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INTRODUCTION

I N THE FIRST MOMENTS after the launch of a space shuttle, many changes occur rapidly. The rocket gains altitude as it accelerates to higher speeds. Its mass decreases as fuel burns up. Inside the shuttle, an astronaut feels increasing force due to the acceleration. As the distance from the earth gets larger, the astronaut's weight decreases. Indeed the values of many variables change dramatically during this time period.

Calculus is the mathematics of change. Wherever there is motion or growth or where variations in one quantity produce alterations in another, calculus helps us understand the changes that occur. We can use calculus, for example, to predict the height and speed of the rocket at each instant of time after launch. We also use calculus to study important geometric properties of curves, such as their tangent lines and the amount of area they enclose.

In this preliminary chapter, we briefly review topics from precalculus mathematics that are essential for the study of calculus, beginning with inequalities, equations, absolute values, and graphs of lines and circles. We then turn our attention to functions and their graphs. We also discuss some very important functions that occur frequently in applications of calculus: trigonometric functions, exponential functions, and logarithmic functions. We conclude the precalculus review with an examination of the elementary geometry of conic sections: parabolas, ellipses, and hyperbolas.

To say that the concept of function is important in calculus is an understatement. It is literally the foundation of calculus and the backbone of the entire subject. You will find the word *function* and the symbol f or $f(x)$ used frequently on many pages of this text.

In precalculus courses, we study properties of functions by using algebra and graphical methods that include plotting points and determining symmetry. These techniques are adequate for obtaining a rough sketch of a graph. Calculus is required, however, to find precisely where graphs of functions rise or fall, exact coordinates of high or low points, slopes of tangent lines, and many other useful facts. We can often successfully attack applied problems in science, engineering, economics, and the social sciences that cannot be solved by means of algebra, geometry, or trigonometry if we represent physical quantities in terms of functions and then apply the tools of calculus.

With the preceding remarks in mind, carefully read Section B on functions and their graphs. You should have a good understanding of this material before beginning your study of calculus in the next chapter.

Planning a complex mission like the launch of a space shuttle requires extensive use of calculus and precalculus mathematics.

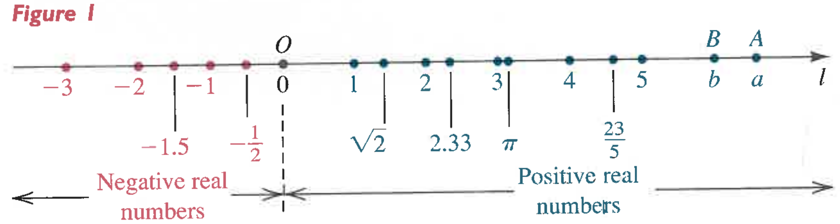
Precalculus Review

A ALGEBRA

This section reviews topics from algebra that are prerequisites for calculus. We shall state important facts and work examples without supplying detailed reasons to justify our work. Texts on precalculus mathematics provide more extensive coverage of this material.

All concepts in calculus are based on properties of the set $\mathbb{R}$ of real numbers. There is a one-to-one correspondence between $\mathbb{R}$ and points on a *coordinate line* (or *real line*) l, as Figure 1 illustrates. The point O is called the *origin* and corresponds to the number 0 (*zero*), which is neither positive nor negative. The real number associated with a point on the line is called a **coordinate** of the point.

Figure 1



If a and b are real numbers, then $a > b$ (**a is greater than b**) if $a - b$ is positive. An equivalent statement is $b < a$ (**b is less than a**). Referring to the coordinate line in Figure 1, we see that $a > b$ if and only if the point A corresponding to a lies to the right of the point B corresponding to b. Other types of inequality symbols include $a \le b$, which means $a < b$ or $a = b$, and $a < b \le c$, which means $a < b$ and $b \le c$.

ILLUSTRATION

- $5 > 3$
- $-7 < -2$
- $(-3)^2 > 0$
- $a^2 \ge 0$ for every real number a

The following properties can be proved for real numbers a, b, and c.

Properties of Inequalities 1

(i) If $a > b$ and $b > c$, then $a > c$.
(ii) If $a > b$, then $a + c > b + c$.
(iii) If $a > b$, then $a - c > b - c$.
(iv) If $a > b$ and c is positive, then $ac > bc$.
(v) If $a > b$ and c is negative, then $ac < bc$.

Analogous properties are true if the inequality signs are reversed. Thus, if $a < b$ and $b < c$, then $a < c$; if $a < b$, then $a + c < b + c$; and so on.

The **absolute value** $|a|$ of a real number a is defined as follows:

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

If a is the coordinate of the point A on the coordinate line in Figure 1, then $|a|$ is the number of units (that is, the distance) between A and the origin O. If a and b are real numbers, then $|a - b|$ represents the distance between a and b.

ILLUSTRATION

- $|3| = 3$
- $|-3| = -(-3) = 3$
- $|0| = 0$
- $|3 - \pi| = -(3 - \pi) = \pi - 3$

The following properties can be proved.

Properties of Absolute Values ($b > 0$) 2

(i) $|a| < b$ if and only if $-b < a < b$
(ii) $|a| > b$ if and only if either $a > b$ or $a < -b$
(iii) $|a| = b$ if and only if $a = b$ or $a = -b$

An **equation** (in x) is a statement such as

$$x^2 = 3x - 4 \quad \text{or} \quad 5x^3 + 2\sin x - \sqrt{x} = 0.$$

A **solution** (or **root**) is a number b that produces a true statement when b is substituted for x. To *solve an equation* means to find all the solutions.

EXAMPLE 1 Solve each equation:

(a) $x^3 + 3x^2 - 10x = 0$
(b) $2x^2 + 5x - 6 = 0$
(c) $7.3x^2 - 31.7x + 15.2 = 0$

SOLUTION

(a) Factoring the left-hand side yields

$$x(x^2 + 3x - 10) = 0, \quad \text{or} \quad x(x - 2)(x + 5) = 0.$$

Setting each factor equal to zero gives us the solutions 0, 2, and −5.

(b) Using the *quadratic formula*,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

with $a = 2$, $b = 5$, and $c = -6$, we obtain

$$x = \frac{-5 \pm \sqrt{(5)^2 - (4)(2)(-6)}}{(2)(2)} = \frac{-5 \pm \sqrt{73}}{4}.$$

Thus, the solutions are $\frac{-5 + \sqrt{73}}{4}$ and $\frac{-5 - \sqrt{73}}{4}$.

(c) Again using the quadratic formula, we obtain

$$x = \frac{31.7 \pm \sqrt{(-31.7)^2 - (4)(7.3)(15.2)}}{(2)(7.3)}$$

$$= \frac{31.7 \pm \sqrt{1004.89 - 443.84}}{14.6}$$

$$= \frac{31.7 \pm \sqrt{561.05}}{14.6}.$$

In this case, the solutions are $\frac{31.7 + \sqrt{561.05}}{14.6}$ and $\frac{31.7 - \sqrt{561.05}}{14.6}$.

COMPUTATIONAL METHOD We may want to use a calculator or a computer to obtain numerical answers in decimal form for results similar to the two algebraic solutions in Example 1(c). In such cases, the ability to estimate roots is a useful skill for checking whether the formula has been keyed in correctly. We can quickly estimate the two roots in (c) by rounding the coefficients to convenient integers or fractions in order to approximate the solutions as follows:

$$x \approx \frac{30 \pm \sqrt{(-30)^2 - 4(7.5)(15)}}{2(7.5)}$$

$$= \frac{30 \pm \sqrt{900 - 30(15)}}{15}$$

$$= \frac{30 \pm \sqrt{450}}{15}$$

$$\approx \frac{30 \pm 20}{15} = \frac{50}{15} \text{ and } \frac{10}{15} \approx 3.3 \text{ and } 0.7$$

To obtain more exact numerical answers with a calculator or a computer, we use the quadratic formula and key in the values in a format similar to the following:

```
(31.7+√(-31.7∧2-4*7.3*15.2))/(2*7.3)
```

We then evaluate the expression, obtaining 3.79359548233 as an approximate answer, which is close to the estimated value of 3.3. A similar calculation (replacing the first + sign with a − sign when keying in the formula values) yields 0.548870271098 for the second root in (c), which is also close to the estimated value.

CAUTION When entering complex numerical formulas on an algebraic calculator, it is essential to use parentheses properly. See the reference manual for your calculator for details.

An **inequality** (in x) is a statement that contains at least one of the symbols $<$, $>$, $\leq$, or $\geq$, such as

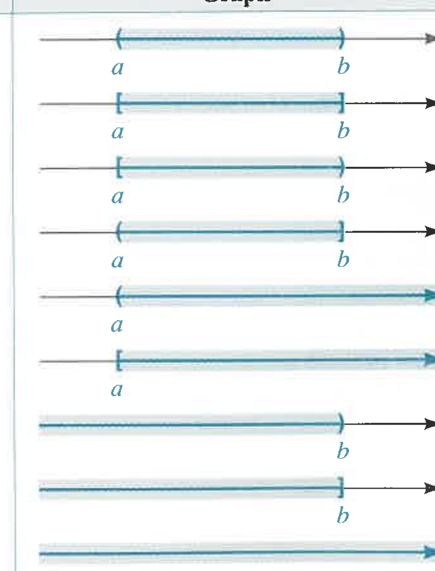
$$5x - 4 > x^2 \quad \text{or} \quad -3 < 4x + 2 \leq 5.$$

The notions of **solution** of an inequality and *solving* an inequality are similar to the analogous concepts for equations.

