Middlebury College Document Delivery

ILLiad TN: 658103

Journal Title: Calculus of a single variable /

ISSN:

Volume:
Issue:
Month/Year:
Pages: 206-245

Article Author:

Article Title: Chapter 2.6-2.9: The Derivative (Part 2)

Imprint:

Deliver to Middlebury College Patron:

-Save to C:/Ariel Scan as a PDF
-Run Odyssey Helper
-Switch to Process Type: Document Delivery
-Process
-Switch back to Lending before closing.

Call #: **Laura's Desk**

Location:

Item #:

Save a copy to Desktop Folder as well:
"Olinick Scan"

2.6 IMPLICIT DIFFERENTIATION

Our objective in this section is to find derivatives of functions that are given in implicit form. If we have an equation such as

$$y = 2x^2 - 3,$$

we sometimes say that y is an **explicit function** of x, since we can write

$$y = f(x) \quad \text{with} \quad f(x) = 2x^2 - 3.$$

The equation

$$4x^2 - 2y = 6$$

determines the same function f, since solving for y gives us

$$-2y = -4x^2 + 6, \quad \text{or} \quad y = 2x^2 - 3.$$

For the case $4x^2 - 2y = 6$, we say that y (or f) is an **implicit function** of x, or that f is determined *implicitly* by the equation. If we substitute $f(x)$ for y in $4x^2 - 2y = 6$, we obtain

$$\begin{aligned} 4x^2 - 2f(x) &= 6 \\ 4x^2 - 2(2x^2 - 3) &= 6 \\ 4x^2 - 4x^2 + 6 &= 6. \end{aligned}$$

The last equation is an identity, since it is true for every x in the domain of f. This is a characteristic of every function f determined implicitly by an equation in x and y; that is, *f is implicit if and only if substitution of $f(x)$ for y leads to an identity.* Since $(x, f(x))$ is a point on the graph of f, the last statement implies that *the graph of the implicit function f coincides with a portion (or all) of the graph of the equation.*

In the next example, we show that an equation in x and y may determine more than one implicit function.

EXAMPLE ■ 1 How many different functions are determined implicitly by the equation $x^2 + y^2 = 1$?

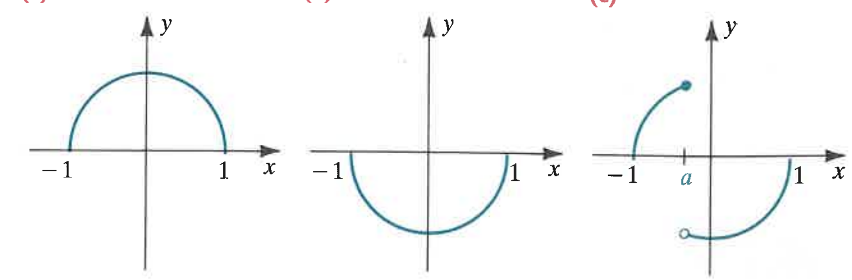
SOLUTION The graph of $x^2 + y^2 = 1$ is the unit circle with center at the origin. Solving the equation for y in terms of x, we obtain

$$y = \pm\sqrt{1 - x^2}.$$

Two functions f and g determined implicitly by the equation are given by

$$f(x) = \sqrt{1 - x^2} \quad \text{and} \quad g(x) = -\sqrt{1 - x^2}.$$

Figure 2.32



The graphs of f and g are the upper and lower halves, respectively, of the unit circle (see Figure 2.32a and b). To find other implicit functions, we may let a be any number between -1 and 1 and then define the function k by

$$k(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } -1 \le x \le a \\ -\sqrt{1 - x^2} & \text{if } a < x \le 1 \end{cases}$$

The graph of k is sketched in Figure 2.32(c). Note that there is a jump discontinuity at $x = a$. The function k is determined implicitly by the equation $x^2 + y^2 = 1$, since

$$x^2 + (k(x))^2 = 1$$

for every x in the domain of k. By letting a take on different values, we can obtain as many implicit functions as desired. Many other functions are determined implicitly by $x^2 + y^2 = 1$, and the graph of each is a portion of the graph of the equation.

If the equation

$$y^4 + 3y - 4x^3 = 5x + 1$$

determines an implicit function f, then

$$(f(x))^4 + 3(f(x)) - 4x^3 = 5x + 1$$

for every x in the domain of f; however, there is no obvious way to solve for y in terms of x to obtain $f(x)$. It is possible to state conditions under which an implicit function exists and is differentiable at numbers in its domain; however, the proof requires advanced methods and hence is omitted. In the examples that follow, we will assume that a given equation in x and y determines a differentiable function f such that if $f(x)$ is substituted for y, the equation is an identity for every x in the domain of f. The derivative of f may then be found by the method of **implicit differentiation,** in which we differentiate each term of the equation with respect to x. In using implicit differentiation, it is often necessary to consider $(d/dx)(y^n)$ for some unknown function y of x, say, $y = f(x)$. By the power rule (2.27) with $y = u$, we can write $(d/dx)(y^n)$ in any of the following forms:

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx} = ny^{n-1}y'$$

Since the dependent variable y represents the expression $f(x)$, it is *essential* to multiply ny^{n-1} by the derivative y' when we differentiate y with respect to x. Thus,

$$\frac{d}{dx}(y^n) \neq ny^{n-1}, \quad \text{unless} \quad y = x.$$

EXAMPLE ■ 2 Assuming that the equation $y^4 + 3y - 4x^3 = 5x + 1$ determines, implicitly, a differentiable function f such that $y = f(x)$, find its derivative.

SOLUTION We regard y as a symbol that denotes $f(x)$ and consider the equation as an identity for every x in the domain of f. Since derivatives of both sides are equal, we obtain the following:

$$\frac{d}{dx}(y^4 + 3y - 4x^3) = \frac{d}{dx}(5x + 1)$$

$$\frac{d}{dx}(y^4) + \frac{d}{dx}(3y) - \frac{d}{dx}(4x^3) = \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$4y^3y' + 3y' - 12x^2 = 5 + 0$$

We now solve for y', obtaining

$$(4y^3 + 3)y' = 12x^2 + 5,$$

or

$$y' = \frac{12x^2 + 5}{4y^3 + 3},$$

provided $4y^3 + 3 \neq 0$. Thus, if $y = f(x)$, then

$$f'(x) = \frac{12x^2 + 5}{4(f(x))^3 + 3}.$$

The last two equations in the solution of Example 2 bring out a disadvantage of using the method of implicit differentiation: The formula for y' (or $f'(x)$) may contain the expression y (or $f(x)$). However, these formulas can still be very useful in analyzing f and its graph.

In the next example, we use implicit differentiation to find the slope of the tangent line at a point $P(a, b)$ on the graph of an equation. In problems of this type, we shall assume that the equation determines an implicit function f whose graph coincides with the graph of the equation for every x in some open interval containing a. Note that since $P(a, b)$ is a point on the graph, the ordered pair (a, b) must be a solution of the equation.

EXAMPLE ■ 3 Find the slope of the tangent line to the graph of

$$y^4 + 3y - 4x^3 = 5x + 1$$

at the point $P(1, -2)$.

SOLUTION The point $P(1, -2)$ is on the graph, since substituting $x = 1$ and $y = -2$ gives us

$$(-2)^4 + 3(-2) - 4(1)^3 = 5(1) + 1, \quad \text{or} \quad 6 = 6.$$

The slope of the tangent line at $P(1, -2)$ is the value of the derivative y' when $x = 1$ and $y = -2$. The given equation is the same as that in Example 2, where we found that $y' = (12x^2 + 5)/(4y^3 + 3)$. Substituting 1 for x and -2 for y gives us the following, where $y']_{(1,-2)}$ denotes the value of y' when $x = 1$ and $y = -2$:

$$y']_{(1,-2)} = \frac{12(1)^2 + 5}{4(-2)^3 + 3} = -\frac{17}{29}.$$

EXAMPLE ■ 4 If $y = f(x)$, where f is determined implicitly by the equation $x^2 + y^2 = 1$, find y'.

SOLUTION In Example 1, we showed that there is an unlimited number of implicit functions determined by $x^2 + y^2 = 1$. As in Example 2, we differentiate both sides of the equation with respect to x, obtaining

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + 2yy' = 0$$

$$yy' = -x$$

$$y' = -\frac{x}{y} \quad \text{if} \quad y \neq 0.$$

The method of implicit differentiation provides the derivative of *any* differentiable function determined by an equation in two variables. For example, the equation $x^2 + y^2 = 1$ determines many implicit functions (see Example 1). From Example 4, the slope of the tangent line at the point (x, y) on any of the graphs in Figure 2.32 is given by $y' = -x/y$, provided the derivative exists.

EXAMPLE ■ 5 Find y' if $4xy^3 - x^2y + x^3 - 5x + 6 = 0$.

SOLUTION Differentiating both sides of the equation with respect to x yields

$$\frac{d}{dx}(4xy^3) - \frac{d}{dx}(x^2y) + \frac{d}{dx}(x^3) - \frac{d}{dx}(5x) + \frac{d}{dx}(6) = \frac{d}{dx}(0).$$

Since y denotes $f(x)$ for some function f, the product rule must be applied to $(d/dx)(4xy^3)$ and $(d/dx)(x^2y)$. Thus,

$$\begin{aligned}\frac{d}{dx}(4xy^3) &= 4x\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(4x)\\ &= 4x(3y^2y') + y^3(4)\\ &= 12xy^2y' + 4y^3\end{aligned}$$

and $$\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2) = x^2y' + y(2x).$$

Substituting these expressions in the first equation of the solution and differentiating the other terms leads to

$$(12xy^2y' + 4y^3) - (x^2y' + 2xy) + 3x^2 - 5 = 0.$$

Collecting the terms containing y' and transposing the remaining terms to the right-hand side of the equation gives us

$$(12xy^2 - x^2)y' = 5 - 3x^2 + 2xy - 4y^3.$$

Consequently, $$y' = \frac{5 - 3x^2 + 2xy - 4y^3}{12xy^2 - x^2},$$

provided $12xy^2 - x^2 \neq 0$.

EXAMPLE ■ 6 Find y' if $y = x^2 \sin y$.

SOLUTION Differentiating both sides of the equation with respect to x and using the product rule, we obtain

$$\frac{dy}{dx} = (x^2)\frac{d}{dx}(\sin y) + \sin y\frac{d}{dx}(x^2).$$

Since $y = f(x)$ for some (implicit) function f, we have, by Theorem (2.28),

$$\frac{d}{dx}(\sin y) = \cos y\frac{dy}{dx}.$$

Using this equation and the fact that $(d/dx)(x^2) = 2x$, we may rewrite the first equation of our solution as

$$\frac{dy}{dx} = (x^2\cos y)\frac{dy}{dx} + \sin y(2x),$$

or $$y' = (x^2\cos y)y' + 2x\sin y.$$

Finally, we solve for y' as follows:

$$\begin{aligned}y' - (x^2\cos y)y' &= 2x\sin y\\ (1 - x^2\cos y)y' &= 2x\sin y\\ y' &= \frac{2x\sin y}{1 - x^2\cos y},\end{aligned}$$

provided $1 - x^2\cos y \neq 0$.

In the next example, we find the second derivative of an implicit function.

EXAMPLE ■ 7 Find y'' if $y^4 + 3y - 4x^3 = 5x + 1$.

SOLUTION The equation was considered in Example 2, where we found that

$$y' = \frac{12x^2 + 5}{4y^3 + 3}.$$

Hence $$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{12x^2 + 5}{4y^3 + 3}\right).$$

We now use the quotient rule, differentiating implicitly as follows:

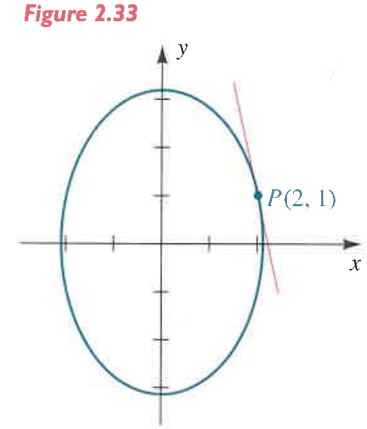
$$\begin{aligned}y'' &= \frac{(4y^3 + 3)\frac{d}{dx}(12x^2 + 5) - (12x^2 + 5)\frac{d}{dx}(4y^3 + 3)}{(4y^3 + 3)^2}\\ &= \frac{(4y^3 + 3)(24x) - (12x^2 + 5)(12y^2y')}{(4y^3 + 3)^2}\end{aligned}$$

Substituting for y' yields

$$\begin{aligned}y'' &= \frac{(4y^3 + 3)(24x) - (12x^2 + 5)\cdot 12y^2\left(\frac{12x^2 + 5}{4y^3 + 3}\right)}{(4y^3 + 3)^2}\\ &= \frac{(4y^3 + 3)^2(24x) - 12y^2(12x^2 + 5)^2}{(4y^3 + 3)^3}.\end{aligned}$$

EXAMPLE ■ 8 Use implicit differentiation to find an equation for the tangent line to the ellipse $9x^2 + 4y^2 = 40$ at the point $P(2, 1)$, as shown in Figure 2.33.

Figure 2.33



SOLUTION We verify first that the point $P(2, 1)$ is on the ellipse by showing that its coordinates satisfy the equation $9x^2 + 4y^2 = 40$:

$$9(2^2) + 4(1^2) = (9)(4) + (4)(1) = 36 + 4 = 40.$$

The slope of the tangent line will be the value of the derivative y' evaluated at $P(2, 1)$. We find y' by differentiating both sides of the equation of the ellipse with respect to x:

$$\begin{aligned}\frac{d}{dx}(9x^2) + \frac{d}{dx}(4y^2) &= \frac{d}{dx}(40)\\ 18x + 8yy' &= 0\\ 8yy' &= -18x\\ y' &= \frac{-18x}{8y} = \frac{-9x}{4y}\end{aligned}$$

Thus, the slope of the tangent line at $P(2, 1)$ is

$$\frac{(-9)(2)}{(4)(1)} = \frac{-9}{2}.$$

We can now find an equation of the tangent line by using the point–slope formula:

$$(y - 1) = \frac{-9}{2}(x - 2), \quad \text{or, equivalently,} \quad 9x + 2y = 20$$

EXERCISES 2.6

Exer. 1–18: Assuming that the equation determines a differentiable function f such that $y = f(x)$, find y'.

1 $8x^2 + y^2 = 10$

2 $4x^3 - 2y^3 = x$

3 $2x^3 + x^2y + y^3 = 1$

4 $5x^2 + 2x^2y + y^2 = 8$

5 $5x^2 - xy - 4y^2 = 0$

6 $x^4 + 4x^2y^2 - 3xy^3 + 2x = 0$

7 $\sqrt{x} + \sqrt{y} = 100$

8 $x^{2/3} + y^{2/3} = 4$

9 $x^2 + \sqrt{xy} = 7$

10 $2x - \sqrt{xy} + y^3 = 16$

11 $\sin^2 3y = x + y - 1$

12 $x = \sin(xy)$

13 $y = \csc(xy)$

14 $y^2 + 1 = x^2 \sec y$

15 $y^2 = x \cos y$

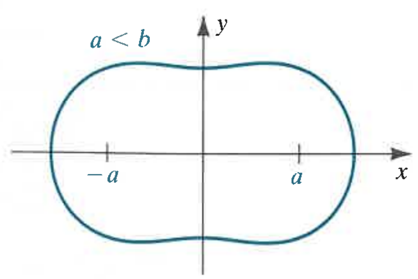
16 $xy = \tan y$

17 $x^2 + \sqrt{\sin y} - y^2 = 1$

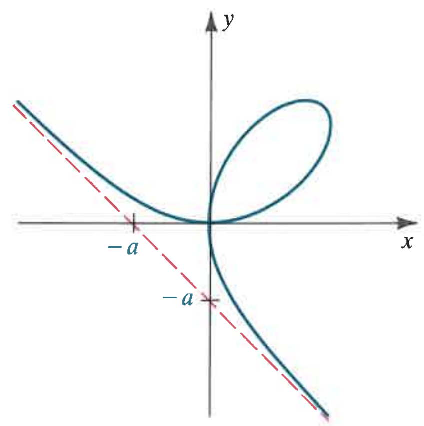
18 $\sin \sqrt{y} - 3x = 2$

Exer. 19–22: The equation of a classical curve and its graph are given for positive constants a and b. (Consult books on analytic geometry for further information.) Find the slope of the tangent line at the point P for the stated values of a and b.

