f **is differentiable on an open interval** if $f'(x)$ exists for every x in that interval. For closed intervals, we use the following convention, which is analogous to the definition of continuity on a closed interval given in Definition (1.22).

Definition 2.13

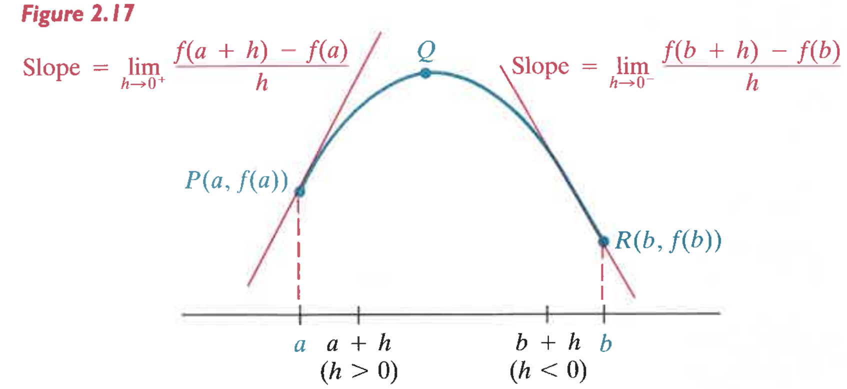
A function f **is differentiable on a closed interval $[a, b]$** if f is differentiable on the open interval (a, b) and if the following limits exist:

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}\quad\text{and}\quad\lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}.$$

The one-sided limits in Definition (2.13) are sometimes referred to as the **right-hand derivative** and the **left-hand derivative** of f at a and b, respectively.

Note that for the right-hand derivative, $h\to 0^+$, and $a+h$ approaches a from the right. In this case, the point $Q(a+h, f(a+h))$ in Figure 2.17 lies to the right of $P(a, f(a))$. The quotient $[f(a+h)-f(a)]/h$ is the slope of the secant line through P and Q on the graph of f to the right of P. The right-hand derivative is the limiting value of the slope of the secant lines through P and Q as Q approaches P from the right.

Figure 2.17



For the left-hand derivative, $h\to 0^-$ and $b+h$ approaches b from the left. The left-hand derivative is the limiting value of the slope of the secant lines through P and Q as Q approaches P from the left.

If a function f is defined only on a closed interval $[a, b]$, then the right-hand and left-hand derivatives define the slopes of the tangent lines at the points $P(a, f(a))$ and $R(b, f(b))$. Figure 2.17 shows the graph of such a function f with the tangent lines at P and R. By using one-sided limits, we can extend Theorem (2.12) to functions that are differentiable on a closed interval.

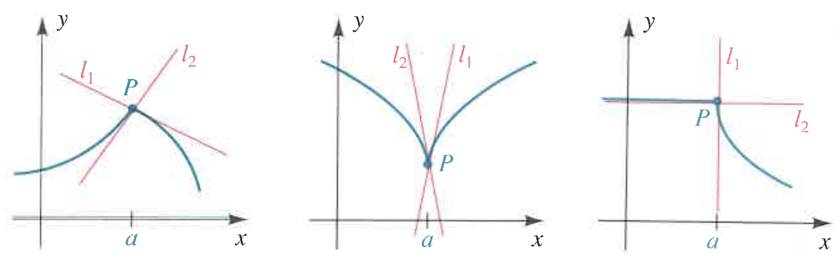
Differentiability on an interval of the form $[a, b)$, $[a,\infty)$, $(a, b]$, or $(-\infty, b]$ is defined in similar fashion, using a one-sided limit at an endpoint.

A number c is in the **domain of the derivative** f' if f is differentiable at c or if c is an endpoint of the domain of f at which the appropriate one-sided limit exists.

If f is defined on an open interval containing a, then $f'(a)$ *exists if and only if both the right-hand and left-hand derivatives at a exist and are equal*. Thus, a function f may fail to be differentiable at a if either one or both one-sided derivatives fail to exist, as we will see in Example 7. The function may also fail to be differentiable at a if both one-sided derivatives exist at a but are not equal to each other. In Example 5, we saw that the absolute value function $f(x) = |x|$ is not differentiable at 0. This function has a right-hand derivative of 1 and a left-hand derivative of -1 at $x = 0$.

The functions whose graphs are sketched in Figure 2.18 have right-hand and left-hand derivatives at a that give the slopes of the lines l_1 and l_2, respectively. Since the slopes of l_1 and l_2 are unequal, $f'(a)$ does not exist. The graph of f has a **corner** at $P(a, f(a))$ if f is continuous at a and if the right-hand and left-hand derivatives at a exist and are unequal or if *one* of those derivatives exists at a and $|f'(x)| \to \infty$ as $x \to a^-$ or $x \to a^+$.

Figure 2.18



As indicated in the next definition, a *vertical tangent line* may occur at $P(a, f(a))$ if $f'(a)$ does not exist. If P is an endpoint of the domain of f, we can state a similar definition using a right-hand or a left-hand derivative, as appropriate.

Definition 2.14

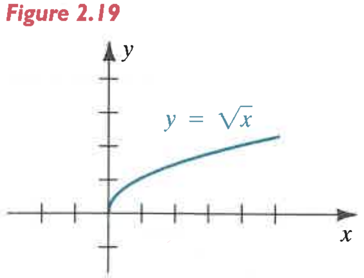
> The graph of a function f has a **vertical tangent line** $x = a$ at the point $P(a, f(a))$ if f is continuous at a and if
> $$\lim_{x\to a} |f'(x)| = \infty.$$

The next example shows a function f with a vertical tangent line at an endpoint of the domain of f.

EXAMPLE 6

(a) sketch the graph of f **(b)** find $f'(x)$ and the domain of f'

Figure 2.19



SOLUTION

(a) The graph of f is sketched in Figure 2.19. Note that the domain of f consists of all nonnegative numbers.

(b) Since $x = 0$ is an endpoint of the domain of f, we shall examine the cases $x > 0$ and $x = 0$ separately.

If $x > 0$, then, by Definition (2.5),

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

To find the limit, we first rationalize the numerator of the quotient and then simplify:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

Since $x = 0$ is an endpoint of the domain of f, we must use a one-sided limit to determine if $f'(0)$ exists. Using Definition (2.13) with $x = 0$, we obtain

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} \\ &= \lim_{h\to 0^+} \frac{\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty. \end{aligned}$$

Since the limit does not exist, the domain of f' is the set of positive real numbers. The last limit shows that the graph of f has a vertical tangent line (the y-axis) at the point (0, 0).

Figure 2.20 illustrates some typical cases of vertical tangent lines. In Figure 2.20(b), $f'(x) \to \infty$ as x approaches a from either side. In contrast, in Figure 2.20(c), $f'(x) \to \infty$ as x approaches a from the left, but $f'(x) \to -\infty$ as x approaches a from the right. The resulting sharp peak (or spike) at P is called a *cusp*, formally defined as follows.

Definition 2.15

> The graph of f has a **cusp** at $P(a, f(a))$ if f is continuous at a and if the following two conditions hold:
>
> **(i)** $f'(x) \to \infty$ as x approaches a from one side
>
> **(ii)** $f'(x) \to -\infty$ as x approaches a from the other side

Figure 2.20

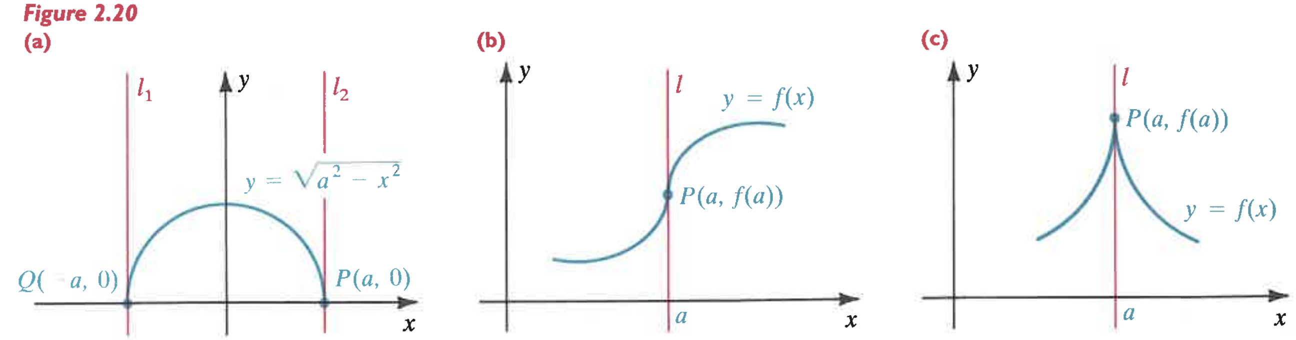
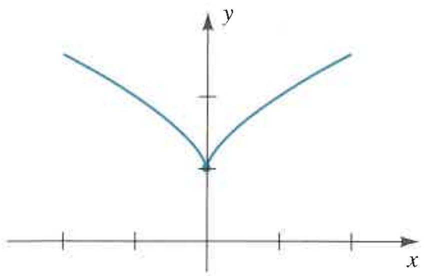


Figure 2.21



EXAMPLE 7 Determine the nature of the graph of the function $f(x) = 1 + x^{2/3}$ near the point (0, 1).

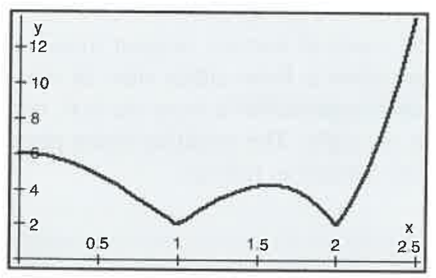
SOLUTION Since $\lim_{x\to 0} f(x) = f(0) = 1$, the function f is continuous at (0, 1). If $x \neq 0$, then we have

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.$$

Since $\lim_{x\to 0^+} f'(x) = \infty$ and $\lim_{x\to 0^-} f'(x) = -\infty$, there is a cusp at the point (0, 1). The function $f(x) = 1 + x^{2/3}$ is not differentiable at 0 because the right-hand and left-hand derivatives do not exist. Figure 2.21 shows a graph of the function f, where the y-axis is a vertical tangent line.

If f is a complicated continuous function, we may have difficulty finding the values of x at which f is not differentiable using only algebraic techniques. We may be able to make good approximations of such values by looking for cusps and corners on the graph of f.

Figure 2.22
$0 \le x \le 2.5, 0 \le y \le 12.5$



EXAMPLE 8 Graph the function

$$f(x) = 2 + |x^4 - 5x^2 + 4|$$

on the interval [0, 2.5], and estimate the values of x at which the function is not differentiable.

SOLUTION We have graphed $f(x) = 2 + |x^4 - 5x^2 + 4|$ in the viewing window shown in Figure 2.22. Note that the graph appears smooth at all values of x on the interval [0, 2.5] except at approximately $x = 1$ and $x = 2$, where there appear to be corners. By zooming in near the point $(1, f(1))$, we are led to believe that the right-hand and left-hand derivatives exist at $x = 1$, but are not equal. A similar situation occurs at $x = 2$, and we conclude that f is not differentiable at $x = 1$ and $x = 2$.

DERIVATIVE NOTATIONS

We conclude this section by considering the various notations for derivatives.

Notations for the Derivative of $y = f(x)$ 2.16

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = D_x f(x) = D_x y$$

All of these notations are used in mathematics and applications, and you should become familiar with the different forms. For example, we can now write

$$D_x f(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}.$$

The letter x in D_x and d/dx denotes the independent variable. If we use a different independent variable, say t, then we write

$$f'(t) = D_t f(t) = \frac{d}{dt}f(t).$$

Each of the symbols D_x and d/dx is called a **differential operator**. Standing alone, D_x or d/dx has no practical significance; however, when either symbol has an expression to its right, it denotes a derivative. We say that D_x or d/dx *operates* on the expression, and we call $D_x y$ or dy/dx **the derivative of y with respect to x**. We shall justify the notation dy/dx in Section 2.8, where the concept of a *differential* is defined.

The next illustration contains some examples of the use of (2.16) and Theorem (2.11).

ILLUSTRATION

- $\frac{d}{dx}(3x^7) = (3 \cdot 7)x^6 = 21x^6$
- $\frac{d}{dt}(\frac{1}{2}t^{12}) = (\frac{1}{2} \cdot 12)t^{11} = 6t^{11}$
- $\frac{d}{dx}(9x^{4/3}) = (9 \cdot \frac{4}{3})x^{1/3} = 12x^{1/3}$
- $\frac{d}{dr}(2r^{-4}) = 2(-4)r^{-5} = -\frac{8}{r^5}$

Note that in (2.16), $D_x y$, y', and dy/dx are used for the derivative of y with respect to x. If we wish to denote the *value* of the derivative $D_x y$, y', or dy/dx at some number $x = a$, we often use a single or double bracket and write

$$D_x y]_{x=a} \qquad \left.\frac{dy}{dx}\right]_{x=a} \qquad [D_x y]_{x=a} \quad \text{or} \quad \left[\frac{dy}{dx}\right]_{x=a}$$

as in the following examples:

$$\left[\frac{d}{dx}(x^3)\right]_{x=5} = [3x^2]_{x=5} = 3(5^2) = 75$$

$$\left[\frac{d}{dx}(9x^{4/3})\right]_{x=8} = [12x^{1/3}]_{x=8} = 12(8^{1/3}) = 24$$

In calculus, we sometimes consider *derivatives of derivatives*. As we have seen, if we differentiate a function f, we obtain another function denoted f'. If f' has a derivative, it is denoted f'' (read f *double prime*) and is called the **second derivative** of f. Thus,

$$f''(x) = [f'(x)]' = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{d}{dx}(f(x))\right) = \frac{d^2}{dx^2}(f(x)),$$

where we use the operator symbol d^2/dx^2 for second derivatives. The **third derivative** of f, denoted f''', is the derivative of the second derivative. Thus,

$$f'''(x) = [f''(x)]' = \frac{d}{dx}(f''(x)) = \frac{d}{dx}\left(\frac{d^2}{dx^2}(f(x))\right) = \frac{d^3}{dx^3}(f(x)).$$

In general, if n is a positive integer, then $f^{(n)}$ denotes the ***n*th derivative** of f and is found by starting with f and differentiating, successively, n times. In operator notation, $f^{(n)}(x) = \frac{d^n}{dx^n}(f(x))$, where the integer n is the **order** of the derivative $f^{(n)}(x)$. The following summarizes various notations used for these **higher derivatives,** with $y = f(x)$.