In calculus, we often use *intervals*. In the definitions that follow, we employ the *set notation* $\{x : \quad\}$, where the space after the colon is used to specify restrictions on the variable x. The notation $\{x : a < x \leq b\}$, for example, denotes the set of all real numbers greater than a and less than or equal to b—the equivalent interval notation for this set is $(a, b]$. In the following chart, we call (a, b) an **open interval**, $[a, b]$ a **closed interval**, $[a, b)$ and $(a, b]$ **half-open intervals**, and intervals defined in terms of ∞ (*infinity*) or $-\infty$ (*minus* or *negative infinity*) **infinite intervals** or **rays**.

Intervals 3

Notation	Definition	Graph
$[a, b]$	$\{x : a \leq x \leq b\}$	
$[a, b)$	$\{x : a \leq x < b\}$	
$(a, b]$	$\{x : a < x \leq b\}$	
(a, ∞)	$\{x : x > a\}$	
$[a, \infty)$	$\{x : x \geq a\}$	
$(-\infty, b)$	$\{x : x < b\}$	
$(-\infty, b]$	$\{x : x \leq b\}$	
$(-\infty, \infty)$	$\mathbb{R}$	



EXAMPLE 2 Solve each inequality, and then sketch the graph of its solution:

(a) $-5 \leq \frac{4 - 3x}{2} < 1$

(b) $x^2 - 10 > 3x$

SOLUTION

(a) $-5 \le \dfrac{4-3x}{2} < 1$ given

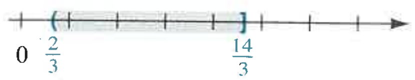
$-10 \le 4 - 3x < 2$ multiply by 2

$-14 \le -3x < -2$ subtract 4

$\frac{14}{3} \ge x > \frac{2}{3}$ divide by -3, reverse the inequality signs

$\frac{2}{3} < x \le \frac{14}{3}$ equivalent inequality

Figure 2



Hence the solutions are the numbers in the half-open interval $(\frac{2}{3}, \frac{14}{3}]$. The graph is sketched in Figure 2.

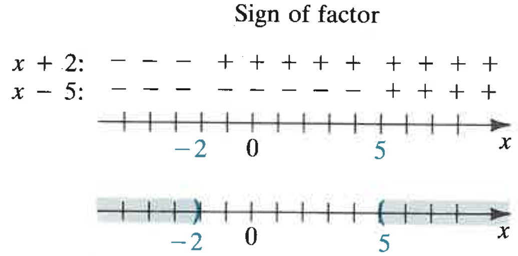
(b) $x^2 - 10 > 3x$ given

$x^2 - 3x - 10 > 0$ subtract $3x$

$(x-5)(x+2) > 0$ factor

We next examine the signs of the factors $x-5$ and $x+2$, as shown in Figure 3. Since $(x-5)(x+2) > 0$ if both factors have the same sign, the solutions are the real numbers in the union $(-\infty, -2) \cup (5, \infty)$, as illustrated in Figure 3.

Figure 3



Inequalities involving absolute value occur frequently in calculus.

EXAMPLE ■ 3 Solve each inequality, and then sketch the graph of its solution:

(a) $|x-3| < 0.5$

(b) $|2x-7| > 3$

SOLUTION

(a) $|x-3| < 0.5$ given

$-0.5 < x - 3 < 0.5$ property (i) of absolute values

$2.5 < x < 3.5$ add 3

Figure 4

The solutions are the real numbers in the open interval (2.5, 3.5), as shown in Figure 4.

(b) $|2x-7| > 3$ given

$2x - 7 < -3$ or $2x - 7 > 3$ property (ii) of absolute values

$2x < 4$ or $2x > 10$ add 7

$x < 2$ or $x > 5$ divide by 2

Figure 5

The solutions are given by $(-\infty, 2) \cup (5, \infty)$. The graph is sketched in Figure 5.

NOTE We can also solve the inequalities in Example 3 graphically (that is, in terms of distance) by observing that $|x-3| < 0.5$ means that x is less than 0.5 unit from 3. Hence, x must lie between $3 - 0.5$ and $3 + 0.5$, or, equivalently, $2.5 < x < 3.5$. Similarly, for $|2x-7| > 3$, we note that $2x$ is more than 3 units away from 7. Thus, if $2x < 7 - 3$ or $2x > 7 + 3$, we obtain the same inequalities: $2x < 4$ or $2x > 10$, or, equivalently, $x < 2$ or $x > 5$.

Inequalities often occur in applications to physical problems, as the next example demonstrates.

EXAMPLE ■ 4 As the altitude of a space shuttle increases, an astronaut's weight decreases until a state of weightlessness is achieved. The weight of a 125-lb astronaut at an altitude of x kilometers above sea level is given by

$$W = 125\left(\frac{6400}{6400+x}\right)^2.$$

At what altitudes is the astronaut's weight less than 5 lb?

SOLUTION We need to find the values of x for which $W < 5$—that is,

$$125\left(\frac{6400}{6400+x}\right)^2 < 5.$$

Dividing each side of the inequality by 125 gives us

$$\left(\frac{6400}{6400+x}\right)^2 < \frac{1}{25}.$$

Taking the square root of each side yields

$$\frac{6400}{6400+x} < \frac{1}{5}.$$

(Since x is positive, the fraction $6400/(6400+x)$ will also be positive. Thus, we can ignore the negative square root.)

Now we can multiply both sides of the last inequality by the positive expression $5(6400 + x)$ to obtain

$$(5)(6400) < (1)(6400 + x)$$

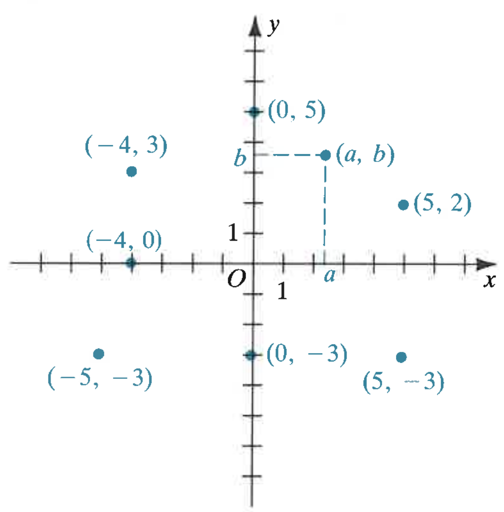
or

$$x > (5)(6400) - 6400 = (4)(6400) = 25{,}600.$$

The astronaut's weight will be less than 5 lb at altitudes greater than 25,600 km.

A **rectangular coordinate system** is an assignment of *ordered pairs* (a, b) to points in a plane, as illustrated in Figure 6. The plane is called a **coordinate plane**, or an ***xy*-plane**. Note that in this context (a, b) is not an open interval. It should always be clear from our discussion whether (a, b) represents a point or an interval.

Figure 6

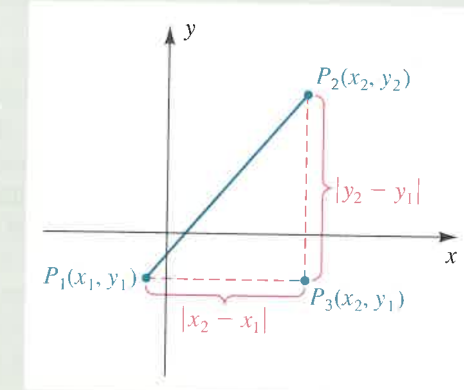


Two important formulas that show how geometric properties of line segments can be obtained from the coordinate-plane representation of points are the *distance formula* and the *midpoint formula*. Both of these formulas can be proved.

Distance Formula 4

The distance between P_1 and P_2 is

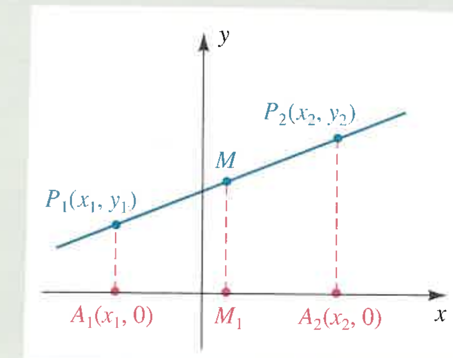
$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



Midpoint Formula 5

The midpoint M of segment P_1P_2 is

$$M(\overline{P_1P_2}) = M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



EXAMPLE 5 Given $A(-2, 3)$ and $B(4, -2)$, find:

(a) the distance between A and B

(b) the midpoint M of segment AB

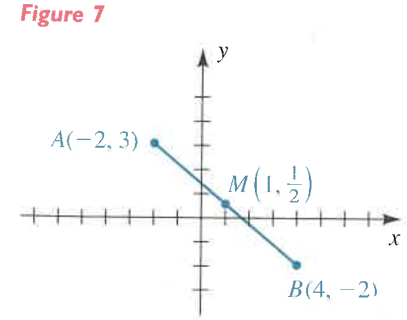
SOLUTION Using the formulas in (4) and (5), we obtain the following:

(a) $d(A, B) = \sqrt{(4 + 2)^2 + (-2 - 3)^2} = \sqrt{36 + 25} = \sqrt{61}$

(b) $M(\overline{AB}) = M\left(\frac{-2 + 4}{2}, \frac{3 + (-2)}{2}\right) = M\left(1, \frac{1}{2}\right)$

The points are plotted in Figure 7.