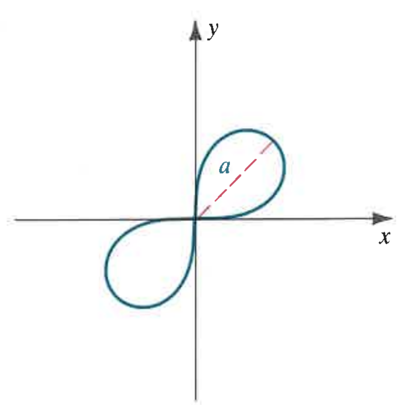
19 *Ovals of Cassini:* $(x^2 + y^2 + a^2)^2 - 4a^2x^2 = b^4$; $a = 2$, $b = \sqrt{6}$, $P(2, \sqrt{2})$



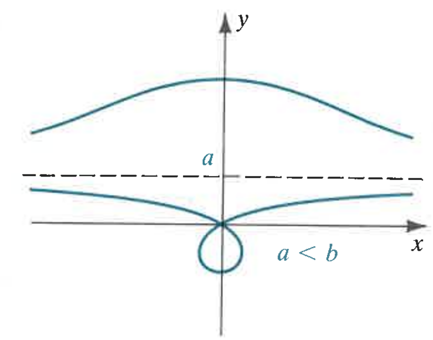
20 *Folium of Descartes:* $x^3 + y^3 - 3axy = 0$; $a = 4$, $P(6, 6)$



21 *Lemniscate of Bernoulli:* $(x^2 + y^2)^2 = 2a^2xy$; $a = \sqrt{2}$, $P(1, 1)$



22 *Conchoid of Nicomedes:* $(y - a)^2(x^2 + y^2) = b^2y^2$; $a = 2$, $b = 4$, $P(\sqrt{15}, 1)$



Exer. 23–28: Find the slope of the tangent line to the graph of the equation at P.

23 $xy + 16 = 0$; $P(-2, 8)$

24 $y^2 - 4x^2 = 5$; $P(-1, 3)$

25 $2x^3 - x^2y + y^3 - 1 = 0$; $P(2, -3)$

26 $3y^4 + 4x - x^2 \sin y - 4 = 0$; $P(1, 0)$

27 $x^2y + \sin y = 2\pi$; $P(1, 2\pi)$

28 $xy^2 + 3y = 27$; $P(2, 3)$

Exer. 29–34: Assuming that the equation determines a function f such that $y = f(x)$, find y'', if it exists.

29 $3x^2 + 4y^2 = 4$

30 $5x^2 - 2y^2 = 4$

31 $x^3 - y^3 = 1$

32 $x^2y^3 = 1$

33 $\sin y + y = x$

34 $\cos y = x$

Exer. 35–38: How many implicit functions are determined by the equation?

35 $x^4 + y^4 - 1 = 0$

36 $x^4 + y^4 = 0$

37 $x^2 + y^2 + 1 = 0$

38 $\cos x + \sin y = 3$

39 Show that the equation $y^2 = x$ determines an infinite number of implicit functions.

40 Use implicit differentiation to show that if P is any point on the circle $x^2 + y^2 = a^2$, then the tangent line at P is perpendicular to OP.

41 If tangent lines to the ellipse $9x^2 + 4y^2 = 36$ intersect the y-axis at the point $(0, 6)$, find the points of tangency.

42 If tangent lines to the hyperbola $9x^2 - y^2 = 36$ intersect the y-axis at the point $(0, 6)$, find the points of tangency.

43 Find an equation of a line through $P(-2, 3)$ that is tangent to the ellipse $5x^2 + 4y^2 = 56$.

44 Find an equation of a line through $P(2, -1)$ that is tangent to the hyperbola $x^2 - 4y^2 = 16$.

Exer. 45–46: Find equations of the tangent line and the normal line to the ellipse at the point P.

45 $5x^2 + 4y^2 = 56$; $P(-2, 3)$

46 $9x^2 + 4y^2 = 72$; $P(2, 3)$

Exer. 47–48: Find equations of the tangent line and the normal line to the hyperbola at the point P.

47 $2x^2 - 5y^2 = 3$; $P(-2, 1)$

48 $3y^2 - 2x^2 = 40$; $P(2, -4)$

49 For the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, where we have $a > b > 0$:

(a) Use implicit differentiation to find a formula for the slope of the tangent line at the point $P(x_1, y_1)$.

(b) Determine at which points the tangent line is horizontal or vertical.

(c) Show that an equation of the tangent line at $P(x_1, y_1)$ is

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

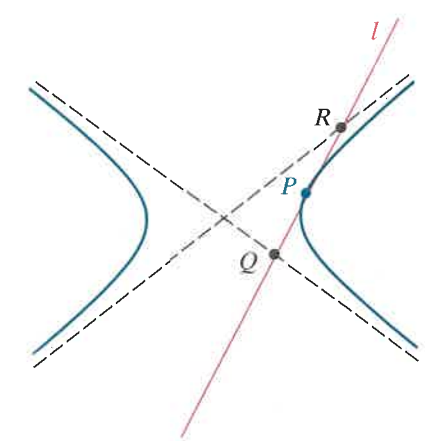
50 Prove that an equation of the tangent line to the graph of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ at the point $P(x_1, y_1)$ is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

51 Prove that if a normal line to each point on an ellipse passes through the center of the ellipse, then the ellipse is a circle.

52 Let l denote the tangent line at a point P on a hyperbola (see figure). If l intersects the asymptotes at Q and R, prove that P is the midpoint of QR.

Exercise 52



2.7 RELATED RATES

In many applications of calculus, we encounter situations in which there are variables—say, x and y—that satisfy some relationship over some interval of time t. We usually have some knowledge of the rate of change of one of these variables with respect to time, and we wish to find the rate of change with respect to time of the other variables. These problems are called *related rate problems* and are the focus of study in this section.

Suppose that two variables x and y are functions of another variable t, say,

$$x = f(t) \quad \text{and} \quad y = g(t).$$

By (2.7)(ii), we may interpret the derivatives dx/dt and dy/dt as the rates of change of x and y with respect to t. As a special case, if f and g are position functions for points moving on coordinate lines, then dx/dt and dy/dt are the velocities of these points (see (2.2)). In other situations, these derivatives may represent rates of change of physical quantities.

In certain applications, x and y may be related by means of an equation, such as

$$x^2 - y^3 - 2x + 7y^2 - 2 = 0.$$

If we differentiate this equation implicitly *with respect to t*, we obtain

$$\frac{d}{dt}(x^2) - \frac{d}{dt}(y^3) - \frac{d}{dt}(2x) + \frac{d}{dt}(7y^2) - \frac{d}{dt}(2) = \frac{d}{dt}(0).$$

Using the power rule (2.27) with t as the independent variable gives us

$$2x\frac{dx}{dt} - 3y^2\frac{dy}{dt} - 2\frac{dx}{dt} + 14y\frac{dy}{dt} = 0.$$

The derivatives dx/dt and dy/dt are called **related rates,** since they are related by means of an equation. This equation can be used to find one of the rates when the other is known. The following examples give several illustrations.

EXAMPLE ■ 1 Two variables x and y are functions of a variable t and are related by the equation

$$x^3 - 2y^2 + 5x = 16.$$

If $dx/dt = 4$ when $x = 2$ and $y = -1$, find the corresponding value of dy/dt.

SOLUTION We differentiate the given equation implicitly with respect to t as follows:

$$\frac{d}{dt}(x^3) - \frac{d}{dt}(2y^2) + \frac{d}{dt}(5x) = \frac{d}{dt}(16)$$

$$3x^2\frac{dx}{dt} - 4y\frac{dy}{dt} + 5\frac{dx}{dt} = 0$$

$$(3x^2 + 5)\frac{dx}{dt} = 4y\frac{dy}{dt}$$

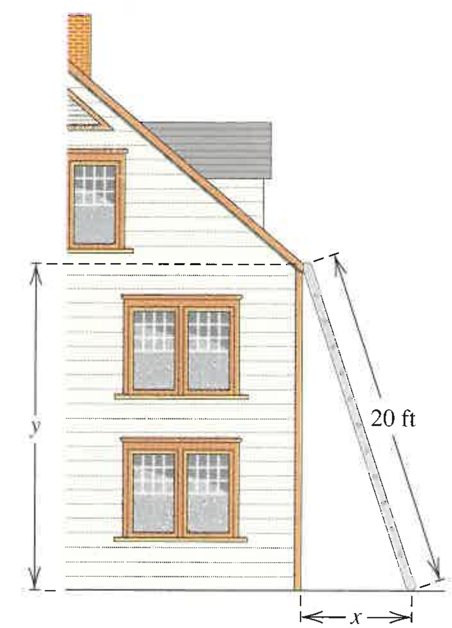
$$\frac{dy}{dt} = \frac{3x^2 + 5}{4y}\frac{dx}{dt}$$

The last equation is a *general* formula relating dy/dt and dx/dt. For the special case $dx/dt = 4$, $x = 2$, and $y = -1$, we obtain

$$\frac{dy}{dt} = \frac{3(2)^2 + 5}{4(-1)} \cdot 4 = -17.$$

EXAMPLE ■ 2 A ladder 20 ft long leans against the wall of a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 2 ft/sec, how fast is the ladder sliding down the building when the top of the ladder is 12 ft above the ground?

Figure 2.34



SOLUTION We begin by sketching a general position of the ladder as in Figure 2.34, where x denotes the distance from the base of the building to the bottom of the ladder and y denotes the distance from the ground to the top of the ladder.

We next consider the following problem involving the rates of change of x and y with respect to t:

Given: $\dfrac{dx}{dt} = 2$ ft/sec

Find: $\dfrac{dy}{dt}$ when $y = 12$ ft

An equation that relates the variables x and y can be obtained by applying the Pythagorean theorem to the right triangle formed by the building, the ground, and the ladder (see Figure 2.34):

$$x^2 + y^2 = 400$$

Differentiating both sides of this equation implicitly with respect to t, we obtain

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(400)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt},$$

provided $y \neq 0$.

The last equation is a general formula relating the two rates of change dx/dt and dy/dt. Let us now consider the special case $y = 12$. The corresponding value of x may be determined from

$$x^2 + 12^2 = 400, \quad \text{or} \quad x^2 = 256.$$

Thus, $x = \sqrt{256} = 16$ when $y = 12$. Substituting these values into the general formula for dy/dt, we obtain

$$\frac{dy}{dt} = -\frac{16}{12}(2) = -\frac{8}{3}\text{ ft/sec.}$$

The following guidelines may be helpful for solving related rate problems of the type illustrated in Example 2.

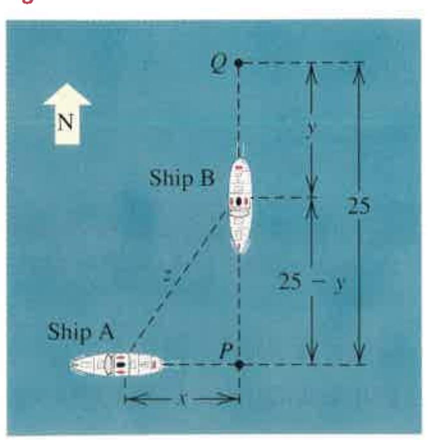
Guidelines for Solving Related Rate Problems 2.29

1. Read the problem carefully several times, and think about the given facts and the unknown quantities that are to be found.
2. Sketch a picture or diagram and label it appropriately, introducing variables for unknown quantities.
3. Write down all the known facts, expressing the given and unknown rates as derivatives of the variables introduced in guideline (2).
4. Formulate a *general* equation that relates the variables.
5. Differentiate the equation formulated in guideline (4) implicitly with respect to t, obtaining a *general* relationship between the rates.
6. Substitute the *known* values and rates, and then find the unknown rate of change.

CAUTION A common error is introducing specific values for the rates and variable quantities *too early* in the solution. Always remember to obtain a *general* formula that involves the rates of change at *any* time t. *Specific values should not be substituted for variables until the final steps of the solution.*

EXAMPLE 3 At 1:00 P.M., ship A is 25 mi due south of ship B. If ship A is sailing west at a rate of 16 mi/hr and ship B is sailing south at a rate of 20 mi/hr, find the rate at which the distance between the ships is changing at 1:30 P.M.

Figure 2.35



SOLUTION Let t denote the number of hours after 1:00 P.M. In Figure 2.35, P and Q are the positions of the ships at 1:00 P.M., x and y are the number of miles they have traveled in t hours, and z is the distance between the ships after t hours. Our problem may be stated as follows:

Given: $\frac{dx}{dt} = 16$ mi/hr and $\frac{dy}{dt} = 20$ mi/hr

Find: $\frac{dz}{dt}$ when $t = \frac{1}{2}$ hr

Applying the Pythagorean theorem to the triangle in Figure 2.35 gives us the following general equation relating the variables x, y, and z:

$$z^2 = x^2 + (25 - y)^2$$

Differentiating implicitly with respect to t and using the power rule and

the chain rule, we obtain

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2) + \frac{d}{dt}(25 - y)^2$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2(25 - y)\left(0 - \frac{dy}{dt}\right)$$

$$z\frac{dz}{dt} = x\frac{dx}{dt} + (y - 25)\frac{dy}{dt}.$$

At 1:30 P.M., the ships have traveled for half an hour and

$$x = \tfrac{1}{2}(16) = 8, \qquad y = \tfrac{1}{2}(20) = 10, \qquad \text{and} \quad 25 - y = 15.$$

Consequently,

$$z^2 = 64 + 225 = 289, \quad \text{or} \quad z = \sqrt{289} = 17.$$

Substituting into the last equation involving dz/dt, we have

$$17\frac{dz}{dt} = 8(16) + (-15)(20), \quad \text{or} \quad \frac{dz}{dt} = -\frac{172}{17} \approx -10.12 \text{ mi/hr.}$$

The negative sign indicates that the distance between the ships is decreasing at 1:30 P.M.

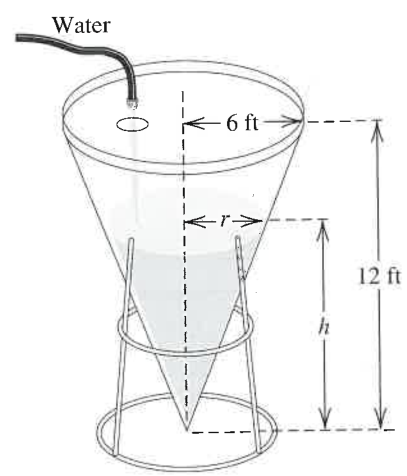
Another method of solution is to write $x = 16t$, $y = 20t$, and

$$z = [x^2 + (25 - y)^2]^{1/2} = [256t^2 + (25 - 20t)^2]^{1/2}.$$

The derivative dz/dt may then be found, and substitution of $\frac{1}{2}$ for t produces the desired rate of change.

EXAMPLE 4 A water tank has the shape of an inverted right circular cone of altitude 12 ft and base radius 6 ft. If water is being pumped into the tank at a rate of 10 gal/min, approximate the rate at which the water level is rising when the water is 3 ft deep (1 gal ≈ 0.1337 ft^3).

Figure 2.36



SOLUTION We begin by sketching the tank as in Figure 2.36, letting r denote the radius of the surface of the water when the depth is h. Note that r and h are functions of time t.

The problem can now be stated as follows:

Given: $\frac{dV}{dt} = 10$ gal/min

Find: $\frac{dh}{dt}$ when $h = 3$ ft

The volume V of water in the tank corresponding to depth h is

$$V = \tfrac{1}{3}\pi r^2 h.$$

This formula for V relates V, r, and h. Before differentiating implicitly with respect to t, let us express V in terms of one variable. Referring to Figure 2.36 and using similar triangles, we obtain

$$\frac{r}{h} = \frac{6}{12}, \quad \text{or} \quad r = \frac{h}{2}.$$

Consequently, at depth h,

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3.$$

Differentiating the last equation implicitly with respect to t gives us the following general relationship between the rates of change of V and h at any time t:

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

If $h \neq 0$, an equivalent formula is

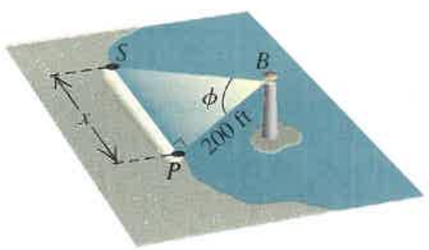
$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}.$$

Finally, we let $h = 3$ and $dV/dt = 10$ gal/min ≈ 1.337 ft^3/min, obtaining

$$\frac{dh}{dt} \approx \frac{4}{\pi(9)}(1.337) \approx 0.189 \text{ ft/min.}$$

EXAMPLE ■ 5 A revolving beacon in a lighthouse makes one revolution every 15 sec. The beacon is 200 ft from the nearest point P on a straight shoreline. Find the rate at which a ray from the light moves along the shore at a point 400 ft from P.