***Notations for Higher Derivatives* 2.17**

$f'(x)$,	$f''(x)$,	$f'''(x)$,	$f^{(4)}(x)$,	$\ldots$,	$f^{(n)}(x)$
y',	y'',	y''',	$y^{(4)}$,	$\ldots$,	y^{n}
$\frac{dy}{dx}$,	$\frac{d^2y}{dx^2}$,	$\frac{d^3y}{dx^3}$,	$\frac{d^4y}{dx^4}$,	$\ldots$,	$\frac{d^ny}{dx^n}$
$D_x y$,	$D_x^2 y$,	$D_x^3 y$,	$D_x^4 y$,	$\ldots$,	$D_x^n y$

EXAMPLE 9 Find the first four derivatives of $f(x) = 4x^{3/2}$.

SOLUTION We use (2.17) and Theorem (2.11) four times:

$$f'(x) = (4 \cdot \tfrac{3}{2})x^{1/2} = 6x^{1/2}$$
$$f''(x) = (6 \cdot \tfrac{1}{2})x^{-1/2} = 3x^{-1/2}$$
$$f'''(x) = 3(-\tfrac{1}{2})x^{-3/2} = -\tfrac{3}{2}x^{-3/2}$$
$$f^{(4)}(x) = -\tfrac{3}{2}(-\tfrac{3}{2})x^{-5/2} = \tfrac{9}{4}x^{-5/2}$$

EXERCISES 2.2

Exer. 1–4: (a) Use Definition (2.5) to find $f'(x)$. (b) Find the domain of f'. (c) Find an equation of the tangent line to the graph of f at P. (d) Find the points on the graph at which the tangent line is horizontal.

1 $f(x) = -5x^2 + 8x + 2$; $P(-1, -11)$

2 $f(x) = 3x^2 - 2x - 4$; $P(2, 4)$

3 $f(x) = x^3 + x$; $P(1, 2)$

4 $f(x) = x^3 - 4x$; $P(2, 0)$

Exer. 5–12: (a) Use (2.8)–(2.11) to find $f'(x)$. (b) Find the domain of f'. (c) Find an equation of the tangent line to the graph of f at P. (d) Find the points on the graph at which the tangent line is horizontal.

5 $f(x) = 9x - 2$; $P(3, 25)$

6 $f(x) = -4x + 3$; $P(-2, 11)$

7 $f(x) = 37$; $P(0, 37)$

8 $f(x) = \pi^2$; $P(5, \pi^2)$

9 $f(x) = 1/x^3$; $P(2, \frac{1}{8})$

10 $f(x) = 1/x^4$; $P(1, 1)$

11 $f(x) = 4x^{1/4}$; $P(81, 12)$

12 $f(x) = 12x^{1/3}$; $P(-27, -36)$

Exer. 13–16: Find the first three derivatives.

13 $f(x) = 3x^6$

14 $f(x) = 6x^4$

15 $f(x) = 9\sqrt[3]{x^2}$

16 $f(x) = 3x^{7/3}$

17 If $z = 25t^{9/5}$, find $D_t^2 z$.

18 If $y = 3x + 5$, find $D_x^3 y$.

19 If $y = -4x + 7$, find $\frac{d^3y}{dx^3}$.

20 If $z = 64\sqrt[4]{t^3}$, find $\frac{d^2z}{dt^2}$.

Exer. 21–22: Is f differentiable on the given interval? Explain.

21 $f(x) = 1/x$ **(a)** $[0, 2]$ **(b)** $[1, 3)$

22 $f(x) = \sqrt[3]{x}$ **(a)** $[-1, 1]$ **(b)** $[-2, -1)$

Exer. 23–24: Use the graph of f to determine if f is differentiable on the given interval.

23 $f(x) = \sqrt{4 - x}$ **(a)** $[0, 4]$ **(b)** $[-5, 0]$

24 $f(x) = \sqrt{4 - x^2}$ **(a)** $[-2, 2]$ **(b)** $[-1, 1]$

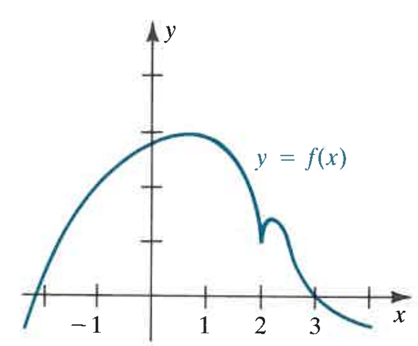
Exer. 25–30: Determine whether f has (a) a vertical tangent line at (0, 0) and (b) a cusp at (0, 0).

25 $f(x) = x^{1/3}$ 26 $f(x) = x^{5/3}$ 27 $f(x) = x^{2/5}$

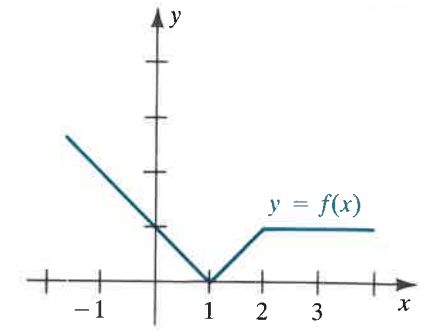
28 $f(x) = x^{1/4}$ 29 $f(x) = 5x^{3/2}$ 30 $f(x) = 7x^{4/3}$

Exer. 31–32: Estimate $f'(-1)$, $f'(1)$, $f'(2)$, and $f'(3)$, whenever they exist.

31



32



Exer. 33–36: Use right-hand and left-hand derivatives to prove that f is not differentiable at a.

33 $f(x) = |x - 5|$; $a = 5$

34 $f(x) = |x + 2|$; $a = -2$

35 $f(x) = [\![x - 2]\!]$; $a = 2$

36 $f(x) = [\![x]\!] - 2$; $a = 2$

Exer. 37–40: Use the graph of f to find the domain of f'.

37 $f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

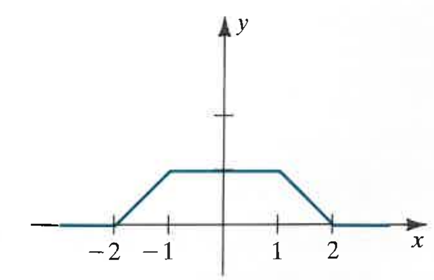
38 $f(x) = \begin{cases} 2x - 1 & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$

39 $f(x) = \begin{cases} -x^2 & \text{if } x < -1 \\ 2x + 3 & \text{if } x \ge -1 \end{cases}$

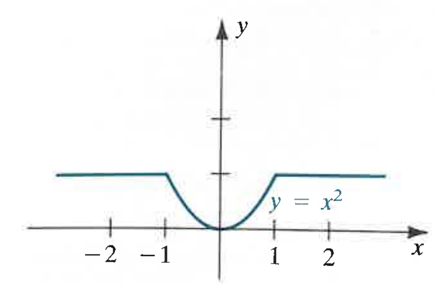
40 $f(x) = \begin{cases} x^2 - 2 & \text{if } x < 0 \\ -3 & \text{if } x \ge 0 \end{cases}$

Exer. 41–42: Each figure is the graph of a function f. Sketch the graph of f' and determine where f is not differentiable.

41



42



Exer. 43–44: (a) Find the first derivative and the second derivative on each of the three subintervals of the domain of the function. (b) Use the right-hand and left-hand derivatives to determine whether f' exists at a and b.

43 $a = 3;\ b = 5$

$$f(x) = \begin{cases} 3x^2 - 6x + 3 & \text{if } 1 \le x < a \\ -7x^2 + 54x - 87 & \text{if } a \le x < b \\ 8x^2 - 96x + 288 & \text{if } b \le x \le 6 \end{cases}$$

44 $a = -1;\ b = 1$

$$f(x) = \begin{cases} 3x^2 + 3x + 1 & \text{if } x < a \\ -x^3 & \text{if } a \le x < b \\ x^3 - 6x^2 + 6x - 2 & \text{if } x \ge b \end{cases}$$

Exer. 45–46: Given the position function s of a point P moving on a coordinate line l, find the times at which the velocity is the given value k.

45 $s(t) = 3t^{2/3};\quad k = 4$

46 $s(t) = 4t^3;\quad k = 300$

47 The relationship between the temperature F on the Fahrenheit scale and the temperature C on the Celsius scale is given by $C = \frac{5}{9}(F - 32)$. Find the rate of change of F with respect to C.

48 Charles's law for gases states that if the pressure remains constant, then the relationship between the volume V that a gas occupies and its temperature T (in °C) is given by $V = V_0(1 + \frac{1}{273}T)$. Find the rate of change of T with respect to V.

49 Show that the rate of change of the volume of a sphere with respect to its radius is numerically equal to the surface area of the sphere.

50 Poiseuille's law states that the velocity v of blood in a small artery with a circular cross section of radius R is given by

$$v(r) = A(R^2 - r^2),$$

where r is the distance from the center of the artery and A is a constant. Find $v'(r)$.

51 An oil spill is increasing such that the surface covered by the spill is always circular. Find the rate at which the area A of the surface is changing with respect to the radius r of the circle at

(a) any value of r **(b)** $r = 500$ ft

52 A spherical balloon is being inflated. Find the rate at which its volume V is changing with respect to the radius r of the balloon at

(a) any value of r **(b)** $r = 10$ ft

53 In some applications, function values $f(x)$ may be known only for several values of x near a. In these situations, $f'(a)$ is frequently approximated by the formula

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}.$$

(a) Interpret this formula graphically.

(b) Show that $\lim_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h} = f'(a)$.

(c) If $f(x) = 1/x^2$, use the approximation formula to estimate $f'(1)$ with $h = 0.1, 0.01$, and 0.001.

(d) Find the exact value of $f'(1)$.

54 (a) Use the approximation formula in Exercise 53 to show that if $h \approx 0$, then

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

(b) If $f(x) = 1/x^2$, use part (a) to estimate $f''(1)$ with $h = 0.1, 0.01$, and 0.001.

(c) Find the exact value of $f''(1)$.

c **Exer. 55–56: Use the following table, which lists the approximate number of feet $s(t)$ that a car travels in t seconds to reach a velocity of 60 mi/hr in 6 sec.**

t	0	1	2	3	4	5	6	7
$s(t)$	0	11.7	42.6	89.1	149.0	220.1	303.7	396.7

55 Use Exercise 53 to approximate the velocity of the car at

(a) $t = 3$ **(b)** $t = 6$

56 Use Exercise 54 to approximate the rate of change of the velocity of the car with respect to t at

(a) $t = 3$ **(b)** $t = 6$

c **57** Graph $f(x) = |x^5 - 2x^4 + 3x^3 - x + 1|$ on the interval $[-1, 1]$ and estimate where f is not differentiable.

c **58** Graph $f(x) = x^4 - 3x^3 + 2x - 1$ on the interval $[-1, 3]$ and estimate the x-coordinates of points at which the tangent line is horizontal.

c **Exer. 59–64: (a) Graph the function on the interval. (b) From the graph, estimate the x-coordinates of points where the function is not differentiable or where it has a horizontal tangent line. (c) Many graphing utilities can display the function and its first and second derivatives without explicitly requiring the formulas for the derivatives. If you have access to such a utility, graph the first and second derivatives on the same viewing window as the function. Discuss the behavior of the derivatives at points where the function has a horizontal tangent or near points where it is not differentiable.**

59 $f(x) = \dfrac{10\cos x}{x^2 + 4}$ on $-2 \le x \le 2$

60 $f(x) = \dfrac{(x - x^2)\sin x}{x^2 + 1}$ on $-1 \le x \le 4$

61 $f(x) = x \sin x$ on $-\pi \le x \le \pi$

62 $f(x) = \sin(\frac{1}{2}x + 1)$ on $0 \le x \le 4\pi$

63 $f(x) = |x^4 - 2x^3 - x^2 + 2x|$ on $-1.5 \le x \le 2.5$

64 $f(x) = \sqrt{4 - x^2}$ on $-2 \le x \le 2$

65 If f is differentiable at a and g is defined by $g(x) = x(f(x))$ for all x in the domain of f, then use Definition (2.5) to show that

$$g'(x) = x(f'(x)) + f(x).$$

66 Use the result of Exercise 65 and mathematical induction to give an alternative proof to the power rule (2.10) for $n \ge 1$. (*Hint:* $x^{n+1} = xx^n$.)

2.3 TECHNIQUES OF DIFFERENTIATION

This section contains some general rules that simplify the task of finding derivatives. The rules are stated in terms of the differential operator d/dx, where $(d/dx)(f(x)) = f'(x)$. In the rules, f and g denote differentiable functions, c, m, and b are real numbers, and n is a rational number. The first three parts of the following theorem were proved in Section 2.2 and are restated here for completeness.

Theorem 2.18

(i) $\dfrac{d}{dx}(c) = 0$

(ii) $\dfrac{d}{dx}(mx + b) = m$

(iii) $\dfrac{d}{dx}(x^n) = nx^{n-1}$

(iv) $\dfrac{d}{dx}(cf(x)) = c\dfrac{d}{dx}(f(x))$

(v) $\dfrac{d}{dx}(f(x) + g(x)) = \dfrac{d}{dx}(f(x)) + \dfrac{d}{dx}(g(x))$

(vi) $\dfrac{d}{dx}(f(x) - g(x)) = \dfrac{d}{dx}(f(x)) - \dfrac{d}{dx}(g(x))$

PROOF

(iv) Applying the definition of the derivative to $cf(x)$, we have

$$\frac{d}{dx}(cf(x)) = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \to 0} c\,\frac{f(x+h) - f(x)}{h}$$

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c\frac{d}{dx}(f(x)).$$

(v) Applying the definition of the derivative to $f(x)+g(x)$ yields

$$\begin{aligned}\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \frac{d}{dx}(f(x))+\frac{d}{dx}(g(x)).\end{aligned}$$

We can prove (vi) in similar fashion, or we can write

$$f(x)-g(x)=f(x)+(-1)g(x)$$

and then use (v) and (iv). ■

If we use the differential operator D_x in place of d/dx, the rules in Theorem (2.18) take on the following forms:

$$\begin{aligned}D_x(c)&=0,\\ D_x(mx+b)&=m,\\ D_x(x^n)&=nx^{n-1},\\ D_x(cf(x))&=cD_x(f(x)),\\ D_x(f(x)\pm g(x))&=D_x(f(x))\pm D_x(g(x)).\end{aligned}$$

Parts (v) and (vi) of Theorem (2.18) may be stated as follows:

(v) *The derivative of a sum is the sum of the derivatives.*

(vi) *The derivative of a difference is the difference of the derivatives.*

These results can be extended to sums or differences involving any finite number of functions. We may use these results to obtain easily the derivative of a polynomial, since a polynomial is a sum of terms of the form cx^n, where c is a real number and n is a nonnegative integer. The next example illustrates this process.