Figure 7



An **equation in x and y** is an equality such as

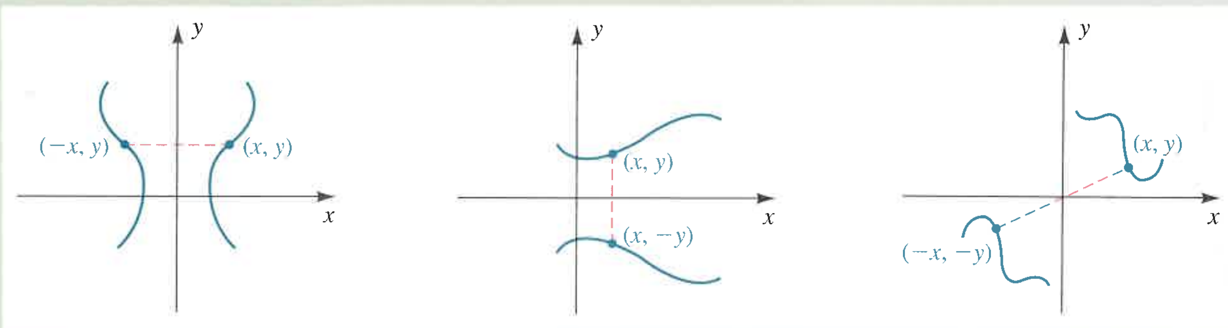
$$2x + 3y = 5, \qquad y = x^2 - 5x + 2, \quad \text{or} \quad y^2 + \sin x = 8.$$

A **solution** is an ordered pair (a, b) that produces a true statement when $x = a$ and $y = b$. The **graph** of the equation consists of all points (a, b) in a plane that correspond to the solutions. We shall assume that you have experience in sketching graphs of basic equations in x and y.

The concept of *symmetry* is useful in calculus. It enables us to sketch only half a graph and then reflect that half through an axis or the origin. Some graphs in the xy-plane are symmetric with respect to the y-axis, the x-axis, or the origin. There are simple tests, given in (6), that we can apply to an equation in x and y in order to determine symmetry.

Symmetries of Graphs 6

(i) y-axis (ii) x-axis (iii) Origin



Test (i): Substitution of $-x$ for x leads to the same equation.

Test (ii): Substitution of $-y$ for y leads to the same equation.

Test (iii): Substitution of *both* $-x$ for x *and* $-y$ for y leads to the same equation.

In the next example, we shall plot several points on each graph to illustrate solutions of the equations. However, *a principal objective in graphing is to obtain an accurate sketch without plotting many (or any) points.*

EXAMPLE 6 Sketch the graph of each of the following equations:

(a) $y = \frac{1}{2}x^2$ **(b)** $y^2 = x$ **(c)** $4y = x^3$

SOLUTION

(a) By symmetry test (i), the graph of $y = \frac{1}{2}x^2$ is symmetric with respect to the y-axis. Some points (x, y) on the graph are listed in the following table.

x	0	1	2	3	4
y	0	$\frac{1}{2}$	2	$\frac{9}{2}$	8

Plotting, drawing a smooth curve through the points, and then using symmetry gives us the sketch in Figure 8. The graph is a *parabola,* with *vertex* (0, 0) and *axis* along the y-axis. We discuss parabolas in more detail in Section E of this chapter.

(b) By symmetry test (ii), the graph of $y^2 = x$ is symmetric with respect to the x-axis. Points above the x-axis are given by $y = \sqrt{x}$. Several such points are (0, 0), (1, 1), (4, 2), and (9, 3). Plotting and using symmetry gives us Figure 9. The graph is a parabola with vertex (0, 0) and axis along the x-axis.

(c) By symmetry test (iii), the graph of $4y = x^3$ is symmetric with respect to the origin. Several points on the graph are (0, 0), $(1, \frac{1}{4})$, and (2, 2). Plotting and using symmetry gives us the sketch in Figure 10.

Figure 8

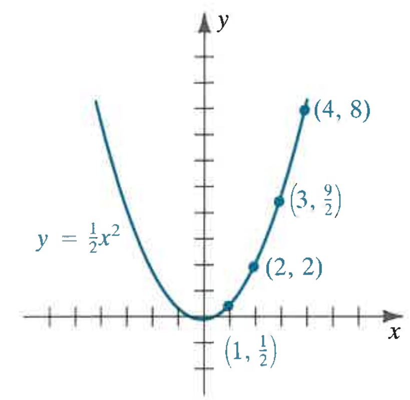


Figure 9

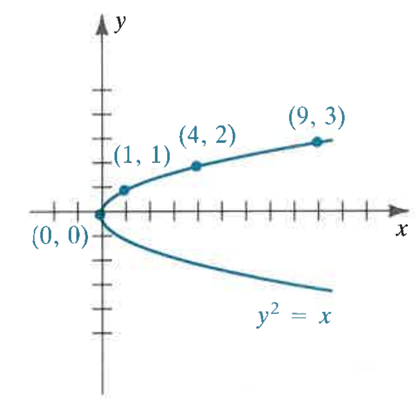


Figure 10

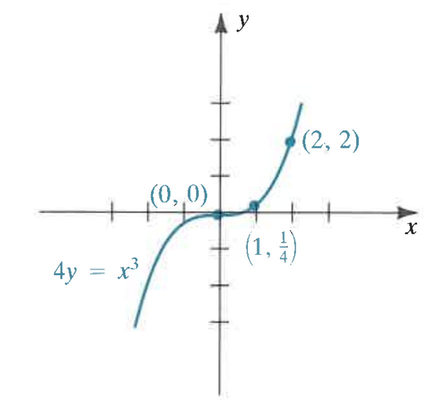
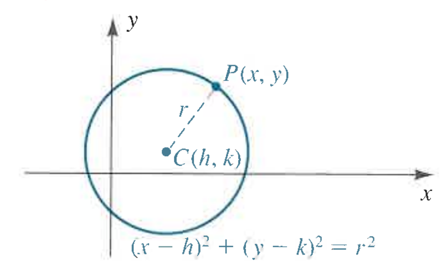


Figure 11



A very symmetric geometric figure in the plane is a circle with its center at the origin, since it is symmetric with respect to both coordinate axes *and* with respect to the origin. A circle with center $C(h, k)$ and radius r is illustrated in Figure 11. If $P(x, y)$ is any point on the circle, then by the distance formula (4), $d(P, C) = r$, or $[d(P, C)]^2 = r^2$, which in turn yields the following equation.

Equation of a Circle 7

$$(x - h)^2 + (y - k)^2 = r^2$$

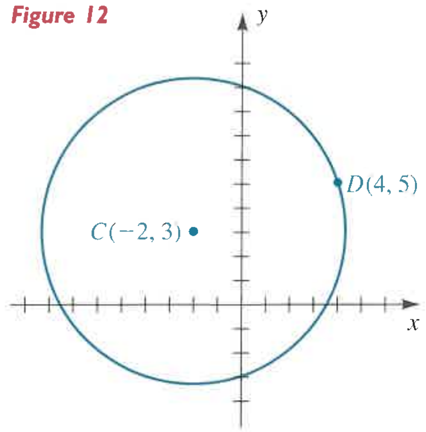
If the radius r is 1, then the circle is called a **unit circle**. A unit circle U with center at the origin has the equation $x^2 + y^2 = 1$.

EXAMPLE 7 Find an equation of the circle that has center $C(-2, 3)$ and contains the point $D(4, 5)$.

SOLUTION The circle is illustrated in Figure 12. Since D is on the circle, the radius r is $d(C, D)$. By the distance formula,

$$r = \sqrt{(4+2)^2 + (5-3)^2} = \sqrt{36+4} = \sqrt{40}.$$

Figure 12



Using the equation of a circle (7) with $h = -2$, $k = 3$, and $r = \sqrt{40}$ gives us

$$(x+2)^2 + (y-3)^2 = 40.$$

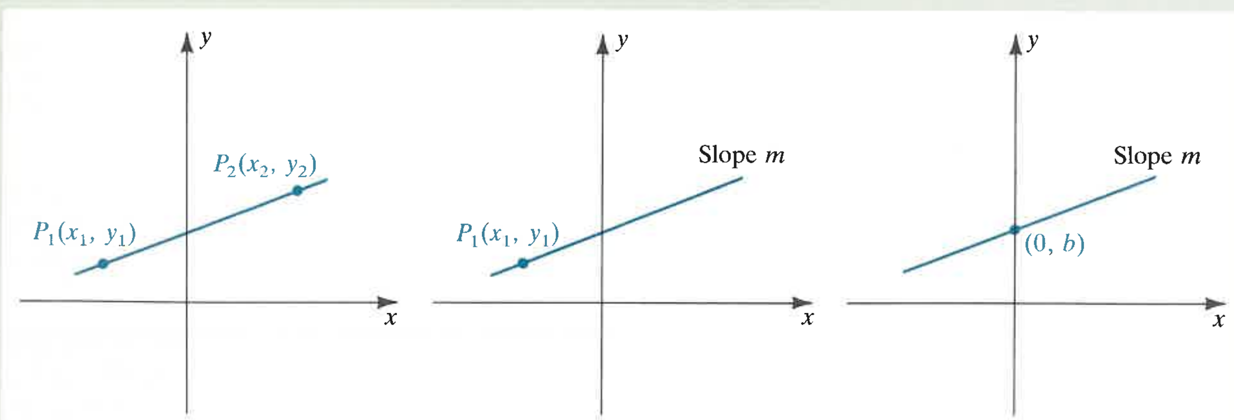
In calculus we often consider lines in a coordinate plane. The following formulas are used for finding their equations.