Figure 2.37



SOLUTION The problem is diagrammed in Figure 2.37, where B denotes the position of the beacon and ϕ is the angle between BP and a light ray to a point S on the shore x units from P.

Since the light revolves four times per minute, the angle ϕ changes at a rate of $4 \cdot 2\pi$ radians per minute; that is, $d\phi/dt = 8\pi$. Using triangle PBS, we see that

$$\tan\phi = \frac{x}{200},$$

or

$$x = 200\tan\phi.$$

The rate at which the ray of light moves along the shore is

$$\frac{dx}{dt} = 200\sec^2\phi\frac{d\phi}{dt} = (200\sec^2\phi)(8\pi) = 1600\pi\sec^2\phi.$$

If $x = 400$, then $BS = \sqrt{200^2 + 400^2} = 200\sqrt{5}$, and

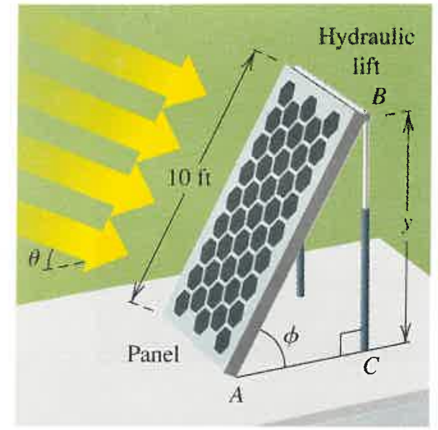
$$\sec\phi = \frac{200\sqrt{5}}{200} = \sqrt{5}.$$

Hence

$$\frac{dx}{dt} = 1600\pi(\sqrt{5})^2 = 8000\pi \approx 25{,}133 \text{ ft/min.}$$

EXAMPLE ■ 6 Figure 2.38, on the following page, shows a solar panel that is 10 ft in width and is equipped with a hydraulic lift. As the sun rises, the panel is adjusted so that the sun's rays are perpendicular to the panel's surface.

Figure 2.38



(a) Find the relationship between the rate dy/dt at which the panel should be lowered and the rate $d\theta/dt$ at which the angle of inclination of the sun increases.

(b) If $d\theta/dt = \pi/12$ radian/hr when $\theta = \pi/6$, find dy/dt.

SOLUTION

(a) If we let ϕ denote angle BAC in Figure 2.38, then, from plane geometry, $\phi = \frac{1}{2}\pi - \theta$. Since $d\phi/dt = -d\theta/dt$, ϕ decreases at the rate that θ increases.

Referring to right triangle BAC, we see that

$$\sin\phi = \frac{y}{10},$$

or

$$y = 10\sin\phi = 10\sin(\tfrac{1}{2}\pi - \theta).$$

Differentiating implicitly with respect to t and using the cofunction identity $\cos(\frac{1}{2}\pi - \theta) = \sin\theta$ yields

$$\frac{dy}{dt} = 10\cos\left(\frac{1}{2}\pi - \theta\right)\left(0 - \frac{d\theta}{dt}\right) = -10\sin\theta\frac{d\theta}{dt}.$$

(b) We substitute $d\theta/dt = \pi/12$ radian/hr and $\theta = \pi/6$ in the formula for dy/dt from part (a), obtaining

$$\frac{dy}{dt} = -10\left(\frac{1}{2}\right)\left(\frac{\pi}{12}\right) = -\frac{5\pi}{12} \approx -1.3 \text{ ft/hr.}$$

EXERCISES 2.7

Exer. 1–8: Assume that all variables are functions of t.

1 If $A = x^2$ and $dx/dt = 3$ when $x = 10$, find dA/dt.

2 If $S = z^3$ and $dz/dt = -2$ when $z = 3$, find dS/dt.

3 If $V = -5p^{3/2}$ and $dV/dt = -4$ when $V = -40$, find dp/dt.

4 If $P = 3/w$ and $dP/dt = 5$ when $P = 9$, find dw/dt.

5 If $x^2 + 3y^2 + 2y = 10$ and $dx/dt = 2$ when $x = 3$ and $y = -1$, find dy/dt.

6 If $2y^3 - x^2 + 4x = -10$ and $dy/dt = -3$ when $x = -2$ and $y = 1$, find dx/dt.

7 If $3x^2y + 2x = -32$ and $dy/dt = -4$ when $x = 2$ and $y = -3$, find dx/dt.

8 If $-x^2y^2 - 4y = -44$ and $dx/dt = 5$ when $x = -3$ and $y = 2$, find dy/dt.

9 As a circular metal griddle is being heated, its diameter changes at a rate of 0.01 cm/min. Find the rate at which the area of one side is changing when the diameter is 30 cm.

10 A fire has started in a dry, open field and spreads in the form of a circle. The radius of the circle increases at a rate of 6 ft/min. Find the rate at which the fire area is increasing when the radius is 150 ft.

11 Gas is being pumped into a spherical balloon at a rate of 5 ft^3/min. Find the rate at which the radius is changing when the diameter is 18 in.

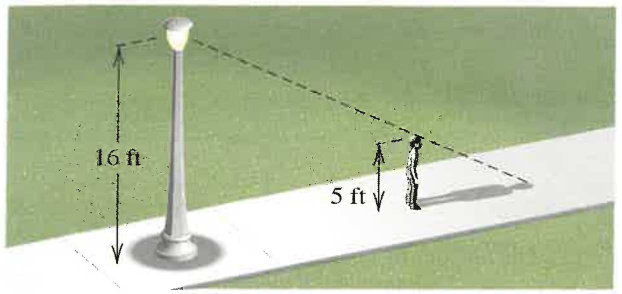
12 Suppose a spherical snowball is melting and the radius is decreasing at a constant rate, changing from 12 in. to 8 in. in 45 min. How fast was the volume changing when the radius was 10 in.?

13 A ladder 20 ft long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 3 ft/sec, how fast is the ladder sliding down the building when the top of the ladder is 8 ft from the ground?

14 A girl starts at a point A and runs east at a rate of 10 ft/sec. One minute later, another girl starts at A and runs north at a rate of 8 ft/sec. At what rate is the distance between them changing 1 min after the second girl starts?

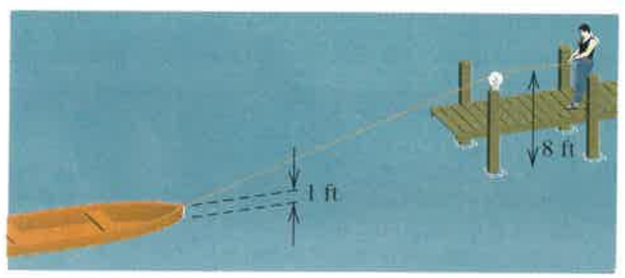
15 A light is at the top of a 16-ft pole. A boy 5 ft tall walks away from the pole at a rate of 4 ft/sec (see figure). At what rate is the tip of his shadow moving when he is 18 ft from the pole? At what rate is the length of his shadow increasing?

Exercise 15



16 A man on a dock is pulling in a boat using a rope attached to the bow of the boat 1 ft above water level and passing through a simple pulley located on the dock 8 ft above water level (see figure). If he pulls in the rope at a rate of 2 ft/sec, how fast is the boat approaching the dock when the bow of the boat is 25 ft from a point that is directly below the pulley?

Exercise 16

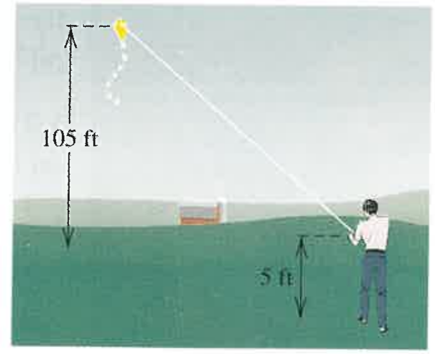


17 The top of a silo has the shape of a hemisphere of diameter 20 ft. If it is coated uniformly with a layer of ice and if the thickness is decreasing at a rate of $\frac{1}{4}$ in./hr, how fast is the volume of the ice changing when the ice is 2 in. thick?

18 As sand leaks out of a hole in a container, it forms a conical pile whose altitude is always the same as its radius. If the height of the pile is increasing at a rate of 6 in./min, find the rate at which the sand is leaking out when the altitude is 10 in.

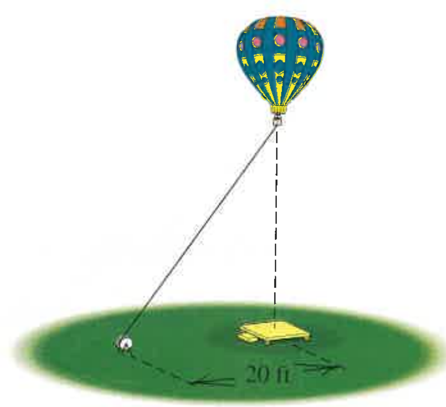
19 A person flying a kite holds the string 5 ft above ground level, and the string is payed out at a rate of 2 ft/sec as the kite moves horizontally at an altitude of 105 ft (see figure). Assuming there is no sag in the string, find the rate at which the kite is moving when 125 ft of string has been payed out.

Exercise 19



20 A hot-air balloon rises vertically as a rope attached to the base of the balloon is released at a rate of 5 ft/sec. The pulley that releases the rope is 20 ft from the platform where passengers board (see figure). At what rate is the balloon rising when 500 ft of rope has been payed out?

Exercise 20

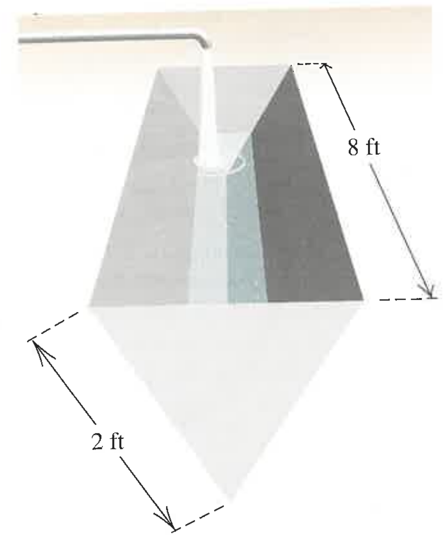


21 Boyle's law for confined gases states that if the temperature is constant, $pv = c$, where p is pressure, v is volume, and c is a constant. At a certain instant, the volume is 75 in^3, the pressure is 30 lb/in^2, and the pressure is decreasing at a rate of 2 lb/in^2 every minute. At what rate is the volume changing at this instant?

22 A 100-ft-long cable of diameter 4 in. is submerged in seawater. Because of corrosion, the surface area of the cable decreases at a rate of 750 in^2/yr. Ignoring the corrosion at the ends of the cable, find the rate at which the diameter is decreasing.

23 The ends of a water trough 8 ft long are equilateral triangles whose sides are 2 ft long (see figure on the following page). If water is being pumped into the trough at a rate of 5 ft^3/min, find the rate at which the water level is rising when the depth of the water is 8 in.

Exercise 23



24 Work Exercise 23 if the ends of the trough have the shape of the graph of $y = 2|x|$ between the points $(-1, 2)$ and $(1, 2)$.

25 The area of an equilateral triangle is decreasing at a rate of 4 cm^2/min. Find the rate at which the length of a side is changing when the area of the triangle is 200 cm^2.

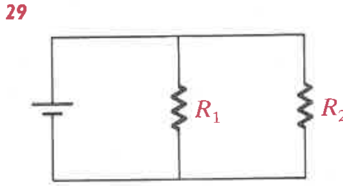
26 Gas is escaping from a spherical balloon at a rate of 10 ft^3/hr. At what rate is the radius changing when the volume is 400 ft^3?

27 A stone is dropped into a lake, causing circular waves whose radii increase at a constant rate of 0.5 m/sec. At what rate is the circumference of a wave changing when its radius is 4 m?

28 A softball diamond has the shape of a square with sides 60 ft long. If a player is running from second base to third base at a speed of 24 ft/sec, at what rate is her distance from home plate changing when she is 20 ft from third base?

29 When two resistors R_1 and R_2 are connected in parallel (see figure), the total resistance R is given by the equation $1/R = (1/R_1) + (1/R_2)$. If R_1 and R_2 are increasing at rates of 0.01 ohm/sec and 0.02 ohm/sec, respectively, at what rate is R changing at the instant that $R_1 = 30$ ohms and $R_2 = 90$ ohms?

Exercise 29



30 The formula for the adiabatic expansion of air is $pv^{1.4} = c$, where p is the pressure, v is the volume, and c is a constant. At a certain instant, the pressure is 40 dyn/cm^2 and is increasing at a rate of 3 dyn/cm^2 per second. If, at that same instant, the volume is 60 cm^3, find the rate at which the volume is changing.

31 If a spherical tank of radius a contains water that has a maximum depth h, then the volume V of water in the tank is given by $V = \frac{1}{3}\pi h^2(3a - h)$. Suppose a spherical tank of radius 16 ft is being filled at a rate of 100 gal/min. Approximate the rate at which the water level is rising when $h = 4$ ft (1 gal ≈ 0.1337 ft^3).

32 A spherical water storage tank for a small community is coated uniformly with a 2-in. layer of ice. As the ice melts, the rate at which the volume of the ice decreases is directly proportional to its surface area. Show that the outside diameter is decreasing at a constant rate.

33 From the edge of a cliff that overlooks a lake 200 ft below, a boy drops a stone and then, 2 sec later, drops another stone from exactly the same position. Discuss the rate at which the distance between the two stones is changing during the next second. (Assume that the distance an object falls in t seconds is $16t^2$ feet.)

34 A metal rod has the shape of a right circular cylinder. As it is being heated, its length is increasing at a rate of 0.005 cm/min and its diameter is increasing at 0.002 cm/min. At what rate is the volume changing when the rod has length 40 cm and diameter 3 cm?

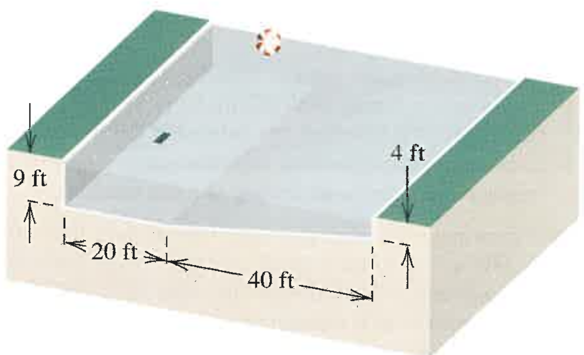
35 An airplane is flying at a constant speed of 360 mi/hr and climbing at an angle of 45°. At the moment the plane's altitude is 10,560 ft, it passes directly over an air traffic control tower on the ground. Find the rate at which the airplane's distance from the tower is changing 1 min later (neglect the height of the tower).

36 A north–south highway A and an east–west highway B intersect at a point P. At 10:00 A.M., an automobile crosses P traveling north on highway A at a speed of 50 mi/hr. At that same instant, an airplane flying east at a speed of 200 mi/hr and an altitude of 26,400 ft is directly above the point on highway B that is 100 mi west of P. If the airplane and the automobile maintain the same speed and direction, at what rate is the distance between them changing at 10:15 A.M.?

37 A paper cup containing water has the shape of a frustum of a right circular cone of altitude 6 in. and lower and upper base radii 1 in. and 2 in., respectively. If water is leaking out of the cup at a rate of 3 in^3/hr, at what rate is the water level decreasing when the depth of the water is 4 in.? (*Note:* The volume V of a frustum of a right circular cone of altitude h and base radii a and b is given by $V = \frac{1}{3}\pi h(a^2 + b^2 + ab)$.)