EXAMPLE ■ 1 If $f(x)=2x^4-5x^3+x^2-4x+1$, find $f'(x)$.

SOLUTION By Theorem (2.18), we have

$$\begin{aligned}f'(x)&=\frac{d}{dx}(2x^4-5x^3+x^2-4x+1)\\ &=\frac{d}{dx}(2x^4)-\frac{d}{dx}(5x^3)+\frac{d}{dx}(x^2)-\frac{d}{dx}(4x)+\frac{d}{dx}(1)\\ &=8x^3-15x^2+2x-4.\end{aligned}$$

To find the equation of a tangent line to the graph of a function f at a particular point $P(a, f(a))$, we need to find the slope of the line, which we know is given by $f'(a)$. We may use the differentiation rules to first determine the general form for the function $f'(x)$ and then substitute a for x to compute $f'(a)$, as in the next example.

EXAMPLE ■ 2

(a) Find an equation of the tangent line to the graph of the function $y = 1.2\sqrt[3]{x^2}-(0.8/\sqrt{x})$ at $P(1, 0.4)$.

(b) Use a graphing utility to graph the function and the tangent line.

SOLUTION

(a) We first express y in terms of rational exponents and then use Theorem (2.18) to find dy/dx:

$$y=1.2x^{2/3}-0.8x^{-1/2}$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}(1.2x^{2/3})-\frac{d}{dx}(0.8x^{-1/2})\\ &=0.8x^{-1/3}-(-0.4x^{-3/2})\\ &=\frac{0.8}{x^{1/3}}+\frac{0.4}{x^{3/2}}\end{aligned}$$

To find the slope of the tangent line at $P(1, 0.4)$, we evaluate dy/dx at $x=1$:

$$\left.\frac{dy}{dx}\right]_{x=1}=\frac{0.8}{1}+\frac{0.4}{1}=1.2.$$

Using the point–slope form, we can express an equation of the tangent line as

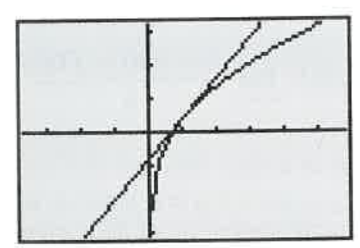
$$y-0.4=1.2(x-1),$$

or

$$y=1.2x-0.8.$$

(b) We use equal scaling and the viewing window shown in Figure 2.23 to graph the function $f(x)=1.2\sqrt[3]{x^2}-(0.8/\sqrt{x})$ and the tangent line $L(x)=1.2x-0.8$. We observe that both the tangent line and the curve are rising as x increases through values near 1. We also note that the tangent line remains close to the curve near the point of tangency.

Figure 2.23
$-4\le x\le 6,\ -3\le y\le 3$



Formulas for derivatives of products or quotients are more complicated than those for sums and differences. In particular, *the derivative of a product generally is not equal to the product of the derivatives*. We may illustrate this fact by using the product $x^2 \cdot x^5$ as follows:

$$\frac{d}{dx}(x^2 \cdot x^5) = \frac{d}{dx}(x^7) = 7x^6$$

$$\frac{d}{dx}(x^2) \cdot \frac{d}{dx}(x^5) = (2x) \cdot (5x^4) = 10x^5$$

Hence $$\frac{d}{dx}(x^2 \cdot x^5) \neq \frac{d}{dx}(x^2) \cdot \frac{d}{dx}(x^5).$$

The derivative of any product $f(x)g(x)$ may be expressed in terms of derivatives of $f(x)$ and $g(x)$ as in the following rule.

Product Rule 2.19

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

PROOF Let $y = f(x)g(x)$. Using the definition of the derivative, we write

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

To change the form of the quotient so that the limit may be evaluated, we subtract and add the expression $f(x+h)g(x)$ in the numerator. Thus,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Since f is differentiable at x, it is continuous at x (see Theorem (2.12)). Hence, $\lim_{h \to 0} f(x+h) = f(x)$. Also, $\lim_{h \to 0} g(x) = g(x)$, since x is fixed in this limiting process. Finally, applying the definition of derivative to $f(x)$ and $g(x)$, we obtain

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x). \quad \blacksquare$$

The product rule may be phrased as follows: *The derivative of a product equals the first factor times the derivative of the second factor, plus the second times the derivative of the first.*

EXAMPLE ■ 3 If $y = (x^3 + 1)(2x^2 + 8x - 5)$, find dy/dx.

SOLUTION Using the product rule (2.19), we have

$$\frac{dy}{dx} = (x^3 + 1)\frac{dy}{dx}(2x^2 + 8x - 5) + (2x^2 + 8x - 5)\frac{dy}{dx}(x^3 + 1)$$
$$= (x^3 + 1)(4x + 8) + (2x^2 + 8x - 5)(3x^2)$$
$$= (4x^4 + 8x^3 + 4x + 8) + (6x^4 + 24x^3 - 15x^2)$$
$$= 10x^4 + 32x^3 - 15x^2 + 4x + 8.$$

EXAMPLE ■ 4 If $f(x) = x^{1/3}(x^2 - 3x + 2)$, find

(a) $f'(x)$

(b) the x-coordinate of the points on the graph of f at which the tangent line is either horizontal or vertical

SOLUTION

(a) By the product rule (2.19),

$$f'(x) = x^{1/3}\frac{d}{dx}(x^2 - 3x + 2) + (x^2 - 3x + 2)\frac{d}{dx}(x^{1/3})$$
$$= x^{1/3}(2x - 3) + (x^2 - 3x + 2)(\tfrac{1}{3}x^{-2/3})$$
$$= \frac{3x(2x - 3) + (x^2 - 3x + 2)}{3x^{2/3}}$$
$$= \frac{7x^2 - 12x + 2}{3x^{2/3}}.$$

(b) The tangent line to the graph of f is horizontal if its slope is zero. Setting $f'(x) = 0$ and using the quadratic formula, we obtain

$$x = \frac{12 \pm \sqrt{144 - 56}}{2(7)} = \frac{12 \pm \sqrt{88}}{14} = \frac{6 \pm \sqrt{22}}{7}.$$

Referring to $f'(x)$, we see that the denominator $3x^{2/3}$ is zero at $x = 0$. Since f is continuous at 0 and $\lim_{x \to 0} |f'(x)| = \infty$, it follows from Definition (2.14) that the graph of f has a vertical tangent line at $x = 0$—that is, the point (0, 0) (the origin).

We shall next obtain a formula for the derivative of a quotient. Note that *the derivative of a quotient generally is not equal to the quotient of the derivatives*. We may illustrate this with the quotient x^5/x^2 as follows:

$$\frac{d}{dx}\left(\frac{x^5}{x^2}\right) = \frac{d}{dx}(x^3) = 3x^2$$

$$\frac{\frac{d}{dx}(x^5)}{\frac{d}{dx}(x^2)} = \frac{5x^4}{2x} = \frac{5}{2}x^3$$

Hence
$$\frac{d}{dx}\left(\frac{x^5}{x^2}\right) \neq \frac{\frac{d}{dx}(x^5)}{\frac{d}{dx}(x^2)}.$$

The derivative of any quotient $f(x)/g(x)$ may be expressed in terms of the derivatives of $f(x)$ and $g(x)$ as in the following rule.

Quotient Rule 2.20

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$$

PROOF Let $y = f(x)/g(x)$. From the definition of derivative,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$
$$= \lim_{h\to 0}\frac{g(x)f(x+h) - f(x)g(x+h)}{hg(x+h)g(x)}.$$

Subtracting and adding $g(x)f(x)$ in the numerator of the last quotient, we obtain

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{g(x)f(x+h) - g(x)f(x) + g(x)f(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$
$$= \lim_{h\to 0}\frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{hg(x+h)g(x)}$$
$$= \lim_{h\to 0}\frac{g(x)\left[\dfrac{f(x+h) - f(x)}{h}\right] - f(x)\left[\dfrac{g(x+h) - g(x)}{h}\right]}{g(x+h)g(x)}.$$

Taking the limit of the numerator and the denominator gives us the quotient rule. ■

The quotient rule may be stated as follows: *The derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, divided by the square of the denominator.*

EXAMPLE ■ 5 Find $\dfrac{dy}{dx}$ if $y = \dfrac{3x^2 - x + 2}{4x^2 + 5}$.

SOLUTION By the quotient rule (2.20),

$$\frac{dy}{dx} = \frac{(4x^2+5)\frac{d}{dx}(3x^2 - x + 2) - (3x^2 - x + 2)\frac{d}{dx}(4x^2+5)}{(4x^2+5)^2}$$
$$= \frac{(4x^2+5)(6x-1) - (3x^2 - x + 2)(8x)}{(4x^2+5)^2}$$
$$= \frac{(24x^3 - 4x^2 + 30x - 5) - (24x^3 - 8x^2 + 16x)}{(4x^2+5)^2}$$
$$= \frac{4x^2 + 14x - 5}{(4x^2+5)^2}.$$

If we let $f(x) = 1$ in the quotient rule (2.20), then, since $(d/dx)(1) = 0$, we obtain the following.

Reciprocal Rule 2.21

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{\frac{d}{dx}(g(x))}{(g(x))^2}$$

ILLUSTRATION

- $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{\frac{d}{dx}(x)}{(x)^2} = -\dfrac{1}{x^2}$
- $\dfrac{d}{dx}\left(\dfrac{1}{3x^2 - 5x + 4}\right) = -\dfrac{\frac{d}{dx}(3x^2 - 5x + 4)}{(3x^2 - 5x + 4)^2} = -\dfrac{6x - 5}{(3x^2 - 5x + 4)^2}$

The differentiation formulas in Theorem (2.18) are stated in terms of the function values $f(x)$ and $g(x)$. If we wish to state such rules without referring to the variable x, we may write

$$(cf)' = cf', \quad (f+g)' = f' + g', \quad \text{and} \quad (f-g)' = f' - g'.$$

Using this notation for the product, quotient, and reciprocal rules and at the same time commuting some of the factors that appear in (2.19) and (2.20), we obtain

$$(fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad \text{and} \quad \left(\frac{1}{g}\right) = \frac{-g'}{g^2}.$$

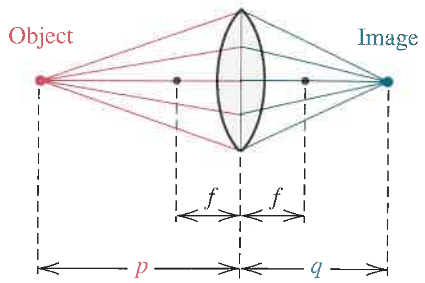
You may find it helpful to memorize these formulas. To obtain the quotient rule, change the + sign in the formula for $(fg)'$ to − and divide by g^2.

The next example gives an application that makes use of the quotient rule for differentiation.

EXAMPLE ■ 6 A convex lens of focal length f is shown in Figure 2.24. If an object is a distance p from the lens as shown, then the distance q from the lens to the image is related to p and f by the *lens equation*

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$

Figure 2.24



If, for a particular lens, $f = 2$ cm and p is increasing, find

(a) a general formula for the rate of change of q with respect to p

(b) the rate of change of q with respect to p if $p = 22$ cm

SOLUTION

(a) By (2.7)(ii), the rate of change of q with respect to p is given by the derivative dq/dp. If $f = 2$, then the lens equation gives us

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{q}, \quad \text{or} \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{p} = \frac{p-2}{2p}.$$

Hence $$q = \frac{2p}{p-2}.$$

Applying the quotient rule (with $x = p$) yields

$$\frac{dq}{dp} = \frac{(p-2)\dfrac{d}{dp}(2p) - (2p)\dfrac{d}{dp}(p-2)}{(p-2)^2}$$

$$= \frac{(p-2)(2) - (2p)(1)}{(p-2)^2}$$

$$= \frac{-4}{(p-2)^2}.$$

(b) Substituting $p = 22$ in the formula obtained in part (a), we get

$$\left.\frac{dq}{dp}\right]_{p=22} = \frac{-4}{(22-2)^2} = \frac{-4}{400} = -\frac{1}{100}.$$

Thus, if $p = 22$ cm, the image distance q is decreasing at a rate of $\frac{1}{100}$ centimeter per centimeter change in p.

We have introduced several rules in this section that ease the task of finding derivatives of functions. These rules permit us to differentiate a complicated function that has been built up from simpler functions in particular ways (addition, subtraction, multiplication, and division) by combining the derivatives of the simpler functions in the correct manner. These formulas and others to be studied later make differentiation a relatively straightforward process of applying the various rules.

In recent years, the possibility of carrying out these rules with the assistance of computational devices has become a reality. We discussed earlier how calculators and computers can provide *numerical* estimates for derivatives. Computer scientists have also created sophisticated programs that accept and operate on algebraic expressions and functions. Such programs can perform a variety of algebraic operations on command, including symbolic differentiation. These programs, called **computer algebra systems (CAS)** or **computer mathematics systems (CMS),** combine the capability for algebraic manipulation with provisions for graphing and numerical evaluations. In effect, a CAS has stored the rules for differentiation. It performs pattern matching on a given symbolic expression to determine which rules to apply, much as you are learning to do.

Since electronic devices can perform symbolic differentiation, you may well ask whether we still need to learn the rules for differentiation. The answer is *yes,* for much the same reasons that the ability of calculators to add and multiply numbers quickly and accurately has not ended the need to learn the rules of arithmetic. In both instances, the emphasis shifts away from developing great skill in applying the rules to very complicated expressions to a greater need for understanding and interpretation. The properties of differentiation reflected in the rules are used throughout calculus and other branches of mathematics to gain new insights.

If you were going to add a few small integers, you would probably do so mentally, instead of searching for a calculator. Similarly, you will find it easier to differentiate a polynomial or a rational function on paper than to type these expressions into the CAS. When you must perform a very important, complicated differentiation, the speed and accuracy of a CAS provides a valuable tool. In summary, the availability of a CAS no more replaces the need to perform basic symbolic manipulation than the availability of a calculator replaces the need to perform basic arithmetic operations.

EXERCISES 2.3

Exer. 1–34: Find the derivative.