Lines 8

(i) Slope m: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

(ii) Point–Slope Form: $y - y_1 = m(x - x_1)$

(iii) Slope–Intercept Form: $y = mx + b$



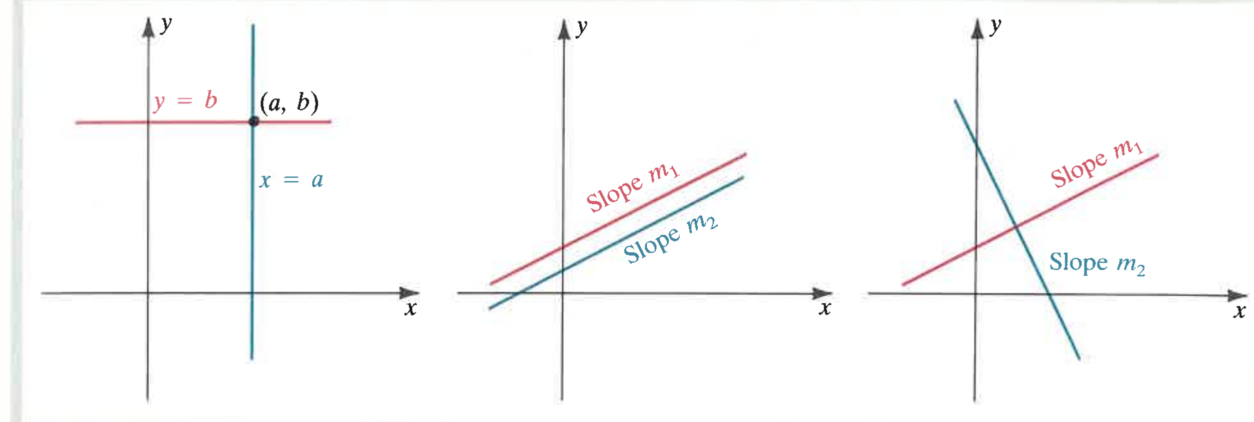
Some special types of lines and properties of their slopes are given in (9).

Special Lines 9

(i) Vertical: m undefined
Horizontal: $m = 0$

(ii) Parallel: $m_1 = m_2$

(iii) Perpendicular: $m_1 m_2 = -1$



EXAMPLE 8 Sketch the line through each pair of points, and find its slope:

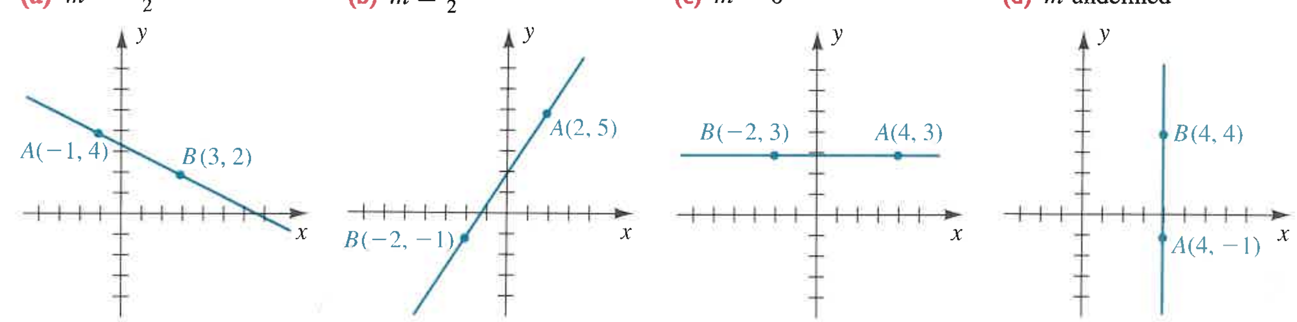
(a) $A(-1, 4)$ and $B(3, 2)$ **(b)** $A(2, 5)$ and $B(-2, -1)$

(c) $A(4, 3)$ and $B(-2, 3)$ **(d)** $A(4, -1)$ and $B(4, 4)$

SOLUTION The lines are sketched in Figure 13.

Figure 13

(a) $m = -\frac{1}{2}$ (b) $m = \frac{3}{2}$ (c) $m = 0$ (d) m undefined



From the slope formula (8)(i),

(a) $m = \dfrac{2-4}{3-(-1)} = \dfrac{-2}{4} = -\dfrac{1}{2}$ **(b)** $m = \dfrac{5-(-1)}{2-(-2)} = \dfrac{6}{4} = \dfrac{3}{2}$

(c) $m = \dfrac{3-3}{4-(-2)} = \dfrac{0}{6} = 0,$ which indicates that the line is horizontal.

(d) $m = \dfrac{4-(-1)}{4-4} = \dfrac{5}{0},$ which is undefined. Note that the line is vertical.

A **linear equation** in x and y is an equation of the form $ax + by = c$ (or $ax + by + d = 0$) with a and b not both zero. The graph of a linear equation is a line.

EXAMPLE ■ 9 Find a linear equation for the line through $A(1, 7)$ and $B(-3, 2)$.

SOLUTION The slope m of the line is

$$m = \frac{7-2}{1-(-3)} = \frac{5}{4}.$$

We may use the coordinates of either A or B for (x_1, y_1) in the point–slope form (8)(ii). Using $A(1, 7)$ gives us

$$y - 7 = \tfrac{5}{4}(x - 1),$$

which is equivalent to

$$4y - 28 = 5x - 5, \quad \text{or} \quad 5x - 4y = -23.$$

EXAMPLE ■ 10

(a) Find the slope of the line l with equation $2x - 5y = 9$.

(b) Find linear equations for the lines through $P(3, -4)$ that are parallel to l and perpendicular to l.

SOLUTION

(a) If we rewrite the equation as $5y = 2x - 9$ and divide both sides by 5, we obtain

$$y = \tfrac{2}{5}x - \tfrac{9}{5}.$$

Comparing this equation with the slope–intercept form $y = mx + b$, we see that the slope is $m = \frac{2}{5}$.

(b) By (ii) and (iii) of (9), the line through $P(3, -4)$ parallel to l has slope $\frac{2}{5}$ and the line perpendicular to l has slope $-\frac{5}{2}$. The corresponding equations are

$$y + 4 = \tfrac{2}{5}(x - 3), \quad \text{or} \quad 2x - 5y = 26,$$

and

$$y + 4 = -\tfrac{5}{2}(x - 3), \quad \text{or} \quad 5x + 2y = 7.$$

EXERCISES A

Exer. 1–8: Rewrite the expression without using the absolute value symbol.

1 **(a)** $(-5)\,|3 - 6|$ **(b)** $|-6|/(-2)$ **(c)** $|-7| + |4|$

2 **(a)** $(4)\,|6 - 7|$ **(b)** $5/|-2|$ **(c)** $|-1| + |-9|$

3 **(a)** $|4 - \pi|$ **(b)** $|\pi - 4|$ **(c)** $|\sqrt{2} - 1.5|$

4 **(a)** $|\sqrt{3} - 1.7|$ **(b)** $|1.7 - \sqrt{3}|$ **(c)** $|\frac{1}{5} - \frac{1}{3}|$

5 $|3 + x|$ if $x < -3$

6 $|5 - x|$ if $x > 5$

7 $|2 - x|$ if $x < 2$

8 $|7 + x|$ if $x \geq -7$

Exer. 9–12: Solve the equation by factoring.

9 $15x^2 - 12 = -8x$

10 $15x^2 - 14 = 29x$

11 $2x(4x + 15) = 27$

12 $x(3x + 10) = 77$

Exer. 13–16: Solve the equation by using the quadratic formula.

13 $x^2 + 4x + 2 = 0$

14 $x^2 - 6x - 3 = 0$

15 $2x^2 - 3x - 4 = 0$

16 $3x^2 + 5x + 1 = 0$

Exer. 17–38: Solve the inequality and express the solution in terms of intervals whenever possible.

17 $2x + 5 < 3x - 7$

18 $x - 8 > 5x + 3$

19 $3 \leq \dfrac{2x - 3}{5} < 7$

20 $-2 < \dfrac{4x + 1}{3} \leq 0$

21 $x^2 - x - 6 < 0$

22 $x^2 + 4x + 3 \geq 0$

23 $x^2 - 2x - 5 > 3$

24 $x^2 - 4x - 17 \leq 4$

25 $x(2x + 3) \geq 5$

26 $x(3x - 1) \leq 4$

27 $\dfrac{x + 1}{2x - 3} > 2$

28 $\dfrac{x - 2}{3x + 5} \leq 4$

29 $\dfrac{1}{x - 2} \geq \dfrac{3}{x + 1}$

30 $\dfrac{2}{2x + 3} \leq \dfrac{2}{x - 5}$

31 $|x + 3| < 0.01$

32 $|x - 4| \leq 0.03$

33 $|x + 2| \geq 0.001$

34 $|x - 3| > 0.002$

35 $|2x + 5| < 4$

36 $|3x - 7| \geq 5$

37 $|6 - 5x| \leq 3$

38 $|-11 - 7x| > 6$

Exer. 39–40: Describe the set of all points $P(x, y)$ in a coordinate plane that satisfy the given condition.