38 The top part of a swimming pool is a rectangle of length 60 ft and width 30 ft. The depth of the pool varies uniformly from 4 ft to 9 ft through a horizontal distance of 40 ft and then is level for the remaining 20 ft, as illustrated by the cross-sectional view in the figure. If the pool is being filled with water at a rate of 500 gal/min, approximate the rate at which the water level is rising when the depth of the water at the deep end of the pool is 4 ft (1 gal $\approx$ 0.1337 ft^3).

Exercise 38



39 An airplane at an altitude of 10,000 ft is flying at a constant speed on a line that will take it directly over an observer on the ground. If, at a given instant, the observer notes that the angle of elevation of the airplane is 60° and is increasing at a rate of 1° per second, find the speed of the airplane.

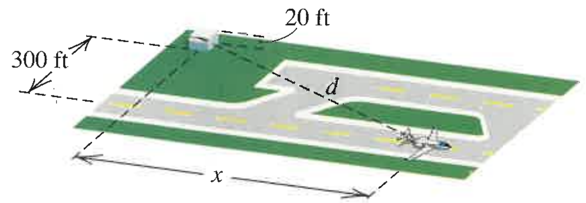
40 In Exercise 16, let θ be the angle that the rope makes with the horizontal. Find the rate at which θ is changing at the instant that $\theta = 30°$.

41 An isosceles triangle has equal sides 6 in. long. If the angle θ between the equal sides is changing at a rate of 2° per minute, how fast is the area of the triangle changing when $\theta = 30°$?

42 A ladder 20 ft long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 2 ft/sec, at what rate is the angle between the ladder and the ground changing when the top of the ladder is 12 ft above the ground?

43 The relative positions of an airport runway and a 20-ft-tall control tower are shown in the figure. The beginning of the runway is at a perpendicular distance of 300 ft from the base of the tower. If an airplane reaches a speed of 100 mi/hr after having traveled 300 ft down the runway, at approximately what rate is the distance between the airplane and the top of the control tower increasing at this time?

Exercise 43

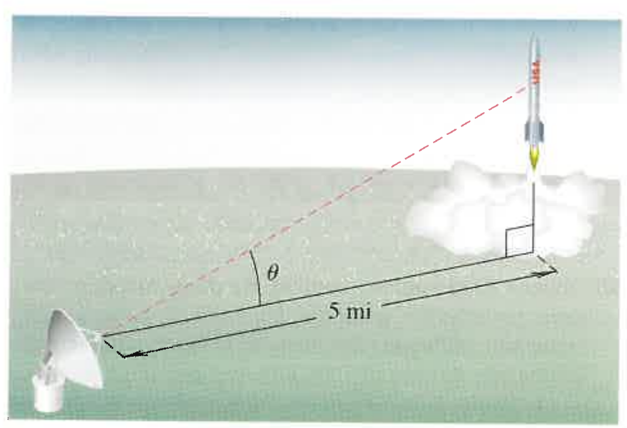


44 The speed of sound in air at 0 °C (or 273 °K) is 1087 ft/sec, but this speed increases as the temperature rises. If T is temperature in °K, the speed of sound v at this temperature is given by $v = 1087\sqrt{T/273}$. If the temperature increases at the rate of 3 °C per hour, approximate the rate at which the speed of sound is increasing when $T = 30$ °C (or 303 °K).

45 An airplane is flying at a constant speed and altitude on a line that will take it directly over a radar station located on the ground. At the instant that the airplane is 60,000 ft from the station, an observer in the station notes that the airplane's angle of elevation is 30° and is increasing at a rate of 0.5° per second. Find the speed of the airplane.

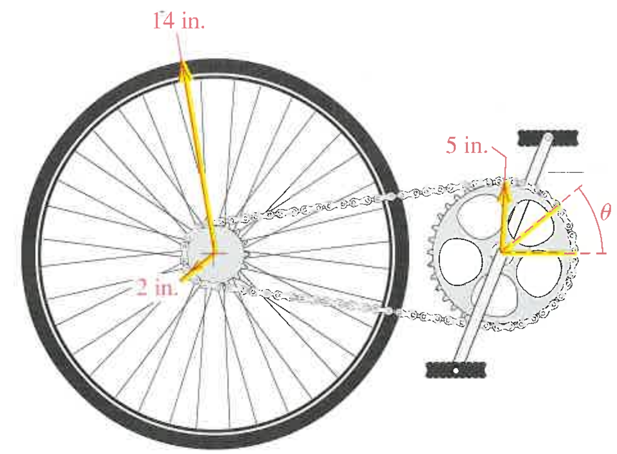
46 A missile is fired vertically from a point that is 5 mi from a tracking station and at the same elevation (see figure). For the first 20 sec of flight, its angle of elevation θ changes at a constant rate of 2° per second. Find the velocity of the missile when the angle of elevation is 30°.

Exercise 46



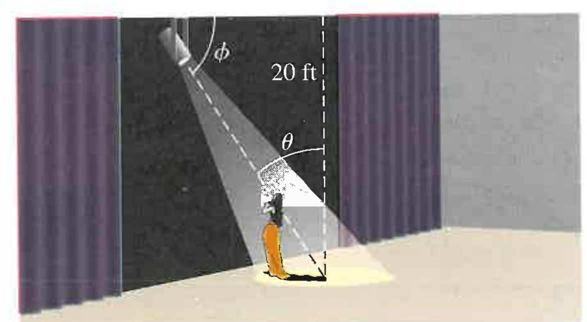
47 The sprocket assembly for a 28-in. bicycle is shown in the figure on the following page. Find the relationship between the angular velocity $d\theta/dt$ (in radians per second) of the pedal assembly and the ground speed of the bicycle (in miles per hour).

Exercise 47



48 A 100-candlepower spotlight is located 20 ft above a stage (see figure). The illuminance E (in footcandles) in the small lighted area of the stage is given by $E = (I\cos\theta)/s^2$, where I is the intensity of the light, s is the distance the light must travel, and θ is the indicated angle. As the spotlight is rotated through ϕ degrees, find the relationship between the rate of change in illumination dE/dt and the rate of rotation $d\phi/dt$.

Exercise 48

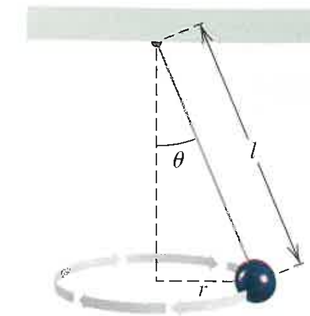


49 A *conical pendulum* consists of a mass m, attached to a string of fixed length l, that travels around a circle of radius r at a fixed velocity v (see figure). As the velocity of the mass is increased, both the radius and the angle θ increase. Given that $v^2 = rg\tan\theta$, where g is a gravitational constant, find a relationship between the related rates

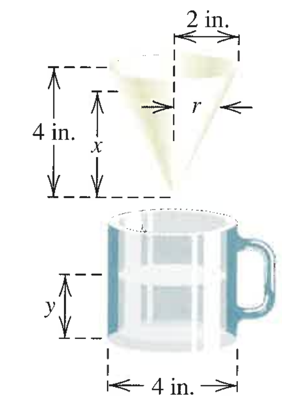
(a) dv/dt and $d\theta/dt$ **(b)** dv/dt and dr/dt

50 Water in a paper conical filter drips into a cup as shown in the figure. Let x denote the height of the water in the filter and y the height of the water in the cup. If 10 in^3 of water is poured into the filter, find the relationship between dy/dt and dx/dt.

Exercise 49



Exercise 50



c **51** Ship A is sailing north, and ship B is sailing east. Using an xy-plane, radar records the coordinates (in miles) of each ship at intervals of 1.25 min as shown in the following tables. Approximate the rate (in miles per hour) at which the distance between the ships is changing at $t = 5$.

Ship A:

t (min)	1.25	2.50	3.75	5.00
x (mi)	1.77	1.77	1.77	1.77
y (mi)	2.71	3.03	3.35	3.67

Ship B:

t (min)	1.25	2.50	3.75	5.00
x (mi)	5.24	5.52	5.80	6.08
y (mi)	1.24	1.24	1.24	1.24

c **52** Two variables x and y are functions of a variable t and are related by the formula

$$1.31\sin(2.56x) + \sqrt{y} = (x-1)^2.$$

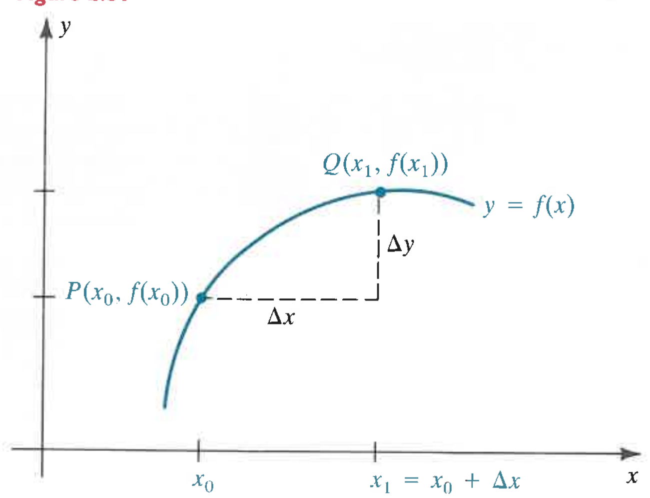
If $dy/dt \approx 3.68$ when $x \approx 1.71$ and $y \approx 3.03$, approximate the corresponding value of dx/dt.

2.8 LINEAR APPROXIMATIONS AND DIFFERENTIALS

We now examine a fundamental geometric property of the tangent line to the graph of a function: The tangent line stays close to the graph near the point of tangency. We will also introduce some additional notation and terminology that scientists and engineers commonly use in approximations involving derivatives.

Let us consider a differentiable function $y = f(x)$ where we know the behavior of the function at a point $P(x_0, f(x_0))$; in particular, assume that we know the values of $f(x_0)$ and $f'(x_0)$. Suppose we wish to approximate the value of f at x_1, where x_1 is close to x_0. Figure 2.39 illustrates this situation with Q being the point $(x_1, f(x_1))$. In some instances, we may not be able to determine $f(x_1)$ exactly because we do not have an explicit formula for f. In other cases, the numbers x_0 and x_1 may be two slightly different measurements of a physical quantity, and we want to estimate quickly the difference between the values of $f(x_0)$ and $f(x_1)$. In such cases, our needs might be met by finding a good approximation for $f(x_1)$. We will consider in this section a simple way to approximate $f(x_1)$ using the known values $f(x_0)$ and $f'(x_0)$.

Figure 2.39



If the variable x has an initial value x_0 and is then assigned a different value x_1, the change or difference $x_1 - x_0$ is called an **increment of x**. In calculus, it is traditional to denote an increment of x by the symbol Δx (read *delta* x). The corresponding change to the value of $y = f(x)$, namely, $f(x_1) - f(x_0)$, is called an **increment of y** and is traditionally labeled Δy. We summarize this notation in the following definition.

Definition 2.30

> If $y = f(x)$ and the variable x has an initial value of x_0 that is changed to x_1, then the **increment Δx of x** is
>
> $$\Delta x = x_1 - x_0$$
>
> and the corresponding **increment Δy of y** is
>
> $$\Delta y = f(x_0 + \Delta x) - f(x_0).$$

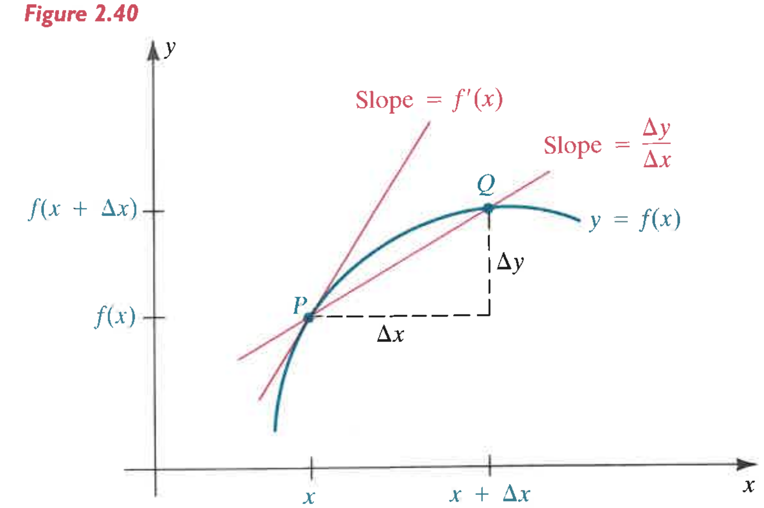
Definition (2.30) implies that Δy is the *change in* y corresponding to the *change in* x of Δx. Since x_1 is not equal to x_0, Δx must be either positive or negative. The resulting Δy may be positive, negative, or zero. In Figure 2.39, both changes, Δx and Δy, are positive.

From Definition (2.30), we have $f(x_0) + \Delta y = f(x_0 + \Delta x) = f(x_1)$. Thus we can obtain an accurate approximation for the value $f(x_1)$ if we can accurately estimate Δy. The ratio $\Delta y/\Delta x$ is the slope m_{PQ} of the secant line through P and Q (see Figure 2.40). Since $\Delta y/\Delta x = m_{PQ}$, we have $\Delta y = m_{PQ}\Delta x$. We already know that the value of Δx is $x_1 - x_0$, so if we can obtain an estimate for m_{PQ}, we can then estimate Δy and $f(x_1)$. In Section 2.1, we defined the slope of the tangent line to $y = f(x)$ at P to be the limit of slopes of secant lines through P and Q. In Section 2.2, we defined $f'(x_0)$ as the notation for this limit. Thus m_{PQ} is approximately equal to $f'(x_0)$ if x_1 is approximately equal to x_0. Thus, we have

$$f(x_1) = f(x_0) + \Delta y = f(x_0) + m_{PQ}\,\Delta x \approx f(x_0) + f'(x_0)\Delta x.$$

This approximation allows us to estimate $f(x_1)$ using the known values $f(x_0)$ and $f'(x_0)$. The approximation $f(x_1) \approx f(x) + f'(x_0)\Delta x$ is particularly useful when x_1 is close to a value x_0 where it is easier to compute $f(x_0)$ and $f'(x_0)$ than to calculate $f(x_1)$ directly.

Figure 2.40



To make this discussion more precise, we use the increment notation to rewrite the definition of the derivative of a function, substituting Δx for h in Definition (2.5) as follows.

Increment Definition of Derivative 2.31

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If f is differentiable, then as Δx approaches 0, the ratio $\Delta y/\Delta x$ approaches $f'(x_0)$, as illustrated in Figure 2.40.

In earlier sections, we used quotients

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

to approximate numerically the derivative $f'(x_0)$. Here we reverse the process and use $f'(x_0)$ to approximate $\Delta y/\Delta x$.

Approximation Formula for Δy 2.32

$$\Delta y \approx f'(x_0)\Delta x \quad \text{if } \Delta x \approx 0$$

It is helpful to consider the *graphical* interpretation of this approximation formula (see Figure 2.41). The slope $f'(x_0)$ is the slope of the line tangent to the graph at $P(x_0, f(x_0))$. An equation for this line is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

This *tangent line* l is the graph of the function L, where

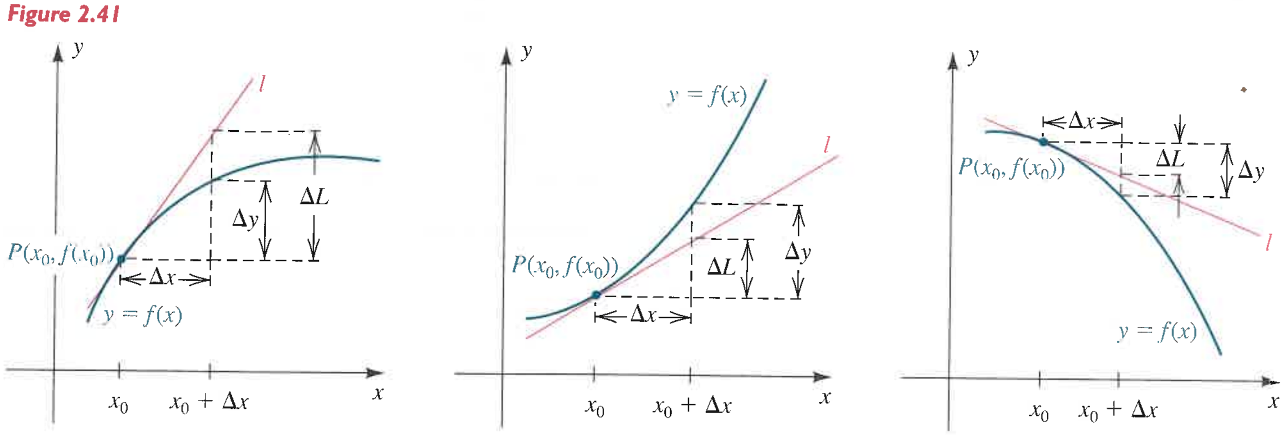
$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

Evaluating L at x_0 and at $x_1 = x_0 + \Delta x$ gives $L(x_0) = f(x_0)$ and $L(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x$, respectively. Thus,

$$\Delta L = L(x_0 + \Delta x) - L(x_0) = f'(x_0)\Delta x.$$

We can interpret (2.32) as stating that the *change in y* is approximately the *change in the tangent function L*. We have replaced the graph of the function f by the line representing L to obtain this estimate, and this function L is described as a **linear approximation** for f at x_0. Figure 2.41 illustrates this approximation.