1 $g(t) = 6t^{5/3}$

2 $h(z) = 8z^{3/2}$

3 $f(s) = 15 - s + 4s^2 - 5s^4$

4 $f(t) = 12 - 3t^4 + 4t^6$

5 $f(x) = 3x^2 + \sqrt[3]{x^4}$

6 $g(x) = x^4 - \sqrt[4]{x^3}$

7 $g(x) = (x^3 - 7)(2x^2 + 3)$

8 $k(x) = (2x^2 - 4x + 1)(6x - 5)$

9 $f(x) = x^{1/2}(x^2 + x - 4)$

10 $h(x) = x^{2/3}(3x^2 - 2x + 5)$

11 $h(r) = r^2(3r^4 - 7r + 2)$

12 $k(v) = v^3(-2v^3 + v - 3)$

13 $g(x) = (8x^2 - 5x)(13x^2 + 4)$

14 $H(z) = (z^5 - 2z^3)(7z^2 + z - 8)$

15 $f(x) = \dfrac{4x-5}{3x+2}$

16 $h(x) = \dfrac{8x^2 - 6x + 11}{x - 1}$

17 $h(z) = \dfrac{8 - z + 3z^2}{2 - 9z}$

18 $f(w) = \dfrac{2w}{w^3 - 7}$

19 $G(v) = \dfrac{v^3 - 1}{v^3 + 1}$

20 $f(t) = \dfrac{8t + 15}{t^2 - 2t + 3}$

21 $g(t) = \dfrac{\sqrt[3]{t^2}}{3t - 5}$

22 $f(x) = \dfrac{\sqrt{x}}{2x^2 - 4x + 8}$

23 $f(x) = \dfrac{1}{1 + x + x^2 + x^3}$

24 $p(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$

25 $h(x) = \dfrac{7}{x^2 + 5}$ **26** $k(z) = \dfrac{6}{z^2 + z - 1}$

27 $F(t) = t^2 + \dfrac{1}{t^2}$ **28** $s(x) = 2x + \dfrac{1}{2x}$

29 $K(s) = (3s)^{-4}$ **30** $W(s) = (3s)^4$

31 $h(x) = (5x - 4)^2$ **32** $S(w) = (2w + 1)^3$

33 $g(r) = (5r - 4)^{-2}$ **34** $S(x) = (3x + 1)^{-2}$

c **Exer. 35–40: (a) Find the derivative. (b) Plot the function and its derivative in the given viewing window.**

35 $f(x) = x^3 - 5x^2 + 8x - 25$,
$-3.5 \le x \le 7, -150 \le y \le 150$

36 $f(x) = x^4 - x^3 - 13x^2 + x + 12$,
$-4 \le x \le 4, -70 \le y \le 110$

37 $f(x) = \left(4 + \dfrac{1}{x}\right)\left(6x - \dfrac{1}{x^2}\right)$,
$-3 \le x \le 3, -50 \le y \le 50$

38 $f(x) = \left(x^2 - \dfrac{1}{x^2}\right)\left(x^2 + \dfrac{1}{x^2}\right)$,
$-3 \le x \le 3, -50 \le y \le 50$

39 $f(x) = \dfrac{(4x + 1)(x - 3)}{3x + 2}$,
$-6 \le x \le 6, -15 \le y \le 10$

40 $f(x) = \left(\dfrac{x + 3}{x + 1}\right)(x^2 - 2x - 1)$,
$-11 \le x \le 5, -25 \le y \le 35$

Exer. 41–44: Solve the equation $dy/dx = 0$.

41 $y = 2x^3 - 3x^2 - 36x + 4$

42 $y = 4x^3 + 21x^2 - 24x + 11$

43 $y = \dfrac{2x^2 + 3x - 6}{x - 2}$ **44** $y = \dfrac{x^2 + 2x + 5}{x + 1}$

Exer. 45–46: Solve the equation $d^2y/dx^2 = 0$.

45 $y = 6x^4 + 24x^3 - 540x^2 + 7$

46 $y = 6x^5 - 5x^4 - 30x^3 + 11x$

Exer. 47–50: Find dy/dx by (a) using the quotient rule, (b) using the product rule, and (c) simplifying algebraically and using Theorem (2.18).

47 $y = \dfrac{3x - 1}{x^2}$ **48** $y = \dfrac{x^2 + 1}{x^4}$

49 $y = \dfrac{x^2 - 3x}{\sqrt[3]{x^2}}$ **50** $y = \dfrac{2x + 3}{\sqrt{x^3}}$

Exer. 51–52: Find d^2y/dx^2.

51 $y = \dfrac{3x + 4}{x + 1}$ **52** $y = \dfrac{x + 3}{2x + 3}$

Exer. 53–54: Find an equation of the tangent line to the graph of f at P.

53 $f(x) = \dfrac{5}{1 + x^2}$; $P(-2, 1)$

54 $f(x) = 3x^2 - 2\sqrt{x}$; $P(4, 44)$

55 Find the x-coordinates of all points on the graph of $y = x^3 + 2x^2 - 4x + 5$ at which the tangent line is
(a) horizontal **(b)** parallel to the line $2y + 8x = 5$

56 Find the point P on the graph of $y = x^3$ such that the tangent line at P has x-intercept 4.

57 Find the points on the graph of $y = x^{3/2} - x^{1/2}$ at which the tangent line is parallel to the line $y - x = 3$.

58 Find the points on the graph of $y = x^{5/3} + x^{1/3}$ at which the tangent line is perpendicular to the line $2y + x = 7$.

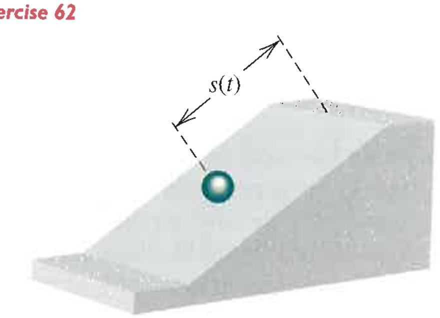
Exer. 59–60: Sketch the graph of the equation and find the vertical tangent lines.

59 $y = \sqrt{x} - 4$ **60** $y = x^{1/3} + 2$

61 A weather balloon is released and rises vertically such that its distance $s(t)$ above the ground during the first 10 sec of flight is given by $s(t) = 6 + 2t + t^2$, where $s(t)$ is in feet and t is in seconds. Find the velocity of the balloon at
(a) $t = 1$, $t = 4$, and $t = 8$
(b) the instant the balloon is 50 ft above the ground

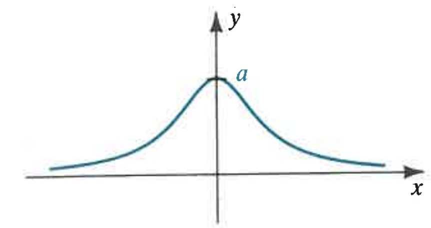
62 A ball rolls down an inclined plane such that the distance (in centimeters) that it rolls in t seconds is given by $s(t) = 2t^3 + 3t^2 + 4$ for $0 \le t \le 3$ (see figure).
(a) Find the velocity of the ball at $t = 2$.
(b) At what time is the velocity 30 cm/sec?

Exercise 62

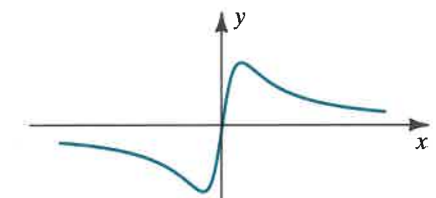


Exer. 63–64: An equation of a classical curve and its graph are given for positive constants a and b. (Consult books on analytic geometry for further information.) Find the slope of the tangent line at the point P.

63 *Witch of Agnesi:* $y = \dfrac{a^3}{a^2 + x^2}$; $P(a, a/2)$



64 *Serpentine curve:* $y = \dfrac{abx}{a^2 + x^2}$; $P(a, b/2)$



Exer. 65–66: Find equations of the lines through P that are tangent to the graph of the equation.

65 $P(5, 9)$; $y = x^2$ **66** $P(3, 1)$; $xy = 4$

Exer. 67–70: If f and g are functions such that $f(2) = 3$, $f'(2) = -1$, $g(2) = -5$, and $g'(2) = 2$, evaluate the expression.

67 **(a)** $(f + g)'(2)$ **(b)** $(f - g)'(2)$ **(c)** $(4f)'(2)$
(d) $(fg)'(2)$ **(e)** $(f/g)'(2)$ **(f)** $(1/f)'(2)$

68 **(a)** $(g - f)'(2)$ **(b)** $(g/f)'(2)$
(c) $(4g)'(2)$ **(d)** $(ff)'(2)$

69 **(a)** $(2f - g)'(2)$ **(b)** $(5f + 3g)'(2)$
(c) $(gg)'(2)$ **(d)** $\left(\dfrac{1}{f + g}\right)'(2)$

70 **(a)** $(3f - 2g)'(2)$ **(b)** $(5/g)'(2)$
(c) $(6f)'(2)$ **(d)** $\left(\dfrac{f}{f + g}\right)'(2)$

71 If f, g, and h are differentiable functions of x, use the product rule to prove that

$$\frac{d}{dx}(fgh) = f'gh + fg'h + fgh'.$$

As a corollary, let $f = g = h$ to prove that

$$\frac{d}{dx}(f(x))^3 = 3[f(x)]^2 f'(x).$$

72 Extend Exercise 71 to the derivative of a product of four functions, and then find a formula for $(d/dx)(f(x))^4$.

Exer. 73–76: Use Exercise 71 to find dy/dx.

73 $y = (8x - 1)(x^2 + 4x + 7)(x^3 - 5)$

74 $y = (3x^4 - 10x^2 + 8)(2x^2 - 10)(6x + 7)$

75 $y = x(2x^3 - 5x - 1)(6x^2 + 7)$

76 $y = 4x(x - 1)(2x - 3)$

77 As a spherical balloon is being inflated, its radius r (in centimeters) after t minutes is given by $r = 3\sqrt[3]{t}$ for $0 \le t \le 10$. Find the rate of change for each of the following with respect to t at $t = 8$:
(a) the radius r **(b)** the volume V of the balloon
(c) the surface area S of the balloon

78 The volume V (in cubic feet) of water in a small reservoir during spring runoff is given by the formula $V = 5000(t + 1)^2$ for t in months and $0 \le t \le 3$. The rate of change of volume with respect to time is the instantaneous *flow rate* into the reservoir. Find the flow rate at times $t = 0$ and $t = 2$. What is the flow rate when the volume is 11,250 ft^3?

79 A stone is dropped into a pond, causing water waves that form concentric circles. If, after t seconds, the radius of one of the waves is $40t$ centimeters, find the rate of change, with respect to t, of the area of the circle caused by the wave at
(a) $t = 1$ **(b)** $t = 2$ **(c)** $t = 3$

80 Boyle's law for confined gases states that if the temperature remains constant, then $pv = c$, where p is the pressure, v is the volume, and c is a constant. Suppose that at time t (in minutes) the pressure is $20 + 2t$ centimeters of mercury for $0 \le t \le 10$. If the volume is 60 cm^3 at $t = 0$, find the rate at which the volume is changing with respect to t at $t = 5$.

81 When a bright light is directed toward the eye, the pupil contracts. Suppose that the relationship between R, the area of the pupil (in square millimeters), and x, the brightness of the light source (in lumens), is given by

$$R = \frac{40 + 23.7x^4}{1 + 3.95x^4}.$$

The rate of change dR/dx is called the *sensitivity at stimulus level* x.
(a) Show that R decreases from 40 to 6 as x increases without bound.

(b) Find a formula for the sensitivity as a function of x.

c **(c)** Using the result of part (b), plot the graph of the sensitivity as a function of x for $x \geq 0$. Approximate the value of x for which the absolute value of the sensitivity is largest.

82 To win a point in racquetball, a player must have the serve and then win a rally. If p is the probability of winning a rally, then the probability S of achieving a shutout (winning 21–0) is given by

$$S = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21},$$

provided each player is equally likely to have the initial serve. Note that a probability is always a number between 0 and 1.

(a) Find the rate of change of S with respect to p.

c **(b)** Estimate this rate of change when $p = \frac{1}{2}$.

c **83** **(a)** If $f(x) = x^3 - 2x + 2$, approximate $f'(1)$ using Exercise 53 of Section 2.2 with $h = 0.1$.

(b) Graph the following on the same coordinate axes: $y = f(x)$, the secant line l_1 through $(1, f(1))$ and $(1.1, f(1.1))$, and the secant line l_2 through $(0.9, f(0.9))$ and $(1.1, f(1.1))$.

(c) Find $f'(1)$ and explain why the slope of l_2 is a better approximation to $f'(1)$ than is the slope of l_1.

c **84** **(a)** If $f(x) = x^{2/3} + 1$, approximate $f'(0)$ using Exercise 53 of Section 2.2 with $h = 0.1$.

(b) Graph the following on the same coordinate axes: $y = f(x)$, the secant line through $(0, f(0))$ and $(0.1, f(0.1))$, and the secant line through $(-0.1, f(-0.1))$ and $(0.1, f(0.1))$.

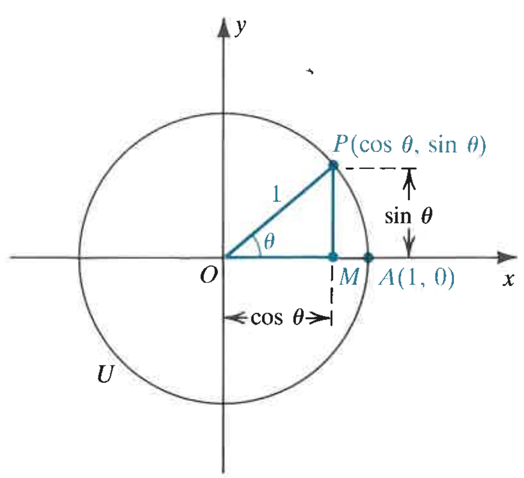
(c) Why don't the slopes of the secant lines in part (b) approximate $f'(0)$?

2.4 DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

In this section, we examine limits and derivatives involving the trigonometric functions. To obtain formulas for the derivatives of these functions, we must first prove several results about limits. Whenever we discuss limits of trigonometric expressions involving $\sin\theta$, $\cos t$, $\tan x$, and so on, *we shall assume that each variable represents the radian measure of an angle or a real number*.