39 **(a)** $x = -2$ **(b)** $y = 3$ **(c)** $x \geq 0$ **(d)** $xy > 0$ **(e)** $y < 0$ **(f)** $|x| \leq 2$ and $|y| \leq 1$

40 **(a)** $y = -2$ **(b)** $x = -4$ **(c)** $x/y < 0$ **(d)** $xy = 0$ **(e)** $y > 1$ **(f)** $|x| \geq 2$ and $|y| \geq 3$

Exer. 41–42: Find (a) $d(A, B)$ and (b) the midpoint of AB.

41 $A(4, -3)$, $B(6, 2)$

42 $A(-2, -5)$, $B(4, 6)$

43 Show that the triangle with vertices $A(8, 5)$, $B(1, -2)$, and $C(-3, 2)$ is a right triangle, and find its area.

44 Show that the points $A(-4, 2)$, $B(1, 4)$, $C(3, -1)$, and $D(-2, -3)$ are vertices of a square.

Exer. 45–56: Sketch the graph of the equation.

45 $y = 2x^2 - 1$

46 $y = -x^2 + 2$

47 $x = \frac{1}{4}y^2$

48 $x = -2y^2$

49 $y = x^3 - 8$

50 $y = -x^3 + 1$

51 $y = \sqrt{x} - 4$

52 $y = \sqrt{x - 4}$

53 $(x + 3)^2 + (y - 2)^2 = 9$

54 $x^2 + (y - 2)^2 = 25$

55 $y = -\sqrt{16 - x^2}$

56 $y = \sqrt{4 - x^2}$

Exer. 57–60: Find an equation of the circle that satisfies the given conditions.

57 Center $C(2, -3)$; radius 5

58 Center $C(-4, 6)$; passing through $P(1, 2)$

59 Tangent to both axes; center in the second quadrant; radius 4

60 Endpoints of a diameter $A(4, -3)$ and $B(-2, 7)$

Exer. 61–66: Find an equation of the line that satisfies the given conditions.

61 Through $A(5, -3)$; slope -4

62 Through $A(-1, 4)$; slope $\frac{2}{3}$

63 x-intercept 4; y-intercept -3

64 Through $A(5, 2)$ and $B(-1, 4)$

65 Through $A(2, -4)$; parallel to the line $5x - 2y = 4$

66 Through $A(7, -3)$; perpendicular to the line $2x - 5y = 8$

c **Exer. 67–70: Use the quadratic formula to solve the equation. Give approximations to two decimal places.**

67 $0.7x^2 + 3.2x + 1.5 = 0$

68 $\sqrt{3}x^2 + \frac{3}{7}x - \frac{5}{13} = 0$

69 $375x^2 - 921x + 47 = 0$

70 $x^4 - 8x^2 + 5 = 0$

71 The cost C (in dollars) of renting a luxury car for one week is given by $C = 0.25m + 150$, where m is the number of miles driven. What range of miles will result in a rental charge that is between \$200 and \$300?

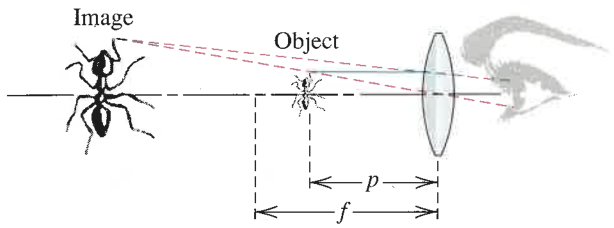
72 A coin is considered fair if it has an equal probability of landing with heads up or tails up when tossed. An experimenter tosses a coin 100 times and counts the number of heads H. From statistical theory, the coin will be considered fair if

$$\left|\frac{H - 50}{5}\right| \leq 1.645.$$

For what range of values of H will the experimenter declare the coin fair?

73 Shown in the figure is a simple magnifier consisting of a convex lens. The object to be magnified is positioned so that its distance p from the lens is less than the focal length f. The linear magnification M is the ratio of the image size to the object size. It is shown in physics that $M = f/(f - p)$. If $f = 6$ cm, how far should the object be placed from the lens so that its image appears at least three times as large?

Exercise 73



c **74** The *escape velocity* is the initial velocity v_0 with which a rocket must leave the surface of a planet so that it can eventually rise as far as desired. The escape velocity satisfies the inequality

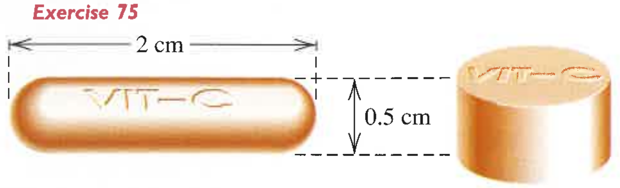
$$\frac{1}{2}v_0^2 \geq \frac{km}{R},$$

where m and R are the mass and the radius of the planet, respectively, and k is a constant. If mass is given in kilograms and radius in meters, then $k = 6.67 \times 10^{-11}$, with units chosen so that v_0 is measured in meters per second. For which initial velocities can a rocket escape the earth, Mars, and the moon? Use the data in the following table.

	m (kg)	R (m)
Earth	6.0×10^{24}	6.2×10^6
Mars	6.4×10^{23}	3.3×10^6
Moon	7.3×10^{22}	1.7×10^6

75 The rate at which a tablet of vitamin C begins to dissolve depends on the surface area of the tablet. One brand of tablet is 2 cm long and is in the shape of a cylinder with hemispheres of diameter 0.5 cm attached to both ends (see figure). A second brand of tablet is to be manufactured in the shape of a right circular cylinder of altitude 0.5 cm.

Exercise 75

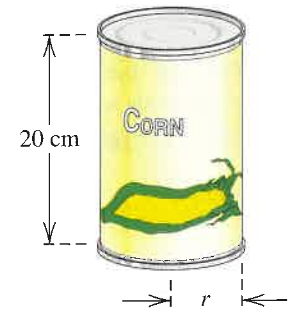


(a) Find the diameter of the second tablet so that its surface area is equal to that of the first tablet.

(b) Find the volume of each tablet.

76 A manufacturer of tin cans wishes to construct a right circular cylindrical can of height 20 cm and of capacity 3000 cm^3 (see figure). Find the inner radius r of the can.

Exercise 76



77 The braking distance d (in feet) of a car traveling v mi/hr is approximated by $d = v + (v^2/20)$. Determine velocities that result in braking distances of less than 75 ft.

78 In order for a drug to have a beneficial effect, its concentration in the bloodstream must exceed a certain value, the *minimum therapeutic level*. Suppose that the concentration c of a drug t hours after it has been taken orally is given by $c = 20t/(t^2 + 4)$ mg/L. If the minimum therapeutic level is 4 mg/L, determine when this level is exceeded.

79 The electrical resistance R (in ohms) for a pure metal wire is related to its temperature T (in °C) by the formula $R = R_0(1 + aT)$ for positive constants a and R_0.

(a) For what temperature is $R = R_0$?

(b) Assuming that the resistance is 0 if $T = -273$ °C (absolute zero), find a.

(c) Silver wire has a resistance of 1.25 ohms at 0 °C. At what temperature is the resistance 2 ohms?

80 Pharmacological products must specify recommended dosages for adults and children. Two formulas for modification of adult dosage levels for young children are

$$\text{Cowling's rule:} \quad y = \tfrac{1}{24}(t + 1)a$$
$$\text{Friend's rule:} \quad y = \tfrac{2}{25}ta,$$

where a denotes the adult dose (in milligrams) and t denotes the age of the child (in years).

(a) If $a = 100$, graph the two linear equations on the same axes for $0 \leq t \leq 12$.

(b) For what age do the two formulas specify the same dosage?

Mathematicians and Their Times

HYPATIA

HYPATIA, THE FIRST WOMAN MATHEMATICIAN whose achievements we know, was a brilliant scholar and gifted teacher who suffered a horrible death at the hands of a mob blinded by religious hatred.

Hypatia was born around A.D. 370. Her father, Theon, was a mathematician at the Alexandrian Museum, a university that Egypt's rulers had founded 700 years earlier. "In an era in which the domains of intellect and politics were almost exclusively male," notes one biographer,* "Theon was an unusually liberated person who taught an unusually gifted daughter and encouraged her to achieve things that, as far as we know, no woman before her did or perhaps even dreamed of doing."

Theon supervised his daughter's education, immersing her in an environment of learning and exploration and passing on his own great love of mathematics. Hypatia's remarkable intellectual skills, combined with great eloquence, modesty, and beauty, attracted many enthusiastic students from Europe, Africa, and Asia. She lectured on philosophy as well as mathematics and was recognized as the leader of the Neoplatonic philosophers. Students gathered in her home or followed her in the streets to hear more of her brilliant philosophical discussions or expositions on mathematics. Hypatia authored commentaries on the *Conics* of Appolonius, the *Arithmetica* of Diophantus, and the astronomical work of Ptolemy. These expositions were designed to help students understand difficult classic texts.