Figure 2.41



Linear Approximation Formula 2.33

If $y = f(x)$, with f differentiable at x_0, then

$$f(x) \approx L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \text{for } x \text{ near } x_0.$$

There are many practical uses for linear approximations to functions. We will see one such application in Section 2.9 when we develop Newton's method for approximating zeros of functions. The next example shows how we can use linear approximations to estimate square roots.

EXAMPLE 1 For the function $y = f(x) = \sqrt{3 + x}$:

(a) Find the linear approximation at $x_0 = 6$.

(b) Use this linear approximation to estimate $\sqrt{8}$, $\sqrt{8.9}$, and $\sqrt{9.3}$.

(c) Compare these approximations to values obtained with a calculator.

SOLUTION

(a) We use the linear approximation formula (2.33). For the function $f(x) = \sqrt{3 + x} = (3 + x)^{1/2}$, we have

$$f'(x) = \frac{1}{2}(3 + x)^{-1/2} = \frac{1}{2\sqrt{3 + x}}.$$

Evaluating f and f' at $x_0 = 6$ gives

$$f(6) = \sqrt{3 + 6} = 3 \quad \text{and} \quad f'(6) = \frac{1}{2\sqrt{3 + 6}} = \frac{1}{6}.$$

Thus the linear approximation to f at $x_0 = 6$ is

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$= 3 + \tfrac{1}{6}(x - 6).$$

(b) For values of x close to 6, $3 + 6$ is close to 9, and so we can use the approximation $f(x) \approx L(x)$ to estimate square roots of numbers close to 9. For example,

$$\sqrt{8} = \sqrt{3 + 5} = f(5) \approx L(5) = 3 + \tfrac{1}{6}(-1) \approx 2.83333333333.$$

Similarly, we have

$$\sqrt{8.9} = \sqrt{3 + 5.9} = f(5.9) \approx L(5.9) = 3 + \tfrac{1}{6}(-0.1) \approx 2.98333333333,$$
$$\sqrt{9.3} = \sqrt{3 + 6.3} = f(6.3) \approx L(6.3) = 3 + \tfrac{1}{6}(0.3) = 3.05.$$

(c) The following table lists the approximated square roots obtained with a linear approximation and with a calculator:

Square root	$\sqrt{8}$	$\sqrt{8.9}$	$\sqrt{9.3}$
Linear approximation	2.83333333333	2.98333333333	3.05
Calculator	2.82842712475	2.98328677804	3.04959013640

Note that the linear approximations are close to the calculator values and do not require the computation of a square root.

EXAMPLE 2 Use a linear approximation to estimate $\sin 0.05$ and compare the result to that obtained with a calculator.

SOLUTION If $f(x) = \sin x$, then $f'(x) = \cos x$. Evaluating f and f' at $x_0 = 0$ gives

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

By Theorem (2.23), the linear approximation to f at $x_0 = 0$ is given by $L(x) = 0 + 1(x - 0)$, or

$$\sin x \approx x \quad \text{for } x \text{ near } 0,$$

and the linear approximation to $\sin 0.05$ is 0.05. Using a calculator, we get

$$\sin 0.05 \approx 0.049979169271.$$

You may be asking yourself: Why use linear approximations for square roots or trigonometric functions when a scientific calculator can do the job more efficiently and accurately? There are several answers. First, the *process of linear approximation* is widely used throughout mathematics and in applications. It is important then to consider the geometric reasoning behind linear approximation. It is also easier to examine this process in elementary problems in which we can check the approximations against more precise answers. Second, computers and calculators themselves use algorithms, such as linear approximation, to produce approximate values of elementary functions. To gain a better understanding of the powers and limitations of calculating devices, we need to study approximation techniques, beginning with linear ones. We will consider other approximation techniques in later chapters.

Linear approximations are closely connected with the idea of *differentials*. It is traditional to use the differential expression dy to denote the actual change in the tangent line corresponding to a change Δx in x. The expression dy is another notation for the quantity we have previously labeled as ΔL.

Definition 2.34

Let $y = f(x)$, where f is a differentiable function. The **differential** $\mathbf{dy}$ is defined by the expression

$$dy = f'(x)\,\Delta x.$$

The vertical change in the graph of the function over an interval is the change in the value of the function over that interval. Note that dy measures the vertical change in the *tangent line* and Δy measures the actual change in the value of the *function* for the same change in x. The next example illustrates this distinction.

EXAMPLE 3 Let $y = 3x^2 - 5$ and let Δx be an increment of x.

(a) Find general formulas for Δy and dy.

(b) If x changes from 2 to 2.1, find the values of Δy and dy.

SOLUTION

(a) If $y = f(x) = 3x^2 - 5$, then, by Definition (2.30) with $x = x_0$,

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= [3(x + \Delta x)^2 - 5] - (3x^2 - 5)\\ &= [3(x^2 + 2x(\Delta x) + (\Delta x)^2) - 5] - (3x^2 - 5)\\ &= 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 - 5 - 3x^2 + 5\\ &= 6x(\Delta x) + 3(\Delta x)^2.\end{aligned}$$

To find dy, we use Definition (2.34):

$$dy = f'(x)\Delta x = 6x\Delta x$$

(b) We wish to find Δy and dy if $x = 2$ and $\Delta x = 0.1$. Substituting in the formula for Δy obtained in part (a) gives us

$$\Delta y = 6(2)(0.1) + 3(0.1)^2 = 1.23.$$

Thus, y changes by 1.23 if x changes from 2 to 2.1. We could also find Δy directly as follows:

$$\begin{aligned}\Delta y &= f(2.1) - f(2)\\ &= [3(2.1)^2 - 5] - [3(2)^2 - 5] = 1.23\end{aligned}$$

Similarly, using the formula $dy = 6x\Delta x$, with $x = 2$ and $\Delta x = 0.1$, yields

$$dy = (6)(2)(0.1) = 1.2.$$

Note that the approximation $dy = 1.2$ is correct to the nearest tenth.

Using differential notation, we can state an alternative form of the linear approximation formula (2.33).

Alternative Linear Approximation Formula 2.35

If $y = f(x)$ is a differentiable function, then

$$f(x + \Delta x) \approx f(x) + dy,$$

where $dy = f'(x)\Delta x$.

The next example illustrates the use of this alternative linear approximation formula.

EXAMPLE 4

(a) Use differentials to approximate the change in $\sin\theta$ if θ changes from $\pi/3 = 60\pi/180$ to $61\pi/180$.

(b) Use a linear approximation to estimate $\sin(61\pi/180)$.

SOLUTION

(a) If $y = \sin\theta = f(\theta)$, then

$$dy = f'(\theta)\Delta\theta = \cos\theta\,\Delta\theta.$$

The change $\Delta\theta$ in θ is $61\pi/180 - 60\pi/180 = \pi/180$. Thus if we let $\theta = \pi/3$ and $\Delta\theta = \pi/180$ in formula (2.34) for dy, we obtain

$$dy = \left(\cos\frac{\pi}{3}\right)\left(\frac{\pi}{180}\right) = \left(\frac{1}{2}\right)\left(\frac{\pi}{180}\right) = \frac{\pi}{360} \approx 0.0087.$$

(b) If we use the linear approximation formula (2.35) with $x = \theta$ and $y = \sin\theta$, we have

$$\sin(\theta + \Delta\theta) \approx \sin\theta + dy\,.$$

Letting $\theta = \pi/3$, $\Delta\theta = \pi/180$, and $dy = 0.0087$ (from part a), we obtain

$$\begin{aligned}\sin\left(\frac{61\pi}{180}\right) &\approx \sin\frac{\pi}{3} + dy\\ &\approx \frac{\sqrt{3}}{2} + 0.0087\\ &\approx 0.8660 + 0.0087 = 0.8747.\end{aligned}$$

Using a calculator and rounding to four decimal places, we get $\sin(61\pi/180) \approx 0.8746$. Thus the error involved in using the linear approximation is roughly 0.0001.

The next example illustrates the use of differentials in estimating errors that may occur because of approximate measurements. As indicated in the solution, *it is important to first consider general formulas involving the variables* that are being considered. Specific values should *not* be substituted for variables until the final steps of the solution.

EXAMPLE ■ 5 The radius of a spherical balloon is measured as 12 in., with a maximum error in measurement of ± 0.06 in. Approximate the maximum error in the calculated volume of the sphere.

SOLUTION We begin by considering *general* formulas involving the radius and the volume. Thus we let

$$x = \text{the measured value of the radius}$$

and

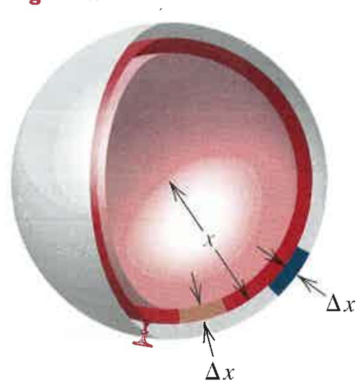
$$\Delta x = \text{the maximum error in } x.$$

Assuming that Δx is positive, we have

$$x - \Delta x \le \text{the exact radius} \le x + \Delta x.$$

If Δx is negative, we may use $|\Delta x|$ in place of Δx. A cross-sectional view of the balloon, indicating the possible error Δx, is shown in Figure 2.42. If the volume V of the balloon is calculated using the measured value x, then $V = \frac{4}{3}\pi x^3$.

Figure 2.42



Let ΔV be the change in V that corresponds to Δx. We may interpret ΔV as *the error in the calculated volume* caused by the error Δx. We approximate ΔV by means of dV as follows:

$$\Delta V \approx dV = \frac{dV}{dx}(\Delta x) = 4\pi x^2 \Delta x$$

Finally, we substitute specific values for x and Δx. If $x = 12$ and if $\Delta x = \pm 0.06$, then

$$dV = 4\pi(12^2)(\pm 0.06) = \pm(34.56)\pi \approx \pm 109.$$

Thus the maximum error in the calculated volume due to the error in measurement of the radius is approximately ± 109 in^3.

The radius of the balloon in Example 5 was measured as 12 in., with a maximum error of ± 0.06 in. The maximum error is also referred to as the **absolute change**. The ratio of ± 0.06 to 12 is called the *relative error* in the measurement of the radius. We may also refer to the relative error as the **average error** or the **relative change**. Thus, for Example 5,

$$\text{relative error} = \frac{\pm 0.06}{12} = \pm 0.005.$$

The significance of this number is that the error in measurement of the radius is, *on average*, ± 0.005 inch per inch. The *percentage error* (*change*) is defined as the average error (change) multiplied by 100%. In this example,

$$\text{percentage error} = (\pm 0.005)(100\%) = \pm 0.5\%.$$

The general definition of these concepts follows.

Definition 2.36

Let $y = f(x)$ and suppose that y changes from y_0 to y_1 as x changes from x_0 to x_1. We can describe the change in y as follows:

	Exact value	Approximate value
Absolute change (error)	$\Delta y = y_1 - y_0$	$dy = f'(x_0)\,\Delta x$
Relative change (error)	$\dfrac{\Delta y}{y_0}$	$\dfrac{dy}{y_0}$
Percentage change (error)	$\dfrac{\Delta y}{y_0} \times 100\%$	$\dfrac{dy}{y_0} \times 100\%$

EXAMPLE ■ 6 The radius of a spherical balloon is measured as 12 in., with a maximum error in measurement of ± 0.06 in. Approximate the relative error and the percentage error for the calculated value of the volume.

SOLUTION As in Figure 2.42, let x denote the measured radius of the balloon and Δx the maximum error in x. Let V denote the calculated volume and ΔV the error in V caused by Δx. Applying Definition (2.36) to the volume $V = \frac{4}{3}\pi x^3$ yields

$$\text{relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi x^2 \Delta x}{\frac{4}{3}\pi x^3} = \frac{3\Delta x}{x}.$$

For the special case $x = 12$ and $\Delta x = \pm 0.06$, we obtain

$$\text{relative error} \approx \frac{3(\pm 0.06)}{12} = \pm 0.015.$$

From Definition (2.36),

$$\text{percentage error} \approx (\pm 0.015) \times (100\%) = \pm 1.5\%.$$

Thus, *on average,* there is an error of ± 0.015 in^3 per in^3 of calculated volume. Note that this leads to a percentage error of $\pm 1.5\%$ for the volume.

EXAMPLE ■ 7 A sperm whale is spotted by a merchant ship, and crew members estimate its length L to be 32 ft, with a possible error of ± 2 ft. Whale research has shown that the weight W (in metric tons) is related to L by the formula $W = 0.000137L^{3.18}$. Use differentials to approximate

(a) the error in estimating the weight of the whale (to the nearest tenth of a metric ton)

(b) the relative error and the percentage error

SOLUTION Let ΔL denote the error in the estimation of L, and let ΔW be the corresponding error in the calculated value of W. This error may be approximated by dW.

(a) Applying Definition (2.34) yields

$$\Delta W \approx dW = (0.000137)(3.18)L^{2.18}\,\Delta L.$$

Substituting $L = 32$ and $\Delta L = \pm 2$, we obtain

$$\Delta W \approx (0.000137)(3.18)(32)^{2.18}(\pm 2) \approx \pm 1.7 \text{ metric tons}.$$

(b) By Definition (2.36),

$$\text{relative error} = \frac{\Delta W}{W} \approx \frac{dW}{W} = \frac{(0.000137)(3.18)L^{2.18}\,\Delta L}{(0.000137)L^{3.18}} = \frac{3.18\,\Delta L}{L}.$$

Substituting $\Delta L = \pm 2$ and $L = 32$, we have

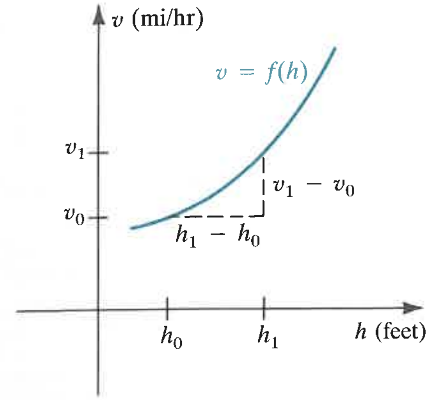
$$\text{relative error} \approx \frac{3.18(\pm 2)}{32} \approx \pm 0.20.$$

By Definition (2.36),

$$\text{percentage error} \approx (\pm 0.20) \times (100\%) = \pm 20\%.$$

Estimates of vertical wind shear are of great importance to pilots during take-offs and landings. If we assume that the wind speed v at a height h above the ground is given by $v = f(h)$, where f is a differentiable

Figure 2.43



function, then **vertical (scalar) wind shear** is defined as dv/dh (the instantaneous rate of change of v with respect to h). Since it is impossible to know the wind speed v at every height h, the wind shear must be estimated by using only a finite number of function values. Consider the situation illustrated in Figure 2.43, where we know only the wind speeds v_0 and v_1 at heights h_0 and h_1, respectively. An estimate of wind shear at height h_1 may be obtained by using the approximation formula

$$\left.\frac{dv}{dh}\right]_{h=h_1} \approx \frac{v_1 - v_0}{h_1 - h_0}.$$

The empirical relation

$$\left(\frac{v_0}{v_1}\right) = \left(\frac{h_0}{h_1}\right)^P$$

may also be employed, where the exponent P is determined by observation and depends on many factors. For strong winds, the value $P = \frac{1}{7}$ is sometimes used.