Let θ denote an angle in standard position on a rectangular coordinate system, and consider the unit circle U in Figure 2.25. According to the definition of the sine and cosine functions, the coordinates of the indicated point P are $(\cos\theta, \sin\theta)$. It appears that if $\theta \to 0$, then $\sin\theta \to 0$ and $\cos\theta \to 1$. This suggests the theorem on the following page.

Figure 2.25



Theorem 2.22

(i) $\lim_{\theta\to 0} \sin\theta = 0$

(ii) $\lim_{\theta\to 0} \cos\theta = 1$

PROOF

(i) Let us first show that $\lim_{\theta\to 0^+} \sin\theta = 0$. If $0 < \theta < \pi/2$, then, referring to Figure 2.25, we see that

$$0 < MP < \widehat{AP},$$

where MP denotes the length of the line segment joining M to P and $\widehat{AP}$ denotes the length of the circular arc between A and P. By the definition of radian measure of an angle, $\widehat{AP} = \theta$, and therefore the preceding inequality can be written

$$0 < \sin\theta < \theta.$$

Since $\lim_{\theta\to 0^+} \theta = 0$ and $\lim_{\theta\to 0^+} 0 = 0$, it follows from the sandwich theorem (1.15) that $\lim_{\theta\to 0^+} \sin\theta = 0$.

To complete the proof of (i), it suffices to show that $\lim_{\theta\to 0^-} \sin\theta = 0$. If $-\pi/2 < \theta < 0$, then $0 < -\theta < \pi/2$ and hence, from the first part of the proof,

$$0 < \sin(-\theta) < -\theta.$$

Using the trigonometric identity $\sin(-\theta) = -\sin\theta$ and then multiplying by -1 gives us

$$\theta < \sin\theta < 0.$$

Since $\lim_{\theta\to 0^-} \theta = 0$ and $\lim_{\theta\to 0^-} 0 = 0$, it follows from the sandwich theorem that $\lim_{\theta\to 0^-} \sin\theta = 0$.

(ii) Using $\sin^2\theta + \cos^2\theta = 1$, we obtain $\cos\theta = \pm\sqrt{1-\sin^2\theta}$. If $-\pi/2 < \theta < \pi/2$, then $\cos\theta$ is positive, and hence $\cos\theta = \sqrt{1-\sin^2\theta}$. Consequently,

$$\begin{aligned}\lim_{\theta\to 0}\cos\theta &= \lim_{\theta\to 0}\sqrt{1-\sin^2\theta}\\ &= \sqrt{\lim_{\theta\to 0}(1-\sin^2\theta)}\\ &= \sqrt{1-0} = 1.\end{aligned}$$ ■

In Section 1.1, we used a calculator and a graph to guess the limit stated in the next theorem (see page 86), which we can now prove.

Theorem 2.23

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

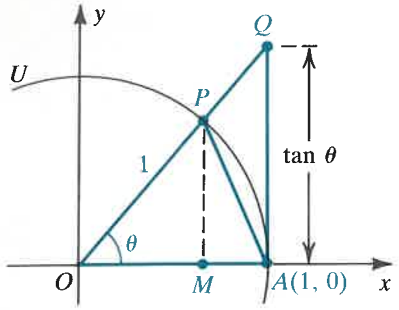


Figure 2.26

PROOF If $0 < \theta < \pi/2$, we have the situation illustrated in Figure 2.26, where U is a unit circle. Note that

$$MP = \sin\theta \quad \text{and} \quad AQ = \tan\theta.$$

From the figure, we see that

$$\text{Area of } \triangle AOP < \text{area of sector } AOP < \text{area of } \triangle AOQ.$$

From geometry and trigonometry,

$$\text{Area of } \triangle AOP = \tfrac{1}{2}bh = \tfrac{1}{2}(1)(MP) = \tfrac{1}{2}\sin\theta,$$
$$\text{Area of sector} AOP = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(1)^2\theta = \tfrac{1}{2}\theta,$$
$$\text{Area of } \triangle AOQ = \tfrac{1}{2}bh = \tfrac{1}{2}(1)(AQ) = \tfrac{1}{2}\tan\theta.$$

Hence the preceding inequality may be written

$$\tfrac{1}{2}\sin\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta.$$

Using the identity $\tan\theta = (\sin\theta)/(\cos\theta)$ and then dividing by $\frac{1}{2}\sin\theta$ leads to the following equivalent inequalities:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

The last inequality is also true if $-\pi/2 < \theta < 0$, for in this case we have $0 < -\theta < \pi/2$ and hence

$$\cos(-\theta) < \frac{\sin(-\theta)}{-\theta} < 1.$$

Using the identities $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, we again obtain

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0}\cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, the statement of the theorem follows from the sandwich theorem. ■

We shall also make use of the following result.

Theorem 2.24

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

PROOF If we let $\theta = 0$ in the expression $(1-\cos\theta)/\theta$, we obtain 0/0. Hence we must change the form of the quotient. Remembering from trigonometry that $1-\cos^2\theta = \sin^2\theta$, we multiply the numerator and the

denominator of the expression by $1+\cos\theta$ and then simplify as follows:

$$\frac{1-\cos\theta}{\theta} = \frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}$$
$$= \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \frac{\sin^2\theta}{\theta(1+\cos\theta)} = \frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}$$

Consequently,

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}\right)$$
$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}$$
$$= 1\cdot\frac{0}{1+1} = 1\cdot 0 = 0. \quad ■$$

The next three examples illustrate the use of Theorems (2.22), (2.23), and (2.24) when finding limits of trigonometric expressions.

EXAMPLE ■ 1 Find $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{2x}$.

SOLUTION We cannot apply Theorem (2.23) directly, since the given expression is not in the form $(\sin t)/t$. However, we may introduce this form (with $t = 5x$) by using the following algebraic manipulation:

$$\lim_{x\to 0}\frac{\sin 5x}{2x} = \lim_{x\to 0}\frac{1}{2}\frac{\sin 5x}{x}$$
$$= \lim_{x\to 0}\frac{5}{2}\frac{\sin 5x}{5x}$$
$$= \frac{5}{2}\lim_{x\to 0}\frac{\sin 5x}{5x}$$

It follows from the definition of limit that $x \to 0$ may be replaced by $5x \to 0$. Hence, by Theorem (2.23), with $t = 5x$, we see that

$$\frac{5}{2}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{2}(1) = \frac{5}{2}.$$

EXAMPLE ■ 2 Find $\displaystyle\lim_{t\to 0}\frac{\tan t}{2t}$.

SOLUTION Using the fact that $\tan t = (\sin t)/(\cos t)$ yields

$$\lim_{t\to 0}\frac{\tan t}{2t} = \lim_{t\to 0}\left(\frac{1}{2}\cdot\frac{\sin t}{t}\cdot\frac{1}{\cos t}\right)$$
$$= \tfrac{1}{2}\cdot 1\cdot\tfrac{1}{1} = \tfrac{1}{2}.$$

EXAMPLE 3 Find $\lim_{x\to 0} \dfrac{2x + 1 - \cos x}{3x}$.

SOLUTION We plan to use Theorem (2.24). With this in mind, we begin by isolating the part of the quotient that involves $(1 - \cos x)/x$ and then proceed as follows:

$$\begin{aligned}\lim_{x\to 0} \frac{2x + 1 - \cos x}{3x} &= \lim_{x\to 0}\left(\frac{2x}{3x} + \frac{1 - \cos x}{3x}\right)\\ &= \lim_{x\to 0}\left(\frac{2x}{3x}\right) + \lim_{x\to 0}\frac{1}{3}\left(\frac{1-\cos x}{x}\right)\\ &= \lim_{x\to 0}\frac{2}{3} + \frac{1}{3}\lim_{x\to 0}\frac{1-\cos x}{x}\\ &= \tfrac{2}{3} + \tfrac{1}{3}\cdot 0 = \tfrac{2}{3}\end{aligned}$$

We may now establish the formulas listed in the following theorem, where x *denotes a real number or the radian measure of an angle.*

Derivatives of the Trigonometric Functions 2.25

$\dfrac{d}{dx}(\sin x) = \cos x$	$\dfrac{d}{dx}(\cos x) = -\sin x$
$\dfrac{d}{dx}(\tan x) = \sec^2 x$	$\dfrac{d}{dx}(\cot x) = -\csc^2 x$
$\dfrac{d}{dx}(\sec x) = \sec x \tan x$	$\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

PROOF Applying Definition (2.5) with $f(x) = \sin x$ and then using the addition formula for the sine function, we obtain

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}\\ &= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right].\end{aligned}$$

By Theorems (2.24) and (2.23),

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \quad\text{and}\quad \lim_{h\to 0}\frac{\sin h}{h} = 1,$$

and hence $\quad \dfrac{d}{dx}(\sin x) = (\sin x)(0) + (\cos x)(1) = \cos x.$

We have shown that the derivative of the sine function is the cosine function. We may obtain the derivative of the cosine function in similar fashion:

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}\\ &= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\\ &= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x \sin h}{h}\\ &= \lim_{h\to 0}\left[\cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)\right]\\ &= (\cos x)(0) - (\sin x)(1) = -\sin x\end{aligned}$$

Thus the derivative of the cosine function is the *negative* of the sine function.

To find the derivative of the tangent function, we begin with the fundamental identity $\tan x = (\sin x)/(\cos x)$ and then apply the quotient rule as follows:

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\dfrac{d}{dx}(\sin x) - \sin x\dfrac{d}{dx}(\cos x)}{\cos^2 x}\\ &= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

For the secant function, we first write $\sec x = 1/\cos x$ and then use the reciprocal rule (2.21):

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= -\frac{\dfrac{d}{dx}(\cos x)}{\cos^2 x}\\ &= -\frac{-\sin x}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x}\\ &= \frac{1}{\cos x}\,\frac{\sin x}{\cos x} = \sec x\tan x\end{aligned}$$

Proofs of the formulas for $(d/dx)(\cot x)$ and $(d/dx)(\csc x)$ are left as exercises. ■

We can use (2.25) to obtain information about the continuity of the trigonometric functions. For example, since the sine and cosine functions are differentiable at every real number, it follows from Theorem (2.12) that these functions are continuous throughout $\mathbb{R}$. Similarly, the tangent function is continuous on the open intervals $(-\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$, and so on, since it is differentiable at each number in these intervals.

EXAMPLE 4 Find y' if $y = \dfrac{\sin x}{1 + \cos x}$.

SOLUTION By the quotient rule and (2.25),

$$\begin{aligned} y' &= \frac{(1+\cos x)\dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(1 + \cos x)}{(1+\cos x)^2} \\ &= \frac{(1+\cos x)\cos x - \sin x(0 - \sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \\ &= \frac{\cos x + 1}{(1+\cos x)^2} \\ &= \frac{1}{1+\cos x}. \end{aligned}$$

In the solution to Example 4, we used the fundamental identity $\cos^2 x + \sin^2 x = 1$. This and other trigonometric identities are often useful in simplifying problems that involve derivatives of trigonometric functions.

EXAMPLE 5 Find $g'(x)$ if $g(x) = \sec x \tan x$.

SOLUTION By the product rule and (2.25),

$$\begin{aligned} g'(x) &= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \\ &= \sec x \sec^2 x + \tan x(\sec x \tan x) \\ &= \sec^3 x + \sec x \tan^2 x \\ &= \sec x(\sec^2 x + \tan^2 x). \end{aligned}$$

The formula for $g'(x)$ can be written in many other ways. For example, because

$$\sec^2 x = \tan^2 x + 1, \quad \text{or} \quad \tan^2 x = \sec^2 x - 1,$$

we can write

$$g'(x) = \sec x(2\tan^2 x + 1), \quad \text{or} \quad g'(x) = \sec x(2\sec^2 x - 1).$$

EXAMPLE 6 Find $dy/d\theta$ if $y = \sec\theta \cot\theta$.

SOLUTION We could use the product rule as in Example 5; however, it is simpler to first change the form of y by using fundamental identities as follows:

$$y = \sec\theta\cot\theta = \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} = \csc\theta$$

Applying (2.25) yields

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\csc\theta) = -\csc\theta\cot\theta.$$

EXAMPLE 7

(a) Find the slopes of the tangent lines to the graph of $y = \sin x$ at the points with x-coordinates $0, \pi/3, \pi/2, 2\pi/3$, and π.

(b) Sketch the graph of $y = \sin x$ and the tangent lines of part (a).

(c) For what values of x is the tangent line horizontal?

SOLUTION

(a) The slope of the tangent line at the point (x, y) on the graph of the equation $y = \sin x$ is given by the derivative $y' = \cos x$. The slopes at the desired points are listed in the following table.

x	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$y' = \cos x$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

(b) A portion of the graph of $y = \sin x$ and the tangent lines of part (a) are sketched in Figure 2.27.

Figure 2.27

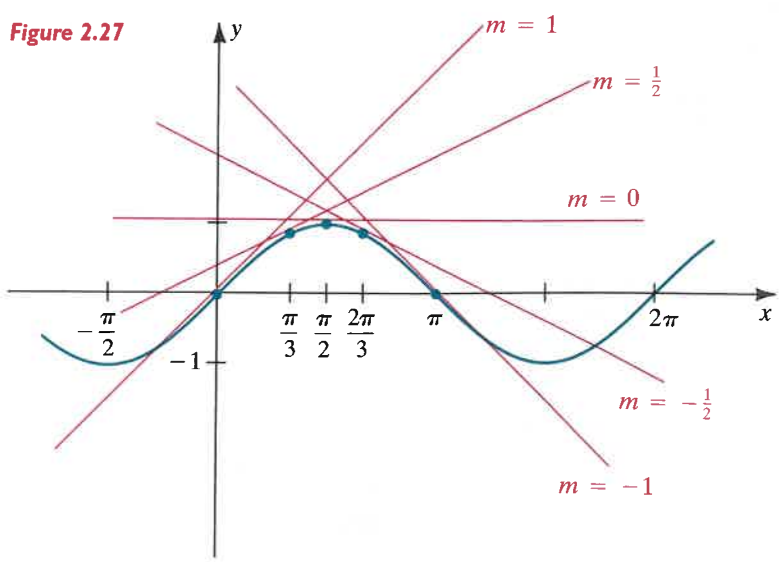
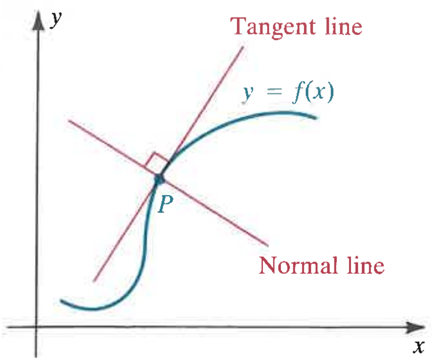


Figure 2.28



(c) A tangent line is horizontal if its slope is zero. Since the slope of the tangent line at the point (x, y) is y', we must solve the equation

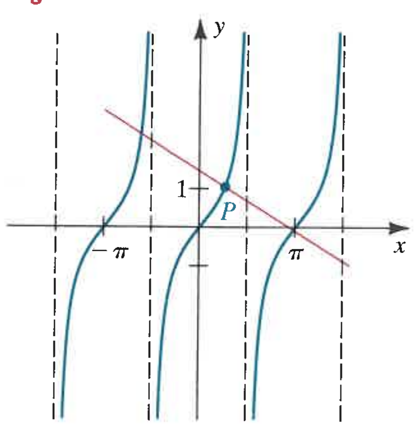
$$y' = 0; \quad \text{that is,} \quad \cos x = 0.$$

Thus the tangent line is horizontal if $x = \pm\pi/2$, $x = \pm 3\pi/2$, and, in general, if $x = (\pi/2) + \pi n$ for any integer n.