The scientific rationalism of the Neoplatonists challenged the more doctrinaire beliefs of the early Christian Church, whose leaders condemned the Greeks as "pagans". When Cyril became Alexandria's Christian patriarch in 412, he began a systematic plan of oppression aimed at all he saw as heretics. He led an attack against the Jews, destroying their synagogue, looting their homes, and finally expelling them from the city. When Orestes, the head of the civil government, complained,

*Ian Mueller, "Hypatia," in Louise S. Grinstein and Paul J. Campbell, eds., *Women of Mathematics: A Bibliographic Sourcebook*. New York: Greenwood Press, 1987.

a band of Cyril's supporters attacked him with stones. Rescued from the mob, Orestes tortured and executed the monk who had wounded him. Cyril, in turn, demanded the sacrifice of a virgin who followed the Greek religion and advised Orestes. Rumors spread that Hypatia was a major force inciting Orestes against Cyril. Cyril's supporters responded swiftly. In March 415, a fanatical mob barbarously murdered Hypatia.

Hypatia's tragic death also brought an eclipse to significant scientific and mathematical thought in the West, one that unfortunately lasted nearly 1000 years.

B FUNCTIONS AND THEIR GRAPHS

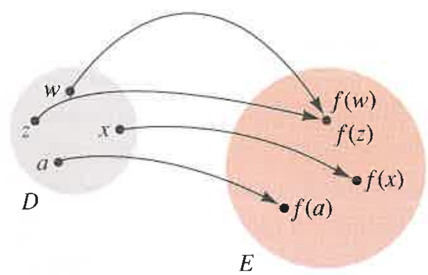
The notion of *function* is basic for much of our work in calculus. We often study the effect that a change in one variable has on the values of a second variable when the second variable is a function of the first.

Definition 10

A **function** f from a set D to a set E is a correspondence that assigns to each element x of the set D exactly one element y of the set E.

The element y of E is the **value** of f at x and is denoted by $f(x)$, read "f of x." The set D is the **domain** of the function f, and the set E is the **codomain** of f. The **range** of f is the subset of the codomain E consisting of all possible function values $f(x)$ for x in D.

Figure 14



We sometimes depict functions as shown in Figure 14, where the sets D and E are represented by points within regions in a plane. The curved arrows indicate that the elements $f(x)$, $f(w)$, $f(z)$, and $f(a)$ of E correspond to the elements x, w, z, and a, respectively, of D. It is important to remember that *to each x in D, there is assigned exactly one function value $f(x)$ in E*. Different elements of D, such as w and z in Figure 14, may yield the same function value in E. Until we reach Chapter 11, the phrase *f is a function* will mean that the domain and the range of f are sets of real numbers. We say that f is a **one-to-one** function if $f(x) \neq f(y)$ whenever $x \neq y$.

We usually define a function f by stating a formula or rule for finding $f(x)$, such as $f(x) = \sqrt{x-2}$. The domain is then assumed to be the set of all real numbers such that $f(x)$ is real. Thus, for $f(x) = \sqrt{x-2}$, the domain is the infinite interval $[2, \infty)$. If x is in the domain, we say that f **is defined at** x, or that $f(x)$ **exists**. If S is a subset of the domain, then f **is defined on** S. The terminology f **is undefined at** x means that x is not in the domain of f.

EXAMPLE ■ 1 Let $f(x) = \dfrac{\sqrt{4+x}}{1-x}$.

(a) Find the domain of f. **(b)** Find $f(5)$, $f(-2)$, $f(-a)$, and $-f(a)$.

SOLUTION

(a) Note that $f(x)$ is a real number if and only if the radicand $4+x$ is nonnegative and the denominator $1-x$ is not equal to 0. Thus, $f(x)$ exists if and only if

$$4 + x \geq 0 \quad \text{and} \quad 1 - x \neq 0$$

or, equivalently,

$$x \geq -4 \quad \text{and} \quad x \neq 1.$$

Hence, the domain is $[-4, 1) \cup (1, \infty)$.

(b) To find values of f, we substitute for x:

$$f(5) = \frac{\sqrt{4+5}}{1-5} = \frac{\sqrt{9}}{-4} = -\frac{3}{4} \qquad f(-2) = \frac{\sqrt{4+(-2)}}{1-(-2)} = \frac{\sqrt{2}}{3}$$

$$f(-a) = \frac{\sqrt{4+(-a)}}{1-(-a)} = \frac{\sqrt{4-a}}{1+a} \qquad -f(a) = -\frac{\sqrt{4+a}}{1-a} = \frac{\sqrt{4+a}}{a-1}$$

In calculus, we often work with the **difference quotient** of a function. If f is a function, then its difference quotient is an expression of the form

$$\frac{f(x+h) - f(x)}{h}, \quad \text{where } h \neq 0.$$

EXAMPLE ■ 2 Simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

using the function $f(x) = x^2 + 6x - 4$.

SOLUTION We have

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 + 6(x+h) - 4] - [x^2 + 6x - 4]}{h} && \text{definition of } f \\
&= \frac{(x^2 + 2xh + h^2 + 6x + 6h - 4) - (x^2 + 6x - 4)}{h} && \text{expand} \\
&= \frac{2xh + h^2 + 6h}{h} && \text{collect like terms} \\
&= \frac{h(2x + h + 6)}{h} && \text{factor out } h \\
&= 2x + h + 6. && \text{cancel } h \neq 0
\end{aligned}$$

Thus, the difference quotient simplifies to $2x + h + 6$.

Many formulas that occur in mathematics and the sciences determine functions. For instance, the formula $A = \pi r^2$ for the area A of a circle of radius r assigns to each positive real number r exactly one value of A. The letter r, which represents an arbitrary number from the domain, is an **independent variable**. The letter A, which represents a number from the range, is a **dependent variable,** since its value *depends* on the number assigned to r. If two variables r and A are related in this manner, we say that A is *a function of* r. As another example, if an automobile travels at a uniform rate of 50 mi/hr, then the distance d (in miles) traveled in time t (in hours) is given by $d = 50t$, and hence the distance d *is a function of time* t.

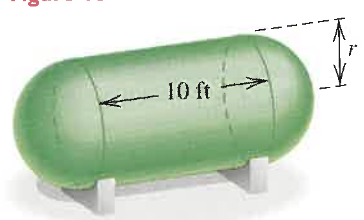
EXAMPLE ■ 3 A steel storage tank for propane gas is to be constructed in the shape of a right circular cylinder of altitude 10 ft with a hemisphere attached to each end. The radius r is yet to be determined. Express the volume V of the tank as a function of r.

SOLUTION The tank is sketched in Figure 15. We may find the volume of the cylindrical part of the tank by multiplying the altitude 10 by the area πr^2 of the base of the cylinder:

$$\text{volume of cylinder} = 10(\pi r^2) = 10\pi r^2$$

The two hemispherical ends, taken together, form a sphere of radius r.

Figure 15



Using the formula for the volume of a sphere, we obtain

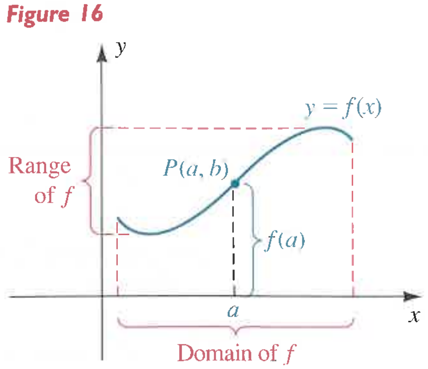
$$\text{volume of the two ends} = \tfrac{4}{3}\pi r^3.$$

Thus, the volume V of the tank is

$$V = \tfrac{4}{3}\pi r^3 + 10\pi r^2 = \tfrac{2}{3}\pi r^2(2r + 15).$$

This formula expresses V as a function of r.

Figure 16



If f is a function, we may use a graph to illustrate the change in the function value $f(x)$ as x varies through the domain of f. The **graph of a function** f with domain D is the graph of the equation $y = f(x)$ for x in D. The graph is the set of all points $(x, f(x))$, where x is in D. If a point $P(a, b)$ is on the graph, then the y-coordinate b is the function value $f(a)$. Figure 16 shows the graph of f and indicates the domain and the range. In this figure, the domain and the range are shown as closed intervals. In other examples, they may be infinite intervals or other sets of real numbers.

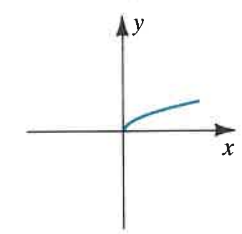
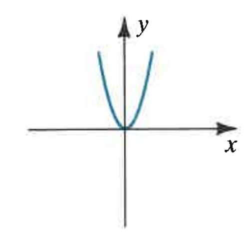
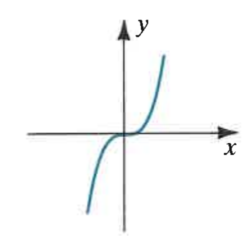
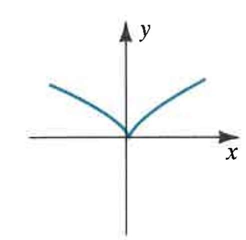
Since there is exactly one value $f(a)$ for each a in the domain, only *one* point on the graph has x-coordinate a. Thus, *every vertical line intersects the graph of a function in at most one point.* Consequently, the graph of a function cannot be a figure such as a circle, which can be intersected by a vertical line in more than one point.

The x-intercepts of the graph of a function f are the solutions of the equation $f(x) = 0$. These numbers are the **zeros** of the function. The y-intercept of the graph is $f(0)$, if it exists.