EXAMPLE ■ 8 Suppose that at a height of 20 ft above the ground the wind speed is 28 mi/hr. On the basis of the preceding discussion (with $P = \frac{1}{7}$), estimate the vertical wind shear 200 ft above the ground.

SOLUTION Using the notation of the preceding discussion, we let

$$h_0 = 20, \quad v_0 = 28, \quad \text{and} \quad h_1 = 200.$$

Solving $(v_0/v_1) = (h_0/h_1)^P$ for v_1 and then substituting values, we obtain

$$v_1 = v_0\left(\frac{h_1}{h_0}\right)^P = 28\left(\frac{200}{20}\right)^{1/7} \approx 39 \text{ mi/hr}.$$

At $h_1 = 200$,

$$\left.\frac{dv}{dh}\right]_{h=h_1} \approx \frac{v_1 - v_0}{h_1 - h_0} \approx \frac{39 - 28}{200 - 20} \approx 0.06.$$

Thus, at a height of 200 ft, the vertical wind shear is approximately 0.06 (mi/hr)/ft, which is a common value. Wind shear values greater than 0.1 are considered high.

EXERCISES 2.8

Exer. 1–8: Use a linear approximation to estimate $f(b)$ if the independent variable changes from a to b.

1 $f(x) = 4x^5 - 6x^4 + 3x^2 - 5$; $a = 1$, $b = 1.03$

2 $f(x) = -3x^3 + 8x - 7$; $a = 4$, $b = 3.96$

3 $f(x) = x^4$; $a = 1$, $b = 0.98$

4 $f(x) = x^4 - 3x^3 + 4x^2 - 5$; $a = 2$, $b = 2.01$

5 $f(\theta) = 2\sin\theta + \cos\theta$; $a = 30°$, $b = 27°$

6 $f(\phi) = \csc\phi + \cot\phi$; $a = 45°$, $b = 46°$

7 $f(\alpha) = \sec\alpha$; $a = 60°$, $b = 62°$

8 $f(\beta) = \tan\beta$; $a = 30°$, $b = 28°$

Exer. 9–12: (a) Find general formulas for Δy and dy. (b) If, for the given values of a and Δx, x changes from a to $a + \Delta x$, find the values of Δy and dy.

9 $y = 2x^2 - 4x + 5$; $a = 2$, $\Delta x = -0.2$

10 $y = x^3 - 4$; $a = -1$, $\Delta x = 0.1$

11 $y = 1/x^2$; $a = 3$, $\Delta x = 0.3$

12 $y = \dfrac{1}{2+x}$; $a = 0$, $\Delta x = -0.03$

Exer. 13–18: Find (a) Δy, (b) dy, and (c) $dy - \Delta y$.

13 $y = 4 - 9x$ 14 $y = 7x + 12$

15 $y = 3x^2 + 5x - 2$ 16 $y = 4 - 7x - 2x^2$

17 $y = 1/x$ 18 $y = 1/x^2$

c 19 (a) If $f(x) = \sin(\tan x - 1)$, find an (approximate) equation of the tangent line to the graph of f at $(2.5, f(2.5))$, using Exercise 53 in Section 2.2.

(b) Use the equation found in part (a) to approximate $f(2.6)$.

(c) Use (2.35) with $x = 2.5$ to approximate $f(2.6)$.

(d) Compare the two approximations in parts (b) and (c).

c 20 (a) If $f(x) = x^3 + 3x^2 - 2x + 5$, find an (approximate) equation of the tangent line to the graph of f at $(0.4, f(0.4))$.

(b) Use the equation found in part (a) to approximate $f(0.43)$.

(c) Use (2.35) with $x = 0.4$ to approximate $f(0.43)$.

(d) Compare the two approximations in parts (b) and (c).

Exer. 21–26: (a) Use differentials to approximate the value.
c (b) Compare the approximation in part (a) with the result obtained from evaluating the number with a calculator.

21 $\sqrt[3]{65}$ (*Hint:* Let $y = \sqrt[3]{x}$.) 22 $\sqrt[5]{35}$

23 $7^{2/3}$ 24 $1/\sqrt{50}$

25 $\cos 59°$ 26 $\tan(\pi/4 + 0.05)$

Exer. 27–30: Let x denote a measurement with a maximum error of Δx. Use differentials to approximate the relative error and the percentage error for the calculated value of y.

27 $y = 3x^4$; $x = 2$, $\Delta x = \pm 0.01$

28 $y = x^3 + 5x$; $x = 1$, $\Delta x = \pm 0.1$

29 $y = 4\sqrt{x} + 3x$; $x = 4$, $\Delta x = \pm 0.2$

30 $y = 6\sqrt[3]{x}$; $x = 8$, $\Delta x = \pm 0.03$

31 If $A = 3x^2 - x$, find dA for $x = 2$ and $dx = 0.1$.

32 If $P = 6t^{2/3} + t^2$, find dP for $t = 8$ and $dt = 0.2$.

33 If $y = 4x^3$ and the maximum percentage error in x is $\pm 15\%$, approximate the maximum percentage error in y.

34 If $z = 40\sqrt[5]{w^2}$ and the maximum relative error in w is ± 0.08, approximate the maximum relative error in z.

35 If $A = 15\sqrt[3]{s^2}$ and the allowable maximum relative error in A is to be ± 0.04, determine the allowable maximum relative error in s.

36 If $S = 10\pi x^2$ and the allowable maximum percentage error in S is to be $\pm 10\%$, determine the allowable maximum percentage error in x.

37 The radius of a circular manhole cover is estimated to be 16 in., with a maximum error in measurement of ± 0.06 in. Use differentials to estimate the maximum error in the calculated area of one side of the cover. Approximate the relative error and the percentage error.

38 The length of a side of a square floor tile is estimated as 1 ft, with a maximum error in measurement of $\pm \frac{1}{16}$ in. Use differentials to estimate the maximum error in the calculated area. Approximate the relative error and the percentage error.

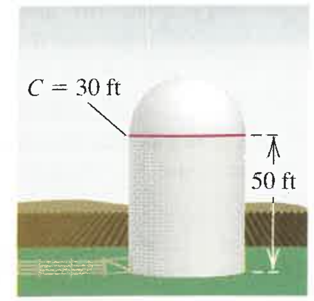
39 Use differentials to approximate the increase in volume of a cube if the length of each edge changes from 10 in. to 10.1 in. What is the absolute change in volume?

40 A spherical balloon is being inflated with gas. Use differentials to approximate the increase in surface area of the balloon if the diameter changes from 2 ft to 2.02 ft.

41 One side of a house has the shape of a square surmounted by an equilateral triangle. If the length of the base is measured as 48 ft, with a maximum error in measurement of ± 1 in., calculate the area of the side. Use differentials to estimate the maximum error in the calculation. Approximate the relative error and the percentage error.

42 Small errors in measurements of dimensions of large containers can have a marked effect on calculated volumes. A silo has the shape of a right circular cylinder surmounted by a hemisphere (see figure on the following page). The altitude of the cylinder is exactly 50 ft. The circumference of the base is measured as 30 ft, with a maximum error in measurement of ± 6 in. Calculate the volume of the silo from these measurements, and use differentials to estimate the maximum error in the calculation. Approximate the relative error and the percentage error.

Exercise 42



43 As sand leaks out of a container, it forms a conical pile whose altitude is always the same as the radius. If, at a certain instant, the radius is 10 cm, use differentials to approximate the change in radius that will increase the volume of the pile by 2 cm^3.

44 An isosceles triangle has equal sides of length 12 in. If the angle θ between these sides is increased from 30° to 33°, use differentials to approximate the change in the area of the triangle.

45 Newton's law of gravitation states that the force F of attraction between two particles having masses m_1 and m_2 is given by $F = Gm_1m_2/s^2$, where G is a constant and s is the distance between the particles. If $s = 20$ cm, use differentials to approximate the change in s that will increase F by 10%.

46 The formula $T = 2\pi\sqrt{l/g}$ relates the length l of a pendulum to its period T, where g is a gravitational constant. What percentage change in the length corresponds to a 30% increase in the period?

47 Constriction of arterioles is a cause of high blood pressure. It has been verified experimentally that as blood flows through an arteriole of fixed length, the pressure difference between the two ends of the arteriole is inversely proportional to the fourth power of the radius. If the radius of an arteriole decreases by 10%, use differentials to find the percentage change in the pressure difference.

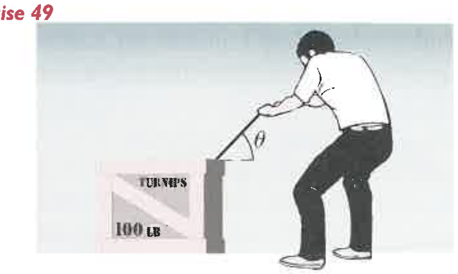
48 The electrical resistance R of a wire is directly proportional to its length and inversely proportional to the square of its diameter. If the length is fixed, how accurately must the diameter be measured (in terms of percentage error) to keep the percentage error in R between −3% and 3%?

49 If an object of weight W pounds is pulled along a horizontal plane by a force applied to a rope that is attached to the object and if the rope makes an angle θ with the horizontal, then the magnitude of the force is given by

$$F(\theta) = \frac{\mu W}{\mu \sin\theta + \cos\theta},$$

where μ is a constant called the *coefficient of friction.* Suppose that a 100-lb box is being pulled along a floor and that $\mu = 0.2$ (see figure). If θ is changed from 45° to 46°, use differentials to approximate the change in the force that must be applied.

Exercise 49



50 It will be shown in Chapter 11 that if a projectile is fired from a cannon with an initial velocity v_0 and at an angle α to the horizontal, then its maximum height h and range R are given by

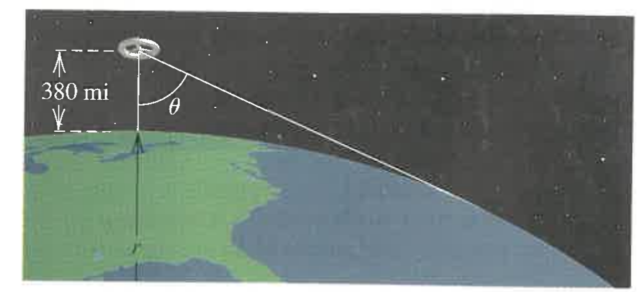
$$h = \frac{v_0^2 \sin^2\alpha}{2g} \quad \text{and} \quad R = \frac{2v_0^2 \sin\alpha\cos\alpha}{g}.$$

Suppose $v_0 = 100$ ft/sec and $g = 32$ ft/sec^2. If α is increased from 30° to 30.5°, use differentials to estimate the changes in h and R.

51 At a point 20 ft from the base of a flagpole, the angle of elevation of the top of the pole is measured as 60°, with a possible error of $\pm 0.25°$. Use differentials to approximate the error in the calculated height of the pole.

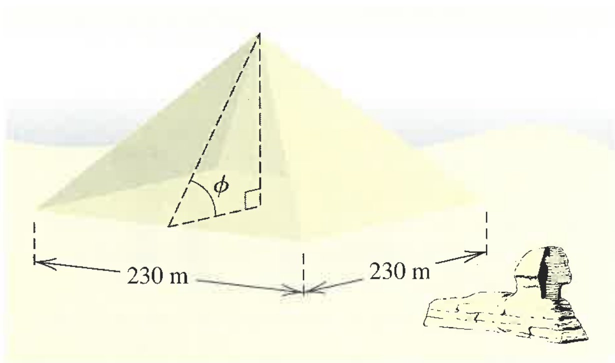
52 A spacelab circles the earth at an altitude of 380 mi. When an astronaut views the horizon, the angle θ shown in the figure is 65.8°, with a possible maximum error of $\pm 0.5°$. Use differentials to approximate the error in the astronaut's calculation of the radius of the earth.

Exercise 52



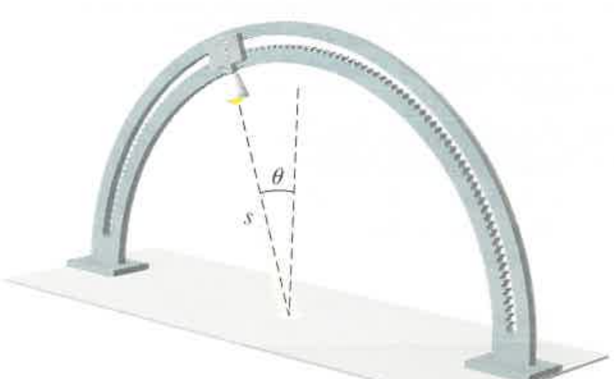
53 The Great Pyramid of Egypt has a square base of 230 m (see figure). To estimate the height h of this massive structure, an observer stands at the midpoint of one of the sides and views the apex of the pyramid. The angle of elevation ϕ is found to be 52°. How accurate must this angle measurement be to keep the error in h between -1 m and 1 m?

Exercise 53



54 As a point light source moves on a semicircular track, as shown in the figure, the illuminance E on the surface is inversely proportional to the square of the distance s from the source and is directly proportional to the cosine of the angle θ between the direction of light flow and the normal to the surface. If θ is decreased from 21° to 20° and s is constant, use differentials to approximate the percentage increase in illuminance.

Exercise 54



55 Boyle's law states that if the temperature is constant, the pressure p and the volume v of a confined gas are related by the formula $pv = c$, where c is a constant or, equivalently, by $p = c/v$ with $v \neq 0$. Show that dp and Δv are related by the formula $p\,\Delta v + v\,dp = 0$.

56 In electrical theory, Ohm's law states that $I = V/R$, where I is the current (in amperes), V is the electromotive force (in volts), and R is the resistance (in ohms). Show that dI and ΔR are related by the formula $R\,dI + I\,\Delta R = 0$.

57 The area A of a square of side s is given by $A = s^2$. If s increases by an amount Δs, illustrate dA and $\Delta A - dA$ geometrically.

58 The volume V of a cube of edge s is given by $V = s^3$. If s increases by an amount Δs, illustrate dV and $\Delta V - dV$ geometrically.

59 The curved surface area S of a right circular cone having altitude h and base radius r is given by $S = \pi r\sqrt{r^2 + h^2}$. For a certain cone, $r = 6$ cm. The altitude is measured as 8 cm, with a maximum error in measurement of ± 0.1 cm.

(a) Calculate S from the measurements and use differentials to estimate the maximum error in the calculation.

(b) Approximate the percentage error.

60 The period T of a simple pendulum of length l may be calculated by means of the formula $T = 2\pi\sqrt{l/g}$, where g is a gravitational constant. Use differentials to approximate the change in l that will increase T by 1%.

61 Suppose that $3x^2 - x^2y^3 + 4y = 12$ determines a differentiable function f such that $y = f(x)$. If $f(2) = 0$, use differentials to approximate the change in $f(x)$ if x changes from 2 to 1.97.

62 Suppose that $x^3 + xy + y^4 = 19$ determines a differentiable function f such that $y = f(x)$. If $P(1, 2)$ is a point on the graph of f, use differentials to approximate the y-coordinate b of the point $Q(1.1, b)$ on the graph.

c **63** Suppose that $x^2 + xy^3 = 4.0764$ determines a differentiable function f such that $y = f(x)$.

(a) If $P(1.2, 1.3)$ and $Q(1.23, b)$ are on the graph of f, use (2.35) to approximate b.

(b) Apply the method in part (a), using $Q(1.23, b)$ to approximate the y-coordinate of $R(1.26, c)$. (This process, called *Euler's method,* can be repeated to approximate additional points on the graph.)

c **64** Suppose that $\sin x + y\cos y = -2.395$ determines a differentiable function f such that $y = f(x)$.

(a) If $P(2.1, 3.3)$ is an approximation to a point on the graph of f, use (2.35) to approximate the y-coordinate of $Q(2.12, b)$.

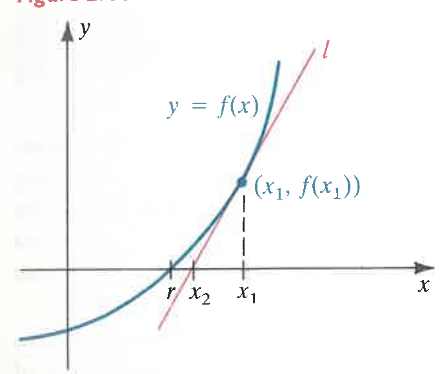
(b) Apply the method in part (a), using $Q(2.12, b)$ to approximate the y-coordinate of $R(2.14, c)$.