If f is a differentiable function, then the **normal line** at a point $P(a, f(a))$ on the graph of f is the line through P that is perpendicular to the tangent line, as illustrated in Figure 2.28. If $f'(a) \neq 0$, then, by (9)(iii) on page 15, the slope of the normal line is $-1/f'(a)$. If $f'(a) = 0$, then the tangent line is horizontal, and in this case the normal line is vertical and has the equation $x = a$.

EXAMPLE 8 Find an equation of the normal line to the graph of $y = \tan x$ at the point $P(\pi/4, 1)$, and illustrate it graphically.

Figure 2.29



SOLUTION Since $y' = \sec^2 x$, the slope m of the tangent line at P is

$$m = \sec^2 \frac{\pi}{4} = (\sqrt{2})^2 = 2$$

and hence the slope of the normal line is $-1/m = -1/2$.

Using the point–slope form, we can express an equation for the normal line as

$$y - 1 = -\frac{1}{2}\left(x - \frac{\pi}{4}\right),$$

or

$$y = -\frac{1}{2}x + \frac{\pi}{8} + 1.$$

The graph of $y = \tan x$ for $-3\pi/2 < x < 3\pi/2$ and the normal line at P are sketched in Figure 2.29.

The next example illustrates an application involving the derivatives of the trigonometric functions.

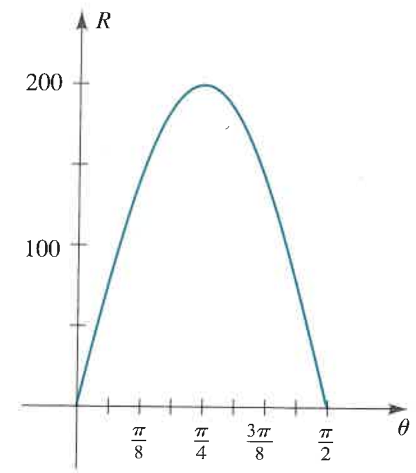
EXAMPLE 9 If a projectile is fired from ground level with an initial velocity of v ft/sec and at an angle of θ, then the range R of the projectile is given by

$$R = \frac{v^2}{16} \sin\theta \cos\theta \quad \text{for } 0 < \theta < \frac{\pi}{2}.$$

(a) If $v = 80$ ft/sec, find the rate of change of R with respect to θ.

(b) Determine the values of θ for which the rate of change in part (a) is positive.

Figure 2.30



SOLUTION

(a) The rate of change of R with respect to θ is given by $dR/d\theta$. With $v = 80$, we have $R = 400 \sin\theta \cos\theta$. Using the product rule (2.19) gives

$$\frac{dR}{d\theta} = 400\left[\sin\theta \frac{d}{d\theta}(\cos\theta) + \cos\theta \frac{d}{d\theta}(\sin\theta)\right]$$
$$= 400[\sin\theta(-\sin\theta) + \cos\theta\cos\theta]$$
$$= 400[-\sin^2\theta + \cos^2\theta],$$

which we can rewrite, using a double-angle formula, as $400\cos 2\theta$. Thus the rate of change of R with respect to θ is $400\cos 2\theta$ feet per radian. For a particular value of θ, $dR/d\theta$ is an estimate of the number of feet the range of the projectile changes for each radian change in θ.

(b) From part (a), $dR/d\theta = 400\cos 2\theta$, which is positive if $\cos 2\theta > 0$; that is, if $0 \le 2\theta < \pi/2$, which corresponds to $0 \le \theta < \pi/4$. If θ is a positive angle less than $\pi/4$, then a small increase in θ means a positive value for the rate of change of the projectile's range. Figure 2.30 shows a graph of the function $R = 400\sin\theta\cos\theta$. We see that R is increasing for values of θ between 0 and $\pi/4$ and R is decreasing for values of θ between $\pi/4$ and $\pi/2$. Do not mistake the graph in Figure 2.30 with the graph of the path of the projectile.

EXERCISES 2.4

Exer. 1–26: Find the limit, if it exists.

1 $\displaystyle\lim_{x\to 0} \frac{x}{\sin x}$

2 $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sqrt[3]{x}}$

3 $\displaystyle\lim_{t\to 0} \frac{\sin^3 t}{(2t)^3}$

4 $\displaystyle\lim_{\theta\to 0} \frac{3\theta + \sin\theta}{\theta}$

5 $\displaystyle\lim_{x\to 0} \frac{2 + \sin x}{3 + x}$

6 $\displaystyle\lim_{t\to 0} \frac{1 - \cos 3t}{t}$

7 $\displaystyle\lim_{\theta\to 0} \frac{2\cos\theta - 2}{3\theta}$

8 $\displaystyle\lim_{x\to 0} \frac{x^2 + 1}{x + \cos x}$

9 $\displaystyle\lim_{x\to 0} \frac{\sin(-3x)}{4x}$

10 $\displaystyle\lim_{x\to 0} \frac{x\sin x}{x^2 + 1}$

11 $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^{2/3}}$

12 $\displaystyle\lim_{x\to 0} \frac{1 - 2x^2 - 2\cos x + \cos^2 x}{x^2}$

13 $\displaystyle\lim_{t\to 0} \frac{4t^2 + 3t\sin t}{t^2}$

14 $\displaystyle\lim_{x\to 0} \frac{x\cos x - x^2}{2x}$

15 $\displaystyle\lim_{t\to 0} \frac{\cos t}{1 - \sin t}$

16 $\displaystyle\lim_{t\to 0} \frac{\sin t}{1 + \cos t}$

17 $\displaystyle\lim_{t\to 0} \frac{1 - \cos t}{\sin t}$

18 $\displaystyle\lim_{x\to 0} \frac{\sin \frac{1}{2}x}{x}$

19 $\displaystyle\lim_{x\to 0} \frac{x + \tan x}{\sin x}$

20 $\displaystyle\lim_{t\to 0} \frac{\sin^2 2t}{t^2}$

21 $\displaystyle\lim_{x\to 0} x\cot x$

22 $\displaystyle\lim_{x\to 0} \frac{\csc 2x}{\cot x}$

23 $\displaystyle\lim_{\alpha\to 0} \alpha^2 \csc^2 \alpha$

24 $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

25 $\displaystyle\lim_{v\to 0} \frac{\cos(v + \frac{1}{2}\pi)}{v}$

26 $\displaystyle\lim_{x\to 0} \frac{\sin^2 \frac{1}{2}x}{\sin x}$

Exer. 27–30: Establish the limit for all nonzero real numbers *a* and *b*.

27 $\displaystyle\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

28 $\displaystyle\lim_{x\to 0} \frac{1 - \cos ax}{bx} = 0$

29 $\displaystyle\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$

30 $\displaystyle\lim_{x\to 0} \frac{\cos ax}{\cos bx} = 1$

Exer. 31–58: Find the derivative.

31 $f(x) = 4\cos x$

32 $H(z) = 7\tan z$

33 $G(v) = 5v \csc v$ **34** $f(x) = 3x \sin x$

35 $k(t) = t - t^2 \cos t$ **36** $p(w) = w^2 + w \sin w$

37 $f(\theta) = \dfrac{\sin\theta}{\theta}$ **38** $g(\alpha) = \dfrac{1 - \cos\alpha}{\alpha}$

39 $g(t) = t^3 \sin t$ **40** $T(r) = r^2 \sec r$

41 $f(x) = 2x \cot x + x^2 \tan x$

42 $f(x) = 3x^2 \sec x - x^3 \tan x$

43 $h(z) = \dfrac{1 - \cos z}{1 + \cos z}$ **44** $R(w) = \dfrac{\cos w}{1 - \sin w}$

45 $g(x) = \dfrac{1}{\sin x \tan x}$ **46** $k(x) = \dfrac{1}{\cos x \cot x}$

47 $g(x) = (x + \csc x)\cot x$

48 $K(\theta) = (\sin\theta + \cos\theta)^2$

49 $p(x) = \sin x \cot x$ **50** $g(t) = \csc t \sin t$

51 $f(x) = \dfrac{\tan x}{1 + x^2}$ **52** $h(\theta) = \dfrac{1 + \sec\theta}{1 - \sec\theta}$

53 $k(v) = \dfrac{\csc v}{\sec v}$ **54** $q(t) = \sin t \sec t$

55 $g(x) = \sin(-x) + \cos(-x)$

56 $s(z) = \tan(-z) + \sec(-z)$

57 $H(\phi) = (\cot\phi + \csc\phi)(\tan\phi - \sin\phi)$

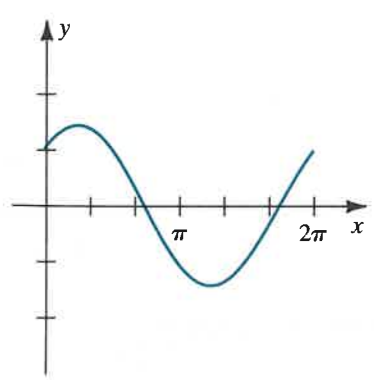
58 $f(x) = \dfrac{1 + \sec x}{\tan x + \sin x}$

Exer. 59–60: Find equations of the tangent line and the normal line to the graph of f at the point $(\pi/4, f(\pi/4))$.

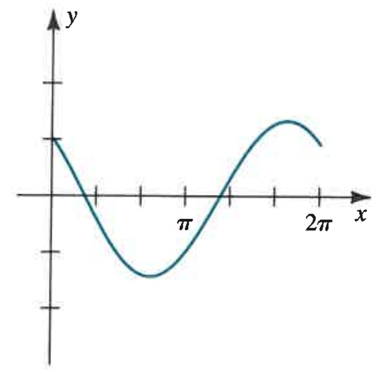
59 $f(x) = \sec x$ **60** $f(x) = \csc x + \cot x$

Exer. 61–64: Shown is a graph of the function f with restricted domain. Find the points at which the tangent line is horizontal.

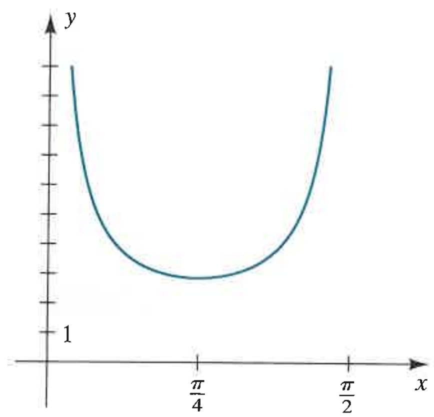
61 $f(x) = \cos x + \sin x, \quad 0 \le x \le 2\pi$



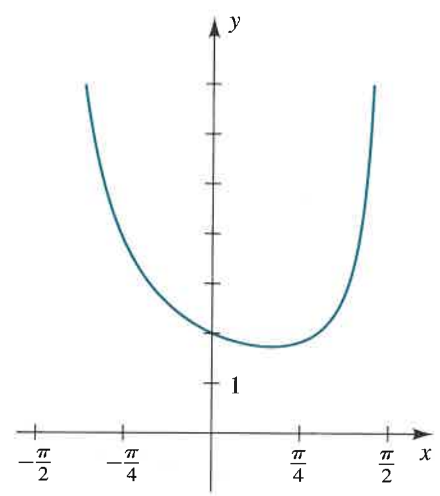
62 $f(x) = \cos x - \sin x, \quad 0 \le x \le 2\pi$



63 $f(x) = \csc x + \sec x, \quad 0 < x < \pi/2$



64 $f(x) = 2\sec x - \tan x, \quad -\pi/2 < x < \pi/2$



Exer. 65–66: (a) Find the x-coordinates of all points on the graph of f at which the tangent line is horizontal. (b) Find an equation of the tangent line to the graph of f at P.

65 $f(x) = x + 2\cos x; \quad P(0, f(0))$

66 $f(x) = x + \sin x; \quad P(\pi/2, f(\pi/2))$

67 If $y = 3 + 2\sin x$, find

(a) the x-coordinates of all points on the graph at which the tangent line is parallel to the line $y = \sqrt{2}x - 5$

(b) an equation of the tangent line to the graph at the point on the graph with x-coordinate $\pi/6$

68 If $y = 1 + 2\cos x$, find

(a) the x-coordinates of all points on the graph at which the tangent line is perpendicular to the line

$$y = \frac{1}{\sqrt{3}}x + 4$$

(b) an equation of the tangent line to the graph at the point where the graph crosses the y-axis

c **69** Graph $f(x) = |\sin^2 x - \cos x \sin(\frac{1}{2}\pi x)|$ on the interval $[0, 5]$ and estimate where f is not differentiable.

c **70** Graph $f(x) = 4/(16\sin 2x - x)$ on the interval $[0, 4]$ and estimate the x-coordinates of points at which the tangent line is horizontal.

Exer. 71–72: A point P moving on a coordinate line l has the given position function s. When is its velocity 0?

71 $s(t) = t + 2\cos t$ **72** $s(t) = t - \sqrt{2}\sin t$

Exer. 73–74: A point $P(x, y)$ is moving from left to right along the graph of the equation. Where is the rate of change of y with respect to x equal to the given number a?

73 $y = x^{3/2} + 2x; \quad a = 8$

74 $y = x^{5/3} - 10x; \quad a = 5$

75 **(a)** Find the first four derivatives of $f(x) = \cos x$.

(b) Find $f^{(99)}(x)$.