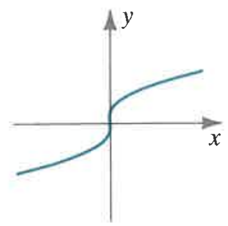
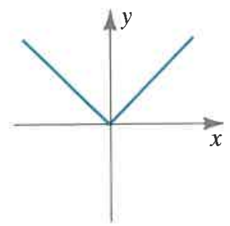
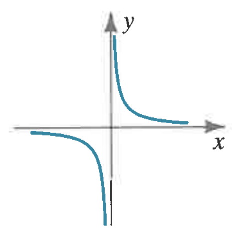
If f is an **even function**—that is, if $f(-x) = f(x)$ for every x in the domain of f—then the graph of f is symmetric with respect to the y-axis, by symmetry test (i) of (6). If f is an **odd function**—that is, if $f(-x) = -f(x)$ for every x in the domain of f—then the graph of f is symmetric with respect to the origin, by symmetry test (iii). Most functions in calculus are neither even nor odd.

The next illustration contains sketches of graphs of some common functions and indicates the symmetry, the domain, and the range for each.

ILLUSTRATION

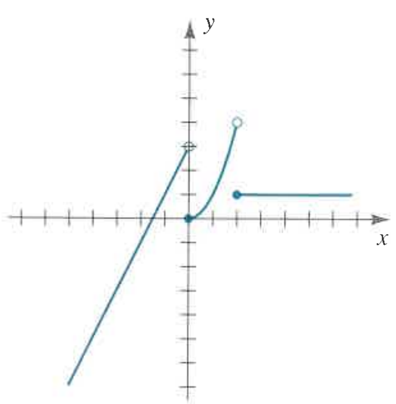
Function f	Graph	Symmetry	Domain D, Range R
■ $f(x) = \sqrt{x}$		none	$D = [0, \infty)$ $R = [0, \infty)$
■ $f(x) = x^2$		y-axis (even function)	$D = (-\infty, \infty)$ $R = [0, \infty)$
■ $f(x) = x^3$		origin (odd function)	$D = (-\infty, \infty)$ $R = (-\infty, \infty)$
■ $f(x) = x^{2/3}$		y-axis (even function)	$D = (-\infty, \infty)$ $R = [0, \infty)$

(continued)

Function f		Symmetry	Domain D, Range R
$f(x) = x^{1/3}$		origin (odd function)	$D = (-\infty, \infty)$ $R = (-\infty, \infty)$
$f(x) = \lvert x \rvert$		y-axis (even function)	$D = (-\infty, \infty)$ $R = [0, \infty)$
$f(x) = \dfrac{1}{x}$		origin (odd function)	$D = (-\infty, 0) \cup (0, \infty)$ $R = (-\infty, 0) \cup (0, \infty)$

Functions that are described by more than one expression, as in the next example, are called **piecewise-defined functions**.

Figure 17



EXAMPLE ■ 4 Sketch the graph of the function f defined as follows:

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

SOLUTION If $x < 0$, then $f(x) = 2x + 3$, and the graph of f is part of the line $y = 2x + 3$, as indicated in Figure 17. The open circle indicates that $(0, 3)$ is not on the graph.

If $0 \leq x < 2$, then $f(x) = x^2$, and the graph of f is part of the parabola $y = x^2$. Note that $(2, 4)$ is not on the graph.

If $x \geq 2$, the function values are always 1, and the graph is a horizontal half-line with endpoint $(2, 1)$.

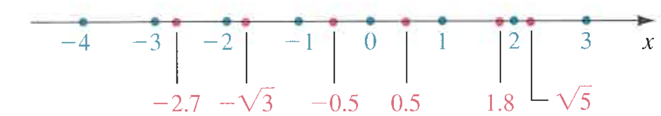
In Example 4, we see a function whose graph is made up of several disconnected pieces. Another function with this property is the **greatest integer function** f defined by $f(x) = [\![x]\!]$, where $[\![x]\!]$ is the greatest integer less than or equal to x. If we identify $\mathbb{R}$ with points on the coordinate line, then $[\![x]\!]$ is the first integer *to the left of* (or *equal to*) x.

The following illustration gives some specific values for the greatest integer function, and Figure 18 graphically illustrates the location of x and $[\![x]\!]$ for each of these values.

ILLUSTRATION

- $[\![0.5]\!] = 0$
- $[\![1.8]\!] = 1$
- $[\![\sqrt{5}]\!] = 2$
- $[\![3]\!] = 3$
- $[\![-3]\!] = -3$
- $[\![-2.7]\!] = -3$
- $[\![-\sqrt{3}]\!] = -2$
- $[\![-0.5]\!] = -1$

Figure 18

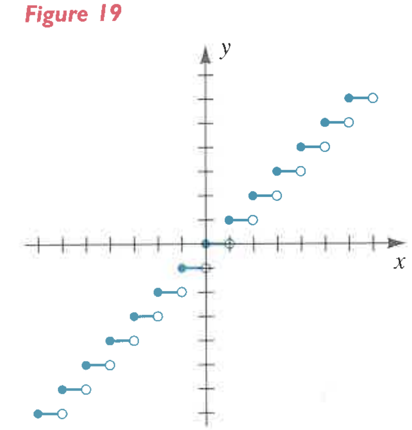


EXAMPLE ■ 5 Sketch the graph of the greatest integer function.

SOLUTION The x- and y-coordinates of some points on the graph may be listed as follows:

Values of x	$f(x) = [\![x]\!]$
⋮	⋮
$-2 \leq x < -1$	-2
$-1 \leq x < 0$	-1
$0 \leq x < 1$	0
$1 \leq x < 2$	1
$2 \leq x < 3$	2
⋮	⋮

Figure 19



Whenever x is between successive integers, the corresponding part of the graph is a segment of a horizontal line. Part of the graph is sketched in Figure 19. The graph continues indefinitely to the right and to the left.

If we know the graph of $y = f(x)$, then it is easy to sketch the graphs of functions obtained from f by *transformations* involving shifts, stretching, compressing, or reflecting. Adding or subtracting a positive constant c to each function value $f(x)$ produces a **vertical shift**. Adding c shifts the graph of f upward a distance of c units, and subtracting c shifts the graph downward, as illustrated in Figure 20. The graphs of $y = f(x + c)$

Figure 20 Vertical shifts, $c > 0$

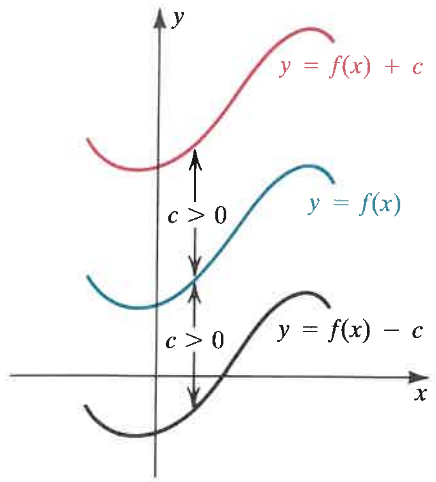


Figure 21 Horizontal shifts, $c > 0$

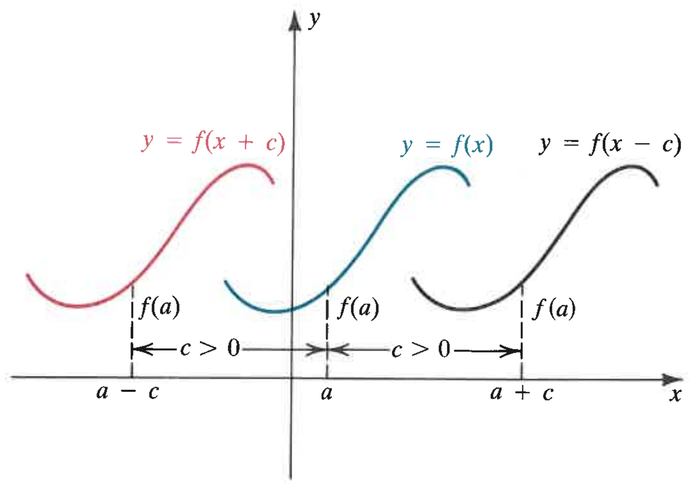


Figure 22 Vertical stretch, $c > 1$

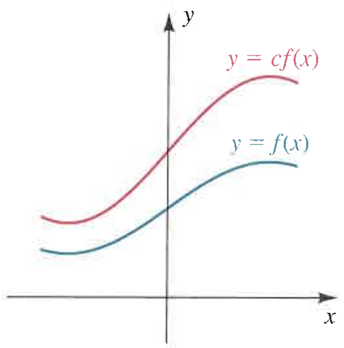


Figure 23 Vertical compression, $0 < c < 1$

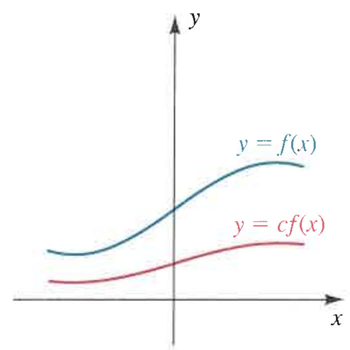
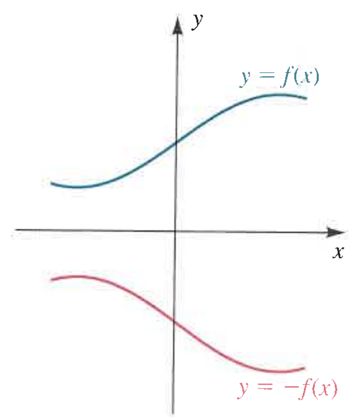


Figure 24 Reflection



and $y = f(x - c)$ are **horizontal shifts** of the graph of $y = f(x)$, c units to the left and c units to the right, respectively, as shown in Figure 21.