2.9 NEWTON'S METHOD

Many problems and applications in mathematics require solving equations. We can write equations involving one variable in the form $f(x) = 0$ for some function f. The cubic equation $2x^3 + x^2 - 16x - 15 = 0$, for example, has this form, where $f(x) = 2x^3 + x^2 - 16x - 15$. Recall that a *zero* of a function f is a value r such that $f(r) = 0$. The zeros of the function are also called the *roots* of the equation.

In this section, we explore methods for finding a real zero of a function f. One such method for a polynomial or a rational function is to *factor* the numerator. Since the cubic polynomial $f(x) = 2x^3 + x^2 - 16x - 15$ factors as $f(x) = (2x + 5)(x + 1)(x - 3)$, the only real numbers r satisfying $f(r) = 0$ are $r = -2.5$, -1, and 3. When f is a quadratic polynomial, the quadratic formula gives the zeros (real and complex). For many functions, it is not possible to use factoring or an exact formula to find the zeros. In such cases, we may be able only to approximate values of the zeros. In this section, we focus on *Newton's method* for carrying out such approximations

Figure 2.44



Newton's method for approximating a zero r of a differentiable function f is based on the idea that the tangent line stays close to the curve near the point of tangency. With this method, we begin with some approximation x_1 for the zero and consider the tangent line l to the graph of $y = f(x)$ at $(x_1, f(x_1))$ (see Figure 2.44). The tangent line and the graph of f should intersect the x-axis near each other since the tangent line remains close to the graph of f. Thus, we can approximate a zero for f by finding a zero for the tangent line. Because the equation of the tangent line is linear, it is easy to determine where it has a zero.

A graph of the function is often very helpful for suggesting a "good" first approximation x_1. Lacking a graphing utility, we might evaluate the function several times. The intermediate value theorem (1.26) guarantees a zero in any interval where the values of a continuous function at the endpoints have opposite signs.

Next, we consider the tangent line l to the graph of f at the point $(x_1, f(x_1))$. If x_1 is sufficiently close to r, then, as illustrated in Figure 2.44, the x-intercept x_2 of l should be a better approximation to r. Since the slope of l is $f'(x_1)$, an equation of the tangent line is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

The x-intercept x_2 of l corresponds to the point $(x_2, 0)$ on l, so

$$0 - f(x_1) = f'(x_1)(x_2 - x_1).$$

If $f'(x_1) \neq 0$, the preceding equation is equivalent to

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Taking x_2 as a second approximation to r, we may repeat the process by using the tangent line at $(x_2, f(x_2))$. If $f'(x_2) \neq 0$, a third approximation x_3 is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

We continue the process until the desired degree of accuracy is obtained. This technique of successive approximations of real zeros is called **Newton's method**.

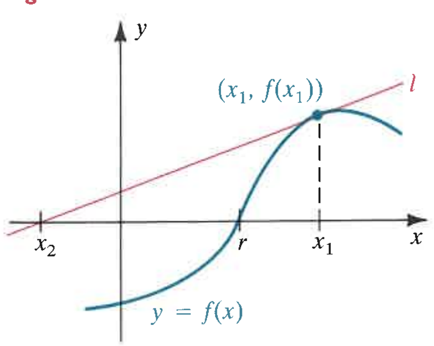
***Newton's Method* 2.37**

Let f be a differentiable function, and suppose r is a real zero of f. If x_n is an approximation to r, then the next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

provided $f'(x_n) \neq 0$.

Figure 2.45

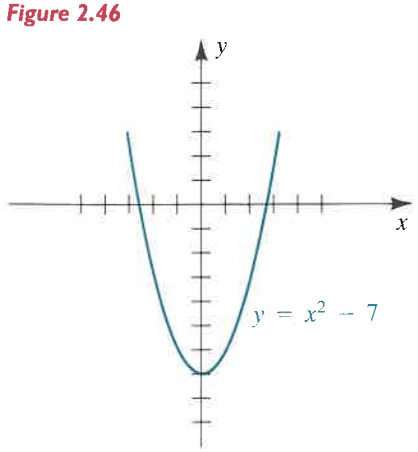


Newton's method does not guarantee that x_{n+1} is a better approximation to r than x_n for every n. In particular, we must be careful in choosing the first approximation x_1. If x_1 is not sufficiently close to r, it is possible for the second approximation x_2 to be worse than x_1. Figure 2.45 shows such a case. It is clear that we should not choose a number x_n such that $f'(x_n)$ is close to 0, for then the tangent line l is almost horizontal.

If $x_1 \approx r$, f'' is continuous near r, and $f'(r) \neq 0$, then we can show that the approximations $x_2, x_3, \ldots$ approach r rapidly, with the number of decimal places of accuracy nearly doubling with each successive approximation. If $f(x)$ has a factor $(x - r)^k$ with $k > 1$ and if $x_n \neq r$ for each n, then the approximations approach r more slowly, because $f'(r) = 0$.

We will use the following rule when applying Newton's method: *If an approximation to k decimal places is required, we continue the process until two consecutive approximations give exactly the same k decimal places.* If we use a computer or calculator, we cannot go beyond the precision of the machine. Working by hand, we can calculate each approximation x_2, $x_3, \ldots$ to at least k decimal places. The following examples illustrate the process.

Figure 2.46



EXAMPLE 1 Use Newton's method to approximate $\sqrt{7}$ to five decimal places.

SOLUTION This problem is equivalent to that of approximating the positive real zero r of $f(x) = x^2 - 7$. Figure 2.46 shows the graph of f. Since $f(2) = -3$ and $f(3) = 2$, it follows from the continuity of f that $2 < r < 3$. From the graph, we believe that f has only one zero in the open interval $(2, 3)$. If x_n is any approximation to r, then Newton's method (2.37) gives the next approximation x_{n+1} as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 7}{2x_n}.$$

Let us choose $x_1 = 2.5$ as a first approximation. Using the formula for x_{n+1} with $n = 1$ gives us

$$x_2 = 2.5 - \frac{(2.5)^2 - 7}{2(2.5)} = 2.65.$$

Again using the formula (with $n = 2$), we obtain the next approximation,

$$x_3 = 2.65 - \frac{(2.65)^2 - 7}{2(2.65)} \approx 2.64575.$$

Repeating the procedure (with $n = 3$) yields

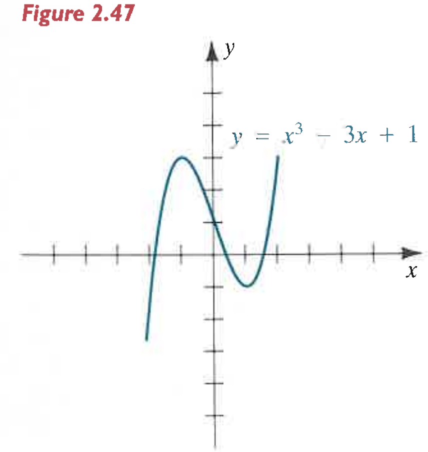
$$x_4 = 2.64575 - \frac{(2.65475)^2 - 7}{2(2.64575)} \approx 2.64575.$$

Since two consecutive values of x_n are the same (to the desired degree of accuracy), we have $\sqrt{7} \approx 2.64575$. Note that $(2.64575)^2 = 6.9999930625$.

NOTE A 13-digit calculator gives $\sqrt{7} \approx 2.645751311065$. Some early computers used a procedure very similar to this one to calculate square roots, but even faster algorithms are now used.

EXAMPLE 2 Find the largest positive real root of $x^3 - 3x + 1 = 0$ to ten decimal places.

Figure 2.47



SOLUTION If we let $f(x) = x^3 - 3x + 1$, then the problem is equivalent to finding the largest real zero of f. Figure 2.47 shows the graph of f. Note that f has three real zeros. We wish to find the zero that lies between 1 and 2. Since $f'(x) = 3x^2 - 3$, the formula for x_{n+1} in Newton's method is

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}.$$

Referring to the graph, we take $x_1 = 1.5$ as a first approximation. On a calculator with which we can store a calculation in a variable memory labeled [X] using an operation symbolized by [→], we first store 1.5 in [X]. Then we create a command line of the form

```
x-(x^3-3x+1)/(3x^2-3)→x
```

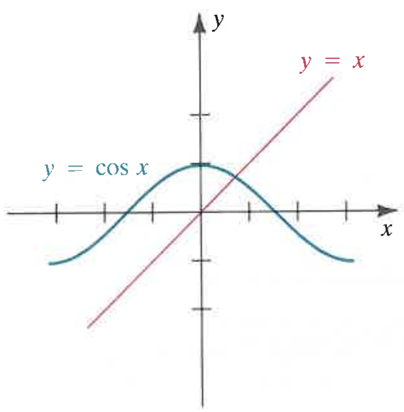
Repeated execution of this line produces the successive approximations of Newton's method:

$$x_2 \approx 1.5333333333$$
$$x_3 \approx 1.5320906433$$
$$x_4 \approx 1.5320888862$$
$$x_5 \approx 1.5320888862$$

Thus the desired approximation is 1.5320888862. The remaining two real roots can be approximated in similar fashion (see Exercise 19).

EXAMPLE ■ 3 Approximate the real root of $x - \cos x = 0$.

Figure 2.48



SOLUTION We wish to find a value of x such that $\cos x = x$. This value coincides with the x-coordinate of the point of intersection of the graphs of $y = \cos x$ and $y = x$. It appears from Figure 2.48 that $x_1 = 0.7$ is a reasonable first approximation. (Note that the figure also indicated that there is only one real root of the given equation.) If we let $f(x) = x - \cos x$, then $f'(x) = 1 + \sin x$ and the formula in Newton's method is

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}.$$

Although we can compute several successive approximations on a calculator, if we use a computer with greater precision, we can see the approximate doubling of the correct digits with each additional step of Newton's method. The following calculations were performed on a computer with 33-digit precision:

$$x_2 \approx \underline{0.739}436497848058195428715911443437$$
$$x_3 \approx \underline{0.739085}160465107398602342091722227$$
$$x_4 \approx \underline{0.73908513321516}0805616473437040986$$
$$x_5 \approx \underline{0.73908513321516064165531208767387}9$$
$$x_6 \approx \underline{0.739085133215160641655312087673873}$$
$$x_7 \approx \underline{0.739085133215160641655312087673873}$$

If the results here are *rounded* to the underscored number of digits, then each result will be correct to that number of digits.

There is seldom a need for 33 decimal places in an answer to a problem in the physical world. Still it is comforting to know that Newton's method often works this well. A final warning about implementing this procedure on a computer is appropriate here. Although we *hope* to continue the process until two successive results agree, it may not happen. To avoid continuing the process indefinitely, we can write code asking for a finite number of approximations. For instance, the code used for Example 3 computed $x_2, x_3, \ldots, x_{10}$. Since the last seven approximations were exactly the same, only two of them are given in the list.

There are many other ways to approximate zeros of functions. Some of these are explored in the exercises of this section and in later chapters. Since approximating zeros of a function is equivalent to the important problem of solving equations in one variable, many calculators and most computer algebra systems provide a simple command to carry out this task. Explore the manual for the device you are using to understand the exact syntax for the "solver" routine. You can be confident that an approximation very much like Newton's method is being used.

EXERCISES 2.9

c **For Exercises 1–26, use Newton's method to at least the accuracy indicated.**

Exer. 1–4: (a) Approximate to four decimal places. (b) Compare the approximation to one obtained from a calculator.

1 $\sqrt{11}$ **2** $\sqrt{57}$

3 $\sqrt[3]{2}$ **4** $\sqrt[5]{3}$

Exer. 5–8: Approximate, to four decimal places, the root of the equation that lies in the interval.

5 $x^4 + 2x^3 - 5x^2 + 1 = 0$; $[1, 2]$

6 $x^4 - 5x^2 + 2x - 5 = 0$; $[2, 3]$

7 $x^5 + x^2 - 9x - 3 = 0$; $[-2, -1]$

8 $\sin x + x \cos x = \cos x$; $[0, 1]$

Exer. 9–10: Approximate the largest zero of $f(x)$ to four decimal places.

9 $f(x) = x^4 - 11x^2 - 44x - 24$

10 $f(x) = x^3 - 36x - 84$

Exer. 11–14: Approximate the real root to two decimal places.

11 The root of $x^3 + 5x - 3 = 0$

12 The largest root of $2x^3 - 4x^2 - 3x + 1 = 0$

13 The positive root of $2x - 3 \sin x = 0$

14 The root of $\cos x + x = 2$

Exer. 15–22: Approximate all real roots of the equation to two decimal places.

15 $x^4 = 125$ **16** $10x^2 - 1 = 0$

17 $x^4 - x - 2 = 0$ **18** $x^5 - 2x^2 + 4 = 0$

19 $x^3 - 3x + 1 = 0$ **20** $x^3 + 2x^2 - 8x - 3 = 0$

21 $2x - 5 - \sin x = 0$ **22** $x^2 - \cos 2x = 0$

Exer. 23–26: Approximate, to two decimal places, the x-coordinates of the points of intersection of the graphs of the equations.

23 $y = x^2$; $y = \sqrt{x + 3}$ **24** $y = x^3$; $y = 7 - x^2$

25 $y = \cos \frac{1}{2}x$; $y = 9 - x^2$ **26** $y = \sin 2x$; $y = 6x - 6$

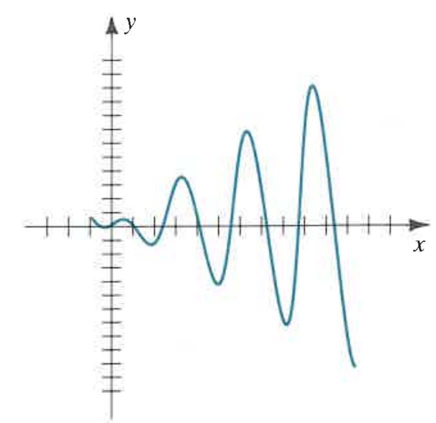
27 Approximations to π may be obtained by applying Newton's method to $f(x) = \sin x$ and letting $x_1 = 3$.

(a) Find the first five approximations to π.

(b) What happens to the approximations if $x_1 = 6$?

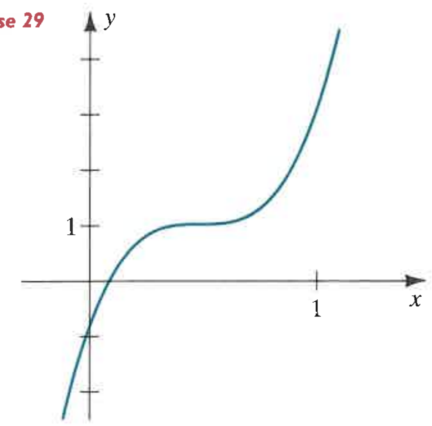
28 A dramatic example of the phenomenon of *resonance* occurs when a singer adjusts the pitch of her voice to shatter a wine glass. Functions having the form $f(x) = ax \cos bx$ occur in the mathematical analysis of such vibrations. Shown in the figure is a graph of $f(x) = x \cos 2x$. Use Newton's method to approximate, to three decimal places, the value of x that lies between 1 and 2 such that $f'(x) = 0$.

Exercise 28



29 The graph of a function f is shown. Explain why Newton's method fails to approximate the zero of f if $x_1 = 0.5$.

Exercise 29



$f(x) = 16x^3 - 24x^2 + 12x - 1$

30 If $f(x) = x^{1/3}$, show that Newton's method fails for any first approximation $x_1 \neq 0$.

c **Exer. 31–32: The functions f and g have a zero at $x = 1$. (a) Let $x_1 = 1.1$ in formula (2.37), and find x_2, x_3, and x_4 for each function. (b) Why are the approximations for the zero of g more accurate than those for the zero of f?**

31 $f(x) = (x-1)^3(x^2 - 3x + 7)$;
$g(x) = (x-1)(x^2 - 3x + 7)$

32 $f(x) = (x-1)^2\sqrt{x+7}$;
$g(x) = (x-1)\sqrt{x+7}$

c **Exer. 33–34: If it is difficult to calculate $f'(x)$, the formula in (2.37) may be replaced by**

$$x_{n+1} = x_n - \frac{f(x_n)}{m},$$

where

$$m = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \approx f'(x_n).$$

Two initial values, x_1 and x_2, are required to use this method (called the *secant method*). Use the secant method to approximate, to three decimal places, the zero of f that is in [0, 1].