76 Find $f'''(x)$ if $f(x) = \cot x$.

77 Find $\dfrac{d^3y}{dx^3}$ if $y = \tan x$.

78 Find $\dfrac{d^3y}{dx^3}$ if $y = \sec x$.

Exer. 79–82: Prove each formula.

79 $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

80 $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

81 $\dfrac{d}{dx}(\sin 2x) = 2\cos 2x$ (*Hint:* $\sin 2x = 2\sin x \cos x$.)

82 $\dfrac{d}{dx}(\cos 2x) = -2\sin 2x$ (*Hint:* $\cos 2x = 1 - 2\sin^2 x$.)

83 Use Theorem (2.22) and the addition formula for the sine to show that the sine function is continuous at $x = a$. (*Hint:* Show $\lim_{h\to 0} \sin(a + h) = \sin a$.)

84 Work Exercise 83 for the cosine rather than the sine.

2.5 THE CHAIN RULE

In this section, we will study perhaps the single most powerful tool for differentiation: the chain rule for differentiating composite functions. The rules for derivatives obtained in previous sections are limited in scope because they can be used only for sums, differences, products, and quotients that involve x^n, $\sin x$, $\cos x$, $\tan x$, and so on. There is no rule that can be applied *directly* to expressions such as $\sin 2x$ or $\sqrt{x^2 + 1}$. Note that

$$\frac{d}{dx}(\sin 2x) \ne \cos 2x,$$

for if we use the identity $\sin 2x = 2\sin x \cos x$ and apply the product rule, as in Example 9 of Section 2.4, we obtain

$$\frac{d}{dx}(\sin 2x) = 2\cos 2x.$$

Since these manipulations are rather cumbersome, we seek a more direct method of finding the derivative of $y = \sin 2x$. The key is to regard y as a composite function of x. Thus, for functions f and g,

$$\text{if} \quad y = f(u) \quad \text{and} \quad u = g(x), \quad \text{then} \quad y = f(g(x)),$$

provided $g(x)$ is in the domain of f. The function given by $y = f(g(x))$ is the composite function $f \circ g$ defined on p. 29. The functions f and g are called the *components* of the composition. Note that $y = \sin 2x$ may be expressed in this way because

$$\text{if} \quad y = \sin u \quad \text{and} \quad u = 2x, \quad \text{then} \quad y = \sin 2x.$$

If we can find a general rule for differentiating $y = f(g(x))$, then, as a special case, we may apply it to $y = \sin 2x$ and, in fact, to $y = \sin g(x)$ for any differentiable function g.

To get an idea of the type of rule to expect, let us consider a composite function that we can easily differentiate by changing its form. If $y = (x^3 - 1)^2$, then we can represent y in a composite function form by letting $u = g(x) = x^3 - 1$ and $f(u) = u^2$ so that $y = f(g(x))$. We can find the derivative of y with respect to x by expanding the original expression for y to obtain

$$y = u^2 = (x^3 - 1)^2 = x^6 - 2x^3 + 1,$$

and then differentiating, term by term, to obtain

$$y' = [f(g(x))]' = \frac{dy}{dx} = 6x^5 - 6x^2,$$

which can be written as

$$6x^5 - 6x^2 = 6x^2(x^3 - 1) = 2(x^3 - 1)(3x^2).$$

Note here that

$$2(x^3 - 1) = 2u = \frac{dy}{du} = f'(g(x)) \quad \text{and} \quad 3x^2 = \frac{du}{dx} = g'(x),$$

so we can write the derivative of $f(g(x))$ in the following equivalent forms:

$$f'(u) = f'(u)g'(x) \quad \text{or} \quad [f(g(x))]' = f'(g(x))g'(x) \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

These results suggest a rule indicating that the derivative of the composite function is a product of derivatives of the component functions.

Note too that this rule also holds true for the derivative $y = \sin 2x$, for if we write

$$y = \sin u \quad \text{and} \quad u = 2x$$

and use the suggested rule, we obtain

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(2) = 2\cos u = 2\cos 2x.$$

The fact that the same rule gives the correct answer for both of these examples of composite functions is no accident. The *chain rule* for differentiation states that the derivative of a composite function is always the product of the derivatives of the component functions, provided they exist.

Chain Rule 2.26

> If $y = f(u)$, $u = g(x)$, and the derivatives dy/du and du/dx both exist, then the composite function defined by $y = f(g(x))$ has a derivative given by
>
> $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)g'(x) = f'(g(x))g'(x).$$

PARTIAL PROOF Using Definition (2.5), we must show that

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = f'(g(x))g'(x).$$

If h is close to 0 and $g(x+h) \neq g(x)$, we can write the left-hand side of the equation as

$$\lim_{h\to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}\right].$$

If each factor has a limit, then by Theorem (1.8)(ii), the above limit can be written as

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}.$$

By Definition (2.5),

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x),$$

so it remains to be shown that

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} = f'(g(x)).$$

To see why this last result is true, first note that since g is differentiable, it is continuous. Thus, as $h \to 0$, $g(x+h) \to g(x)$. Note too that Alternative Definition (2.6) can be written in the form

$$f'(a) = \lim_{t\to a} \frac{f(t) - f(a)}{t - a}.$$

If we now let $a = g(x)$ and $t = g(x+h)$,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} &= \lim_{h\to 0} \frac{f(t) - f(a)}{t - a} \\
&= \lim_{g(x+h)\to g(x)} \frac{f(t) - f(a)}{t - a} \\
&= \lim_{t\to a} \frac{f(t) - f(a)}{t - a} \\
&= f'(a) = f'(g(x)),
\end{aligned}$$

which is what we needed to prove. ■

In many applications of the chain rule, the function $u = g(x)$ has the property that $g(x+h) \neq g(x)$ for values of h sufficiently close to 0, a property that we assumed at the beginning of the proof. If g does not satisfy this property, then every open interval containing x contains a number $x+h$ for which $g(x+h) = g(x)$, so that $g(x+h) - g(x) = 0$. In such cases, our proof is invalid, since the expression $g(x+h) - g(x)$ occurs in the denominator. To construct a proof that takes functions of this type into account requires additional techniques. A complete proof of the chain rule is given in Appendix I.

EXAMPLE ▪ 1 Find $\dfrac{dy}{dx}$ if $y = \sqrt{u}$ and $u = x^2 + 1$.

SOLUTION If we substitute x^2+1 for u in $y = \sqrt{u} = u^{1/2}$, we obtain

$$y = \sqrt{x^2+1} = (x^2+1)^{1/2}.$$

We cannot find dy/dx by using previous differentiation formulas; however, using the chain rule (2.26), we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{2}u^{-1/2}\right)2x = \frac{x}{\sqrt{u}}$$

and hence $$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}.$$

In Example 1, the composite function was given by a power of x^2+1. Since powers of functions occur frequently in calculus, it will save us time to state a general differentiation rule that can be applied to such special cases. In the following, we assume that n is any rational number, g is a differentiable function, and zero denominators do not occur. We shall see later that the rule can be used for any *real* number n.

***Power Rule for Functions* 2.27**

If $y = u^n$ and $u = g(x)$, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx},$$

or, equivalently, $$\frac{d}{dx}(g(x))^n = n[g(x)]^{n-1}\frac{d}{dx}(g(x)).$$

PROOF By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}\frac{d}{dx}(g(x)).$$ ▪

Note that if $u = x$, then $du/dx = 1$ and (2.27) reduces to (2.10).

EXAMPLE ▪ 2 Find $f'(x)$ if $f(x) = (x^5 - 4x + 8)^7$.

SOLUTION Using the power rule (2.27) with $u = x^5 - 4x + 8$ and $n = 7$ yields

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^5-4x+8)^7 \\ &= 7(x^5-4x+8)^6\frac{d}{dx}(x^5-4x+8) \\ &= 7(x^5-4x+8)^6(5x^4-4). \end{aligned}$$

EXAMPLE ▪ 3 Find $\dfrac{dy}{dx}$ if $y = \dfrac{1}{(4x^2+6x-7)^3}$.

SOLUTION Writing $y = (4x^2+6x-7)^{-3}$ and using the power rule with $u = 4x^2+6x-7$ and $n = -3$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(4x^2+6x-7)^{-3} \\ &= -3(4x^2+6x-7)^{-4}\frac{d}{dx}(4x^2+6x-7) \\ &= -3(4x^2+6x-7)^{-4}(8x+6) \\ &= \frac{-6(4x+3)}{(4x^2+6x-7)^4}. \end{aligned}$$

EXAMPLE ▪ 4 Find $f'(x)$ if $f(x) = \sqrt[3]{5x^2-x+4}$.

SOLUTION Writing $f(x) = (5x^2-x+4)^{1/3}$ and using the power rule with $u = 5x^2-x+4$ and $n = \frac{1}{3}$, we obtain

$$\begin{aligned} f'(x) &= \frac{1}{3}(5x^2-x+4)^{-2/3}\frac{d}{dx}(5x^2-x+4) \\ &= \left(\frac{1}{3}\right)\frac{1}{(5x^2-x+4)^{2/3}}(10x-1) = \frac{10x-1}{3\sqrt[3]{(5x^2-x+4)^2}}. \end{aligned}$$

EXAMPLE ▪ 5 Find $F'(z)$ if $F(z) = (2z+5)^3(3z-1)^4$.

SOLUTION Using first the product rule, second the power rule, and then factoring the result gives us

$$\begin{aligned} F'(z) &= (2z+5)^3\frac{d}{dz}(3z-1)^4 + (3z-1)^4\frac{d}{dz}(2z+5)^3 \\ &= (2z+5)^3\cdot 4(3z-1)^3(3) + (3z-1)^4\cdot 3(2z+5)^2(2) \\ &= 6(2z+5)^2(3z-1)^3[2(2z+5)+(3z-1)] \\ &= 6(2z+5)^2(3z-1)^3(7z+9). \end{aligned}$$

EXAMPLE 6 Find y' if $y = (3x+1)^6\sqrt{2x-5}$.

SOLUTION Since $y = (3x+1)^6(2x-5)^{1/2}$, we have, by the product and power rules,

$$\begin{aligned} y' &= (3x+1)^6\tfrac{1}{2}(2x-5)^{-1/2}(2) + (2x-5)^{1/2}6(3x+1)^5(3) \\ &= \frac{(3x+1)^6}{\sqrt{2x-5}} + 18(3x+1)^5\sqrt{2x-5} \\ &= \frac{(3x+1)^6 + 18(3x+1)^5(2x-5)}{\sqrt{2x-5}} = \frac{(3x+1)^5(39x-89)}{\sqrt{2x-5}}. \end{aligned}$$

The next example is of interest because it illustrates the fact that after the power rule is applied to $[g(x)]^r$, it may be necessary to apply it again in order to find $g'(x)$.

EXAMPLE 7 Find $f'(x)$ if $f(x) = (7x + \sqrt{x^2+6})^4$.

SOLUTION Applying the power rule yields

$$\begin{aligned} f'(x) &= 4(7x+\sqrt{x^2+6})^3\frac{d}{dx}(7x+\sqrt{x^2+6}) \\ &= 4(7x+\sqrt{x^2+6})^3\left[\frac{d}{dx}(7x) + \frac{d}{dx}(\sqrt{x^2+6})\right]. \end{aligned}$$

Again applying the power rule, we have

$$\begin{aligned} \frac{d}{dx}(\sqrt{x^2+6}) &= \frac{d}{dx}(x^2+6)^{1/2} = \frac{1}{2}(x^2+6)^{-1/2}\frac{d}{dx}(x^2+6) \\ &= \frac{1}{2\sqrt{x^2+6}}(2x) = \frac{x}{\sqrt{x^2+6}}. \end{aligned}$$

Therefore,

$$f'(x) = 4(7x+\sqrt{x^2+6})^3\left(7 + \frac{x}{\sqrt{x^2+6}}\right).$$

As another application of the chain rule, we can prove the following.

Theorem 2.28

If $u = g(x)$ and g is differentiable, then

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx} \qquad \frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \sec u\tan u\frac{du}{dx} \qquad \frac{d}{dx}(\csc u) = -\csc u\cot u\frac{du}{dx}$$

PROOF If we let $y = \sin u$, then, by (2.25),

$$\frac{dy}{du} = \cos u.$$

Applying the chain rule (2.26) yields

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos u\frac{du}{dx}.$$

The remaining formulas may be obtained in similar fashion. ■

Note that Theorem (2.25) is the special case of Theorem (2.28) in which $u = x$.

EXAMPLE 8 If $y = \cos(5x^3)$, find dy/dx and d^2y/dx^2.

SOLUTION Using the formula for $(d/dx)(\cos u)$ in Theorem (2.28) with $u = 5x^3$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\cos(5x^3)) \\ &= -\sin(5x^3)\frac{d}{dx}(5x^3) \\ &= -\sin(5x^3)(15x^2) \\ &= -15x^2\sin(5x^3). \end{aligned}$$

To find d^2y/dx^2, we differentiate $dy/dx = -15x^2\sin(5x^3)$. Using the product rule and Theorem (2.28) gives us

$$\begin{aligned} \frac{d^2y}{dx^2} &= -15x^2\frac{d}{dx}(\sin(5x^3)) + \sin(5x^3)\frac{d}{dx}(-15x^2) \\ &= -15x^2\cos(5x^3)\frac{d}{dx}(5x^3) + \sin(5x^3)(-30x) \\ &= -15x^2\cos(5x^3)(15x^2) - 30x\sin(5x^3) \\ &= -225x^4\cos(5x^3) - 30x\sin(5x^3). \end{aligned}$$

EXAMPLE 9 Find $f'(x)$ if $f(x) = \tan^3 4x$.

SOLUTION First note that $f(x) = \tan^3 4x = (\tan 4x)^3$. Applying the power rule with $u = \tan 4x$ and $n = 3$ yields

$$f'(x) = 3(\tan 4x)^2\frac{d}{dx}(\tan 4x) = (3\tan^2 4x)\frac{d}{dx}(\tan 4x).$$

Next, by Theorem (2.28),

$$\frac{d}{dx}(\tan 4x) = \sec^2 4x\frac{d}{dx}(4x) = (\sec^2 4x)(4) = 4\sec^2 4x.$$

Thus

$$f'(x) = (3\tan^2 4x)(4\sec^2 4x) = 12\tan^2 4x\sec^2 4x.$$

EXAMPLE ▪ 10 If $f(x) = \sin(\sin x)$, find

(a) $f'(x)$

(b) the slope of the tangent line to the graph of f at $P(1, f(1))$.