33 $f(x) = \tan^2(\cos^2 x - x + 0.25) - 0.5x$; $x_1 = 0.5$, $x_2 = 0.55$

34 $f(x) = \dfrac{1}{\cos^2 x - x} - 5\sqrt{x}$; $x_1 = 0.4$, $x_2 = 0.5$

c **Exer. 35–36: Graph f and g on the same coordinate axes. (a) Estimate, to one decimal place, the x-coordinate x_1 of the point of intersection of the graphs. (b) Use Newton's method to approximate the x-coordinate in part (a) to two decimal places.**

35 $f(x) = \frac{1}{4}x^3 + x - 1$;
$g(x) = \sin^2 x$

36 $f(x) = x^3 - x^2 + x - 1$;
$g(x) = -x^3 - 1.1x^2 - x - 1.9$

c **37** Many of the equations we solved can be rewritten in the form $g(x) = x$. A simple way to try to approximate the solution is to find a close initial x_1 as we did for Newton's method. Then proceed to find successive approximations by using $x_{n+1} = g(x_n)$. Use this method (called a *Picard iteration*) to approximate, to five decimal places, the desired solutions.

(a) $\sin x + 1 = x$ (b) $\frac{1}{2}\cos x = x$

CHAPTER 2 REVIEW EXERCISES

Exer. 1–2: Find $f'(x)$ directly from the definition of the derivative.

1 $f(x) = \dfrac{4}{3x^2 + 2}$

2 $f(x) = \sqrt{5 - 7x}$

Exer. 3–24: Find the first derivative.

3 $f(x) = 2x^3 - 7x + 2$

4 $k(x) = \dfrac{1}{x^4 - x^2 + 1}$

5 $g(t) = \sqrt{6t + 5}$

6 $h(t) = \dfrac{1}{\sqrt{6t + 5}}$

7 $F(z) = \sqrt[3]{7z^2 - 4z + 3}$

8 $f(w) = \sqrt[5]{3w^2}$

9 $G(x) = \dfrac{6}{(3x^2 - 1)^4}$

10 $H(x) = \dfrac{(3x^2 - 1)^4}{6}$

11 $F(r) = (r^2 - r^{-2})^{-2}$

12 $h(z) = \left[(z^2 - 1)^5 - 1\right]^5$

13 $g(x) = \sqrt[5]{(3x + 2)^4}$

14 $P(x) = (x + x^{-1})^2$

15 $r(s) = \left(\dfrac{8s^2 - 4}{1 - 9s^3}\right)^4$

16 $g(w) = \dfrac{(w-1)(w-3)}{(w+1)(w+3)}$

17 $F(x) = (x^6 + 1)^5(3x + 2)^3$

18 $k(z) = [z^2 + (z^2 + 9)^{1/2}]^{1/2}$

19 $k(s) = (2s^2 - 3s + 1)(9s - 1)^4$

20 $p(x) = \dfrac{2x^4 + 3x^2 - 1}{x^2}$

21 $f(x) = 6x^2 - \dfrac{5}{x} + \dfrac{2}{\sqrt[3]{x^2}}$

22 $F(t) = \dfrac{5t^2 - 7}{t^2 + 2}$

23 $f(w) = \sqrt{\dfrac{2w + 5}{7w - 9}}$

24 $S(t) = \sqrt{t^2 + t + 1}\sqrt[3]{4t - 9}$

Exer. 25–32: Find the limit, if it exists.

25 $\displaystyle\lim_{x\to 0} \frac{x^2}{\sin x}$

26 $\displaystyle\lim_{x\to 0} \frac{x^2 + \sin^2 x}{4x^2}$

27 $\displaystyle\lim_{x\to 0} \frac{\sin^2 x + \sin 2x}{3x}$

28 $\displaystyle\lim_{x\to 0} \frac{2 - \cos x}{1 + \sin x}$

29 $\displaystyle\lim_{x\to 0} \frac{2\cos x + 3x - 2}{5x}$

30 $\displaystyle\lim_{x\to 0} \frac{3x + 1 - \cos^2 x}{\sin x}$

31 $\displaystyle\lim_{x\to 0} \frac{x \sin x}{1 - \cos x}$

32 $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{2x}$

Exer. 33–50: Find the first derivative.

33 $g(r) = \sqrt{1 + \cos 2r}$

34 $g(z) = \csc\left(\dfrac{1}{z}\right) + \dfrac{1}{\sec z}$

35 $f(x) = \sin^2(4x^3)$

36 $H(t) = (1 + \sin 3t)^3$

37 $h(x) = (\sec x + \tan x)^5$

38 $K(r) = \sqrt[3]{r^3 + \csc 6r}$

39 $f(x) = x^2 \cot 2x$

40 $P(\theta) = \theta^2 \tan^2(\theta^2)$

41 $K(\theta) = \dfrac{\sin 2\theta}{1 + \cos 2\theta}$

42 $g(v) = \dfrac{1}{1 + \cos^2 2v}$

43 $g(x) = (\cos\sqrt[3]{x} - \sin\sqrt[3]{x})^3$

44 $f(x) = \dfrac{x}{2x + \sec^2 x}$

45 $G(u) = \dfrac{\csc u + 1}{\cot u + 1}$

46 $k(\phi) = \dfrac{\sin\phi}{\cos\phi - \sin\phi}$

47 $F(x) = \sec 5x \tan 5x \sin 5x$

48 $H(z) = \sqrt{\sin\sqrt{z}}$

49 $g(\theta) = \tan^4(\sqrt[4]{\theta})$

50 $f(x) = \csc^3 3x \cot^2 3x$

Exer. 51–56: Assuming that the equation determines a differentiable function f such that $y = f(x)$, find y'.

51 $5x^3 - 2x^2y^2 + 4y^3 - 7 = 0$

52 $3x^2 - xy^2 + y^{-1} = 1$

53 $\dfrac{\sqrt{x} + 1}{\sqrt{y} + 1} = y$

54 $y^2 - \sqrt{xy} + 3x = 2$

55 $xy^2 = \sin(x + 2y)$

56 $y = \cot(xy)$

Exer. 57–58: Find equations of the tangent line and the normal line to the graph of f at P.

57 $y = 2x - \dfrac{4}{\sqrt{x}}$; $P(4, 6)$

58 $x^2y - y^3 = 8$; $P(-3, 1)$

59 Find the x-coordinates of all points on the graph of the equation $y = 3x - \cos 2x$ at which the tangent line is perpendicular to the line $2x + 4y = 5$.

60 If $f(x) = \sin 2x - \cos 2x$ for $0 \le x \le 2\pi$, find the x-coordinates of all points on the graph of f at which the tangent line is horizontal.

Exer. 61–62: Find y', y'', and y'''.

61 $y = 5x^3 + 4\sqrt{x}$

62 $y = 2x^2 - 3x - \cos 5x$

63 If $x^2 + 4xy - y^2 = 8$, find y'' by implicit differentiation.

64 If $f(x) = x^3 - x^2 - 5x + 2$, find

(a) the x-coordinates of all points on the graph of f at which the tangent line is parallel to the line through $A(-3, 2)$ and $B(1, 14)$

(b) the value of f'' at each zero of f'

65 If $y = 3x^2 - 7$, find

(a) Δy (b) dy (c) $dy - \Delta y$

66 If $y = 5x/(x^2 + 1)$, find dy and use it to approximate the change in y if x changes from 2 to 1.98. What is the exact change in y?

67 The side of an equilateral triangle is estimated to be 4 in., with a maximum error of ± 0.03 in. Use differentials to estimate the maximum error in the calculated area of the triangle. Approximate the percentage error.

68 If $s = 3r^2 - 2\sqrt{r + 1}$ and $r = t^3 + t^2 + 1$, use the chain rule to find the value of ds/dt at $t = 1$.

69 If $f(x)=2x^3+x^2-x+1$ and $g(x)=x^5+4x^3+2x$, use differentials to approximate the change in $g(f(x))$ if x changes from -1 to -1.01.

70 Use differentials to find a linear approximation of $\sqrt[3]{64.2}$.

71 Suppose f and g are functions such that $f(2)=-1$, $f'(2)=4$, $f''(2)=-2$, $g(2)=-3$, $g'(2)=2$, and $g''(2)=1$. Find the value of each of the following at $x=2$.

(a) $(2f-3g)'$ **(b)** $(2f-3g)''$ **(c)** $(fg)'$
(d) $(fg)''$ **(e)** $(f/g)'$ **(f)** $(f/g)''$

72 Refer to Exercise 85 in Section 2.5. Let f be an odd function and g an even function such that $f(3)=-3$, $f'(3)=7$, $g(3)=-3$, and $g'(3)=-5$. Find $(f\circ g)'(3)$ and $(g\circ f)'(3)$.

73 Determine where the graph of f has a vertical tangent line or a cusp.

(a) $f(x)=3(x+1)^{1/3}-4$

(b) $f(x)=2(x-8)^{2/3}-1$

74 Let $f(x)=\begin{cases}(2x-1)^3 & \text{if } x\ge 2\\ 5x^2+34x-61 & \text{if } x<2\end{cases}$

Determine if f is differentiable at 2.

75 The Stefan–Boltzmann law states that the radiant energy emitted from a unit area of a black surface is given by $R=kT^4$, where R is the rate of emission per unit area, T is the temperature (in °K), and k is a constant. If the error in the measurement of T is 0.5%, find the resulting percentage error in the calculated value of R.

76 Let V and S denote the volume and surface area, respectively, of a spherical balloon. If the diameter is 8 cm and the volume increases by 12 cm^3, use differentials to approximate the change in S.

77 A right circular cone has height $h=8$ ft, and the base radius r is increasing. Find the rate of change of its surface area S with respect to r when $r=6$ ft.

78 The intensity of illumination from a source of light is inversely proportional to the square of the distance from the source. If a student works at a desk that is a certain distance from a lamp, use differentials to find the percentage change in distance that will increase the intensity by 10%.

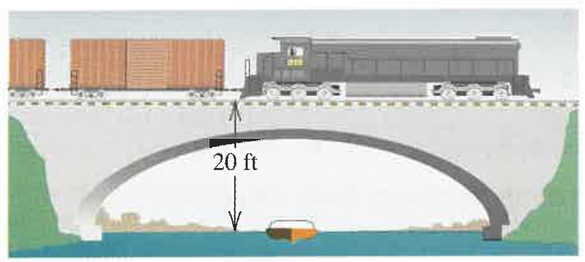
79 The ends of a horizontal water trough 10 ft long are isosceles trapezoids with lower base 3 ft, upper base 5 ft, and altitude 2 ft. If the water level is rising at a rate of $\frac{1}{4}$ in./min when the depth of the water is 1 ft, how fast is water entering the trough?

80 Two cars are approaching the same intersection along roads that run at right angles to each other. Car A is traveling at 20 mi/hr, and car B is traveling at 40 mi/hr. If, at a certain instant, A is $\frac{1}{4}$ mi from the intersection and B is $\frac{1}{2}$ mi from the intersection, find the rate at which they are approaching each other at that instant.

81 Boyle's law states that $pv=c$, where p is pressure, v is volume, and c is a constant. Find a formula for the rate of change of p with respect to v.

82 A railroad bridge is 20 ft above, and at right angles to, a river. A man in a train traveling 60 mi/hr passes over the center of the bridge at the same instant that a man in a motorboat traveling 20 mi/hr passes under the center of the bridge (see figure). How fast are the two men moving away from each other 10 sec later?

Exercise 82

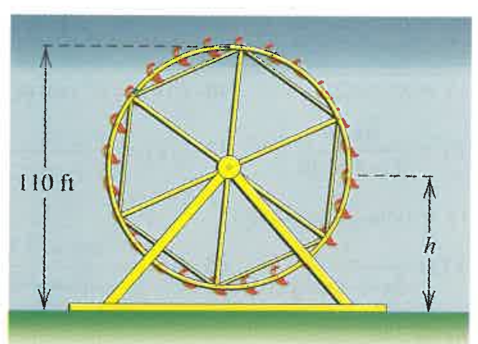


83 A large ferris wheel is 100 ft in diameter and rises 110 ft off the ground, as illustrated in the figure. Each revolution of the wheel takes 30 sec.

(a) Express the distance h of a seat from the ground as a function of time t (in seconds) if $t=0$ corresponds to a time when the seat is at the bottom.

(b) If a seat is rising, how fast is the distance changing when $h=55$ ft?

Exercise 83



84 A piston is attached to a crankshaft, as shown in the figure. The connecting rod AB has length 6 in., and the radius of the crankshaft is 2 in.

(a) If the crankshaft rotates counterclockwise 2 times per second, find formulas for the position of point A at t seconds after A has coordinates $(2,0)$.

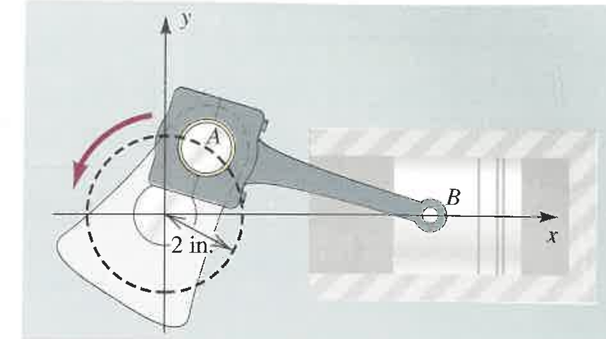
(b) Find a formula for the position of point B at time t.

(c) How fast is B moving when A has coordinates $(0,2)$?

c **85** Use Newton's method to approximate, to three decimal places, the root of the equation $\sin x - x\cos x = 0$ between π and $3\pi/2$.

c **86** Use Newton's method to approximate $\sqrt[4]{5}$ to three decimal places.

Exercise 84



EXTENDED PROBLEMS AND GROUP PROJECTS

1 How far can you proceed in determining the derivative of the function $f(x)=2^x$? Some suggestions follow.

(a) Assuming that $f(x)=2^x$ is defined and continuous for all real values of x, show that $f'(0)=\lim_{h\to 0}(2^h-1)/h$ if this limit exists.

c **(b)** Evaluate the difference quotient $(2^h-1)/h$ for a number of different values of h close to 0. Does the difference quotient appear to reach a limit as h approaches 0? What is that limit?

c **(c)** If you have a graphing calculator or access to graphing software, examine the graph of $(2^h-1)/h$ for $h\neq 0$.

(d) Suppose $\lim_{h\to 0}(2^h-1)/h$ does exist and has value M. Use the definition of the derivative to show that $f'(x)$ would equal $M(2^x)$ for all real values of x. Thus, if we can show that 2^x is differentiable at $x=0$, then we know it is differentiable at all x.

(e) Can you determine analytically what

$$\lim_{h\to 0}\frac{2^h-1}{h}$$

really is?

(f) What can you say about the differentiability of other exponential functions such as 3^x, 10^x, and $(\frac{1}{2})^x$?

(g) Write up your methods of investigation and conclusions.

2 Suppose that the function f is differentiable at every value x in an interval I, and let f' be its derivative.

(a) Give several examples of functions f for which f' is itself differentiable at every x in I.

(b) Give an example in which f' is continuous at every x in I, but not differentiable everywhere.

(c) Can you construct an example of a function f for which f' is continuous at every x in I, but has a derivative at no point in I?

(d) Find a function f such that $f'(x)=|x|$.

(e) Prove that it is impossible to have a differentiable function f with the property that

$$f'(x)=\begin{cases}0 & \text{if } x<0\\ 1 & \text{if } x\ge 0\end{cases}$$

(f) If f' is the derivative of f, can the function f' have any simple discontinuities?

3 **(a)** Show that for each rational number $r\neq -1$, there is a rational number q such that the derivative of $f(x)=(1/q)x^q$ is x^r.

(b) Show that there is no rational number q such that the function $f(x)=(1/q)x^q$ has $f'(x)=1/x$.

(c) Try to construct a function f such that $f'(x)=1/x$ for all positive values of x. Can you build such a function f out of polynomials or rational functions or trigonometric functions? What properties must such a function f have?