SOLUTION

(a) Let $u = \sin x$ and apply Theorem (2.28) to obtain

$$f'(x) = \frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} = \cos(\sin x)\cos x.$$

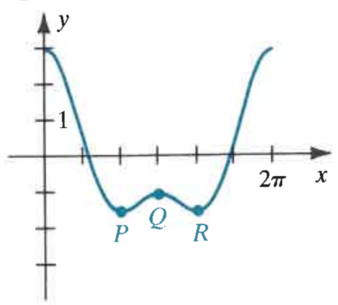
(b) By Definition (2.1), the slope of the tangent line at P is $f'(1)$, which by part (a) is

$$\begin{aligned} f'(1) &= \cos(\sin 1)\cos(1) \\ &\approx \cos(0.84147)\cos(1) \\ &\approx (0.66637)(0.54030) \approx 0.36004. \end{aligned}$$

Compare this result with the numerical approximation discussed in Example 5 of Section 2.1.

EXAMPLE ▪ 11 A graph of $y = \cos 2x + 2\cos x$ for $0 \le x \le 2\pi$ is shown in Figure 2.31. Find the points at which the tangent line is horizontal.

Figure 2.31



SOLUTION Differentiating, we obtain

$$\begin{aligned} \frac{dy}{dx} &= -\sin 2x \frac{d}{dx}(2x) + 2(-\sin x) \\ &= -2\sin 2x - 2\sin x. \end{aligned}$$

The tangent line is horizontal if its slope dy/dx is 0 — that is, if

$$-2\sin 2x - 2\sin x = 0, \quad \text{or} \quad \sin 2x + \sin x = 0.$$

Using the double-angle formula, $\sin 2x = 2\sin x\cos x$, gives us

$$2\sin x\cos x + \sin x = 0,$$

or, equivalently,

$$\sin x(2\cos x + 1) = 0.$$

Thus, either

$$\sin x = 0 \quad \text{or} \quad 2\cos x + 1 = 0;$$

that is,

$$\sin x = 0 \quad \text{or} \quad \cos x = -\tfrac{1}{2}.$$

The solutions of these equations for $0 \le x \le 2\pi$ are

$$0, \quad \pi, \quad 2\pi, \quad 2\pi/3, \quad 4\pi/3.$$

The two solutions $x = 0$ and $x = 2\pi$ tell us that there are horizontal tangent lines at the endpoints of the interval $[0, 2\pi]$. The remaining solutions $2\pi/3$, π, and $4\pi/3$ are the x-coordinates of the points P, Q, and R shown in Figure 2.31. Using $y = \cos 2x + 2\cos x$, we see that horizontal tangent

lines occur at the points

$$(0, 3), \quad (2\pi/3, -1.5), \quad (\pi, -1), \quad (4\pi/3, -1.5), \quad (2\pi, 3).$$

If only approximate solutions are desired, then, to the nearest tenth, we obtain

$$(0, 3), \quad (2.1, -1.5), \quad (3.1, -1), \quad (4.2, -1.5), \quad (6.3, 3).$$

CAUTION Failure to use the chain rule properly is a common error that can be avoided if you write the function to be differentiated as a composition of simpler functions.

The chain rule is vitally important in calculus. It provides the power to differentiate complicated expressions involving many layers of composition. In the vast majority of calculus exercises or in applications of calculus to real-world problems, differentiation plays a critical role. In virtually every instance, the functions you will encounter will be compositions of simpler functions, and you will need to use the chain rule to complete the differentiation correctly.

EXERCISES 2.5

Exer. 1–6: Use the chain rule to find dy/dx, and express the answer in terms of x.

1 $y = u^2$; $u = x^3 - 4$

2 $y = \sqrt[3]{u}$; $u = x^2 + 5x$

3 $y = 1/u$; $u = \sqrt{3x - 2}$

4 $y = 3u^2 + 2u$; $u = 4x$

5 $y = \tan 3u$; $u = x^2$

6 $y = u \sin u$; $u = x^3$

Exer. 7–62: Find the derivative.

7 $f(x) = (x^2 - 3x + 8)^3$

8 $f(x) = (4x^3 + 2x^2 - x - 3)^2$

9 $g(x) = (8x - 7)^{-5}$

10 $k(x) = (5x^2 - 2x + 1)^{-3}$

11 $f(x) = \dfrac{x}{(x^2 - 1)^4}$

12 $g(x) = \dfrac{x^4 - 3x^2 + 1}{(2x + 3)^4}$

13 $f(x) = (8x^3 - 2x^2 + x - 7)^5$

14 $g(w) = (w^4 - 8w^2 + 15)^4$

15 $F(v) = (17v - 5)^{1000}$

16 $s(t) = (4t^5 - 3t^3 + 2t)^{-2}$

17 $N(x) = (6x - 7)^3(8x^2 + 9)^2$

18 $f(w) = (2w^2 - 3w + 1)(3w + 2)^4$

19 $g(z) = \left(z^2 - \dfrac{1}{z^2}\right)^6$

20 $S(t) = \left(\dfrac{3t + 4}{6t - 7}\right)^3$

21 $k(r) = \sqrt[3]{8r^3 + 27}$

22 $h(z) = (2z^2 - 9z + 8)^{-2/3}$

23 $F(v) = \dfrac{5}{\sqrt[5]{v^5 - 32}}$

24 $k(s) = \dfrac{1}{\sqrt{3s - 4}}$

25 $g(w) = \dfrac{w^2 - 4w + 3}{w^{3/2}}$

26 $K(x) = \sqrt{4x^2 + 2x + 3}$

27 $H(x) = \dfrac{2x + 3}{\sqrt{4x^2 + 9}}$

28 $f(x) = (7x + \sqrt{x^2 + 3})^6$

29 $k(x) = \sin(x^2 + 2)$

30 $f(t) = \cos(4 - 3t)$

31 $H(\theta) = \cos^5 3\theta$ **32** $g(x) = \sin^4(x^3)$

33 $g(z) = \sec(2z+1)^2$ **34** $k(z) = \csc(z^2+4)$

35 $H(s) = \cot(s^3 - 2s)$ **36** $f(x) = \tan(2x^2+3)$

37 $f(x) = \cos(3x^2) + \cos^2 3x$

38 $g(w) = \tan^3 6w$

39 $F(\phi) = \csc^2 2\phi$ **40** $M(x) = \sec(1/x^2)$

41 $K(z) = z^2 \cot 5z$ **42** $G(s) = s\csc(s^2)$

43 $h(\theta) = \tan^2\theta \sec^3\theta$ **44** $H(u) = u^2 \sec^3 4u$

45 $N(x) = (\sin 5x - \cos 5x)^5$

46 $p(v) = \sin 4v \csc 4v$

47 $T(w) = \cot^3(3w+1)$ **48** $g(r) = \sin(2r+3)^4$

49 $h(w) = \dfrac{\cos 4w}{1 - \sin 4w}$ **50** $f(x) = \dfrac{\sec 2x}{1 + \tan 2x}$

51 $f(x) = \tan^3 2x - \sec^3 2x$

52 $h(\phi) = (\tan 2\phi - \sec 2\phi)^3$

53 $f(x) = \sin\sqrt{x} + \sqrt{\sin x}$

54 $f(x) = \tan\sqrt[3]{5-6x}$

55 $k(\theta) = \cos^2\sqrt{3-8\theta}$

56 $r(t) = \sqrt{\sin 2t - \cos 2t}$

57 $g(x) = \sqrt{x^2+1}\tan\sqrt{x^2+1}$

58 $h(\phi) = \dfrac{\cot 4\phi}{\sqrt{\phi^2+4}}$ **59** $M(x) = \sec\sqrt{4x+1}$

60 $F(s) = \sqrt{\csc 2s}$ **61** $h(x) = \sqrt{4 + \csc^2 3x}$

62 $f(t) = \sin^2 2t\sqrt{\cos 2t}$

Exer. 63–68: (a) Find equations of the tangent line and the normal line to the graph of the equation at *P*. (b) Find the *x*-coordinates on the graph at which the tangent line is horizontal.

63 $y = (4x^2 - 8x + 3)^4$; $P(2, 81)$

64 $y = (2x-1)^{10}$; $P(1, 1)$

65 $y = \left(x + \dfrac{1}{x}\right)^5$; $P(1, 32)$

66 $y = \sqrt{2x^2+1}$; $P(-1, \sqrt{3})$

67 $y = 3x + \sin 3x$; $P(0, 0)$

68 $y = x + \cos 2x$; $P(0, 1)$

Exer. 69–74: Find the first and second derivatives.

69 $g(z) = \sqrt{3z+1}$ **70** $k(s) = (s^2+4)^{2/3}$

71 $k(r) = (4r+7)^5$ **72** $f(x) = \sqrt[5]{10x+7}$

73 $f(x) = \sin^3 x$ **74** $G(t) = \sec^2 4t$

75 If an object of mass m has velocity v, then its *kinetic energy* K is given by $K = \frac{1}{2}mv^2$. If v is a function of time t, use the chain rule to find a formula for dK/dt.

76 As a spherical weather balloon is being inflated, its radius r is a function of time t. If V is the volume of the balloon, use the chain rule to find a formula for dV/dt.

77 When a space shuttle is launched into space, an astronaut's body weight decreases until a state of weightlessness is achieved. The weight W of a 150-lb astronaut at an altitude of x kilometers above sea level is given by

$$W = 150\left(\frac{6400}{6400 + x}\right)^2.$$

If the space shuttle is moving away from the earth's surface at the rate of 6 km/sec, at what rate is W decreasing when $x = 1000$ km?

78 The length–weight relationship for Pacific halibut is well described by the formula $W = 10.375L^3$, where L is the length (in meters) and W is the weight (in kilograms). The rate of growth in length dL/dt is given by $0.18(2 - L)$, where t is time (in years).

(a) Find a formula for the rate of growth in weight dW/dt in terms of L.

(b) Use the formula in part (a) to estimate the rate of growth in weight of a halibut weighing 20 kg.

79 If $k(x) = f(g(x))$ and if $f(2) = -4$, $g(2) = 2$, $f'(2) = 3$, and $g'(2) = 5$, find $k(2)$ and $k'(2)$.

80 Let p, q, and r be functions such that $p(z) = q(r(z))$. If $r(3) = 3$, $q(3) = -2$, $r'(3) = 4$, and $q'(3) = 6$, find $p(3)$ and $p'(3)$.

81 If $f(t) = g(h(t))$ and if $f(4) = 3$, $g(4) = 3$, $h(4) = 4$, $f'(4) = 2$, and $g'(4) = -5$, find $h'(4)$.

82 If $u(x) = v(w(x))$ and if $v(0) = -1$, $w(0) = 0$, $u(0) = -1$, $v'(0) = -3$, and $u'(0) = 2$, find $w'(0)$.

c **83** Let $h = f \circ g$ be a differentiable function. The following tables list some values of f and g. Use Exercise 53 of Section 2.2 to approximate $h'(1.12)$.

x	2.2210	2.2320	2.2430
$f(x)$	4.9328	4.9818	5.0310

x	1.1100	1.1200	1.1300
$g(x)$	2.2210	2.2320	2.2430

c **84** Let $h = f \circ g$ be a differentiable function. The following tables list some values of f and g. Use Exercise 53 of Section 2.2 to approximate $h'(-2)$.

x	−8.48092	−8.46000	−8.43908
$f(x)$	−2.03930	−2.03762	−2.03594

x	−2.00400	−2.00000	−1.99600
$g(x)$	−8.48092	−8.46000	−8.43908

85 Let f be differentiable. Use the chain rule to prove that

(a) if f is even, then f' is odd

(b) if f is odd, then f' is even

Use polynomial functions to give examples of parts (a) and (b).

86 Use the chain rule, the derivative formula for $(d/dx)(\sin u)$, together with the identities

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

and

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

to obtain the formula $(d/dx)(\cos x)$.

87 Pinnipeds are a suborder of aquatic carnivorous mammals, such as seals and walruses, whose limbs are modified into flippers. The length–weight relationship during fetal growth is well described by the formula $W = (6 \times 10^{-5})L^{2.74}$, where L is the length (in centimeters) and W is the weight (in kilograms).

(a) Use the chain rule to find a formula for the rate of growth in weight with respect to time t.

(b) If the weight of a seal is 0.5 kg and is changing at a rate of 0.4 kg per month, how fast is the length changing?

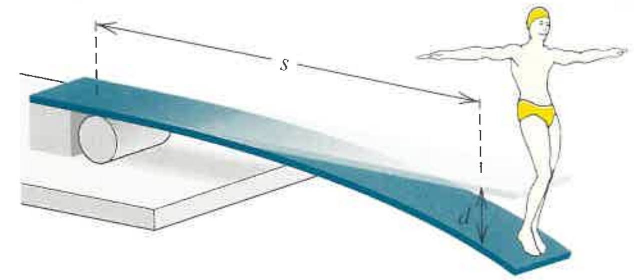
88 The formula for the adiabatic expansion of air is $pv^{1.4} = c$, where p is the pressure, v is the volume, and c is a constant. Find a formula for the rate of change of pressure with respect to volume.

89 The deflection d of a diving board at a position s feet from the stationary end is given by

$$d = cs^2(3L - s) \quad \text{for} \quad 0 \le s \le L,$$

where L is the length of the board and c is a positive constant that depends on the weight of the diver and on the physical properties of the board. If the board is 10 ft long, find the rate of change of d with respect to s.

Exercise 89

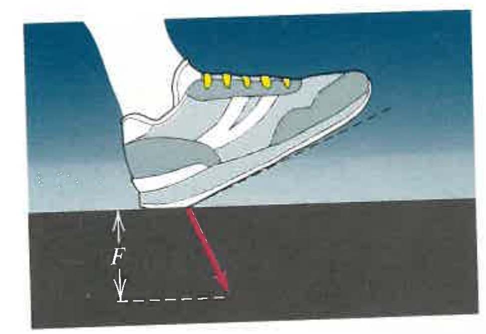


90 When an individual is walking, the magnitude F of the vertical force of one foot on the ground (see figure) can be described by

$$F = A(\cos bt - a\cos 3bt),$$

where t is the time (in seconds), $A > 0$, $b > 0$, and $0 < a < 1$. Use the chain rule to find the rate of change of F with respect to time t.

Exercise 90



91 A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. In the figure, the line through A and D bisects θ and is perpendicular to the line through B and C.

(a) Show that the length L of the artery from A to B is given by

$$L = a + \frac{b}{2}\tan\frac{\theta}{4}.$$

(b) Use the chain rule to find the rate of change of L with respect to θ.

Exercise 91