If we multiply each function value $f(x)$ by a positive constant c to obtain $y = cf(x)$, then we have a **vertical stretch** if $c > 1$ (Figure 22) and a **vertical compression** if $0 < c < 1$ (Figure 23). The graphs of $y = f(x)$ and $y = -f(x)$ are **reflections** of each other across the x-axis, as shown in Figure 24. It should be noted that the x-intercepts of the graph of $y = cf(x)$ are the same as those of $y = f(x)$.

In the graphs of functions we have seen thus far, the two coordinate axes have had equal scales: One unit along the x-axis represents the same length as one unit along the y-axis. We assume equal scales on all coordinate graphs that have no scale markings or numbered "tics."

It is often desirable, however, to use graphs with unequal scales. For some functions f, a relatively small x-value may give a relatively large value for $f(x)$. For example, if $f(x) = x^4 + 100$, as x increases from 0 to 5, $f(x)$ has values between 100 and 725. If we were to use equal scales to graph this function, we would have a large amount of wasted space in which no part of the graph appears.

Figure 25
$-10 \le x \le 10$, $x_{scl} = 2$
$-62 \le y \le 17{,}529$, $y_{scl} = 1000$

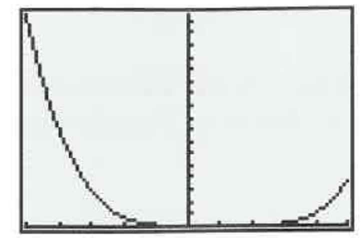


Figure 26
$-10 \le x \le 20$, $x_{scl} = 2$
$-62 \le y \le 17{,}529$, $y_{scl} = 1000$

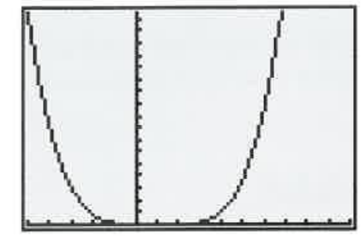


Figure 27
$-2 \le x \le 6$, $x_{scl} = 1$
$-62 \le y \le 89$, $y_{scl} = 10$

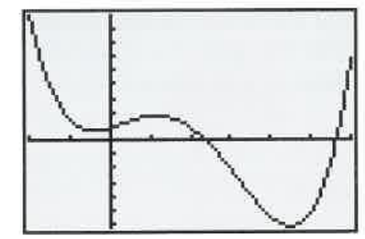
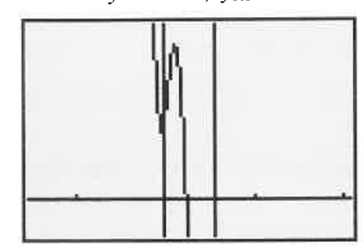


Figure 28
$-15.3 \le x \le 20.8$, $x_{scl} = 10$
$-4.3 \le y \le 19.3$, $y_{scl} = 10$



To minimize such wasted space on the screen, most computer and calculator graphs use unequal scaling. While many graphing utilities set the scales automatically, some also permit the user to set the scales x_{scl} and y_{scl} and thus designate specific units between the tic marks on each axis.

For many functions, the domain or the range of the function may be all real numbers. The computer or calculator, however, can display only a finite rectangle called the *viewing window*. The user must specify the x-interval for the viewing window by giving the left-endpoint x_{min} and the right-endpoint x_{max}. The user may also specify the y-interval, or the graphing utility may automatically calculate y_{min} and y_{max} so that the graph fits within the viewing window.

EXAMPLE ■ 6 Let $f(x) = x^4 - 7x^3 + 6x^2 + 8x + 9$. Use a graphing utility to

(a) view f with x-interval $[-10, 10]$ and y-interval $[y_{min}, y_{max}]$, where y_{min} and y_{max} represent the smallest and largest values of f, respectively, on the given x-interval

(b) estimate, without changing the y-interval from part (a), the number b such that the graph of f on the x-interval $[-10, b]$ stays within the viewing window

(c) investigate the behavior of the function near the origin

(d) view the graph with equal scales near the origin

SOLUTION

(a) To view the graph with $[-10, 10]$ as the x-interval, we set the x-range at $-10 \le x \le 10$. If your graphing utility has an automatic scaling feature, utilize it to determine that the smallest y-value is approximately -62 (at $x \approx 4.5$) and that the largest y-value is 17,529 (at $x = -10$). If this feature is not available, find these values by examining several viewing windows and then tracing to the low point (the high point at $x = -10$ should be easy to detect). Now set the y-range to $-62 \le y \le 17{,}529$ and graph f to obtain a figure similar to Figure 25.

(b) To obtain the same view as shown in Figure 26, we change the x-range to $-10 \le x \le 20$ while leaving the y-range alone. We then use the tracing feature to follow along on the curve until the graph of f leaves the viewing window in the first quadrant at $x = b \approx 13.5$.

(c) To place the origin in the viewing window, as in Figure 27, we change the x- and y-ranges and both scales and then trace the curve for x between -2 and 6. We determine that the function is negative for the (approximate) x-interval $[2.37, 5.63]$.

(d) Figure 28 shows the origin in the viewing window when the scales are set equal to 10. Note that much of this viewing window contains no part of the curve while part of the graph of the function, including its lowest point, falls outside this viewing window.

A function f is a **polynomial function** if $f(x)$ is a polynomial—that is, if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are real numbers and the exponents are nonnegative integers. If $a_n \neq 0$, then f has **degree** n. The following are special cases, where $a \neq 0$:

degree 0:	$f(x) = a$	constant function
degree 1:	$f(x) = ax + b$	linear function
degree 2:	$f(x) = ax^2 + bx + c$	quadratic function

A **rational function** is a quotient of two polynomial functions. Later in the text, we shall use methods of calculus to investigate graphs of polynomial and rational functions.

An **algebraic function** is a function that can be expressed in terms of sums, differences, products, quotients, or rational powers of polynomials. For example, if

$$f(x) = 5x^4 - 2\sqrt[3]{x} + \frac{x(x^2+5)}{\sqrt{x^3 + \sqrt{x}}},$$

then f is an algebraic function. Functions that are not algebraic are termed **transcendental**. The trigonometric, exponential, and logarithmic functions considered later are examples of transcendental functions.

In calculus, we often build complicated functions from simpler functions by combining them in various ways, using arithmetic operations and composition. If f and g are functions, we define the **sum** $f + g$, the **difference** $f - g$, the **product** fg, and the **quotient** f/g as follows:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of $f + g$, $f - g$, and fg is the *intersection* of the domains of f and g—that is, the numbers that are *common* to both domains. The domain of f/g consists of all numbers x in the intersection such that $g(x) \neq 0$.

EXAMPLE ▪ 7 Let $f(x) = \sqrt{4 - x^2}$ and $g(x) = 3x + 1$. Find the sum, difference, product, and quotient of f and g, and specify the domain of each.

SOLUTION The domain of f is the closed interval $[-2, 2]$, and the domain of g is $\mathbb{R}$. Consequently, the intersection of their domains is $[-2, 2]$, and we obtain the following:

$$(f+g)(x) = \sqrt{4-x^2} + (3x+1) \qquad -2 \le x \le 2$$
$$(f-g)(x) = \sqrt{4-x^2} - (3x+1) \qquad -2 \le x \le 2$$
$$(fg)(x) = \sqrt{4-x^2}(3x+1) \qquad -2 \le x \le 2$$
$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{4-x^2}}{3x+1} \qquad -2 \le x \le 2 \text{ and } x \neq -\tfrac{1}{3}$$

We can also combine two functions to form a new function by the process of composition—that is, by applying one function to the result obtained from the other. Starting with functions f and g, we obtain *composite functions* $f \circ g$ and $g \circ f$ (read "f circle g" and "g circle f," respectively). The function $f \circ g$ is defined as follows.

Definition 11

> The **composite function** $f \circ g$ of f and g is defined by
>
> $$(f \circ g)(x) = f(g(x)).$$
>
> The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Figure 29

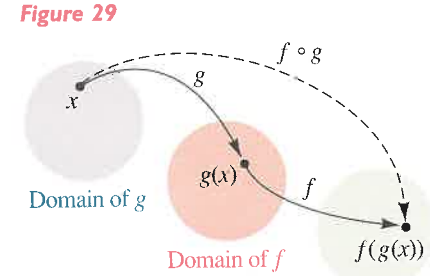


Figure 29 illustrates relationships between f, g, and $f \circ g$. Note that for x in the domain of g, we *first find* $g(x)$ (which must be in the domain of f) and then, *second, find* $f(g(x))$.

For the composite function $g \circ f$, we reverse this order, first finding $f(x)$ and then finding $g(f(x))$. The domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.

EXAMPLE ▪ 8 If $f(x) = x^2 - 1$ and $g(x) = 3x + 5$, find

(a) $(f \circ g)(x)$ and the domain of $f \circ g$

(b) $(g \circ f)(x)$ and the domain of $g \circ f$

SOLUTION

(a)
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{definition of } f \circ g\\ &= f(3x+5) && \text{definition of } g\\ &= (3x+5)^2 - 1 && \text{definition of } f\\ &= 9x^2 + 30x + 24 && \text{simplifying}\end{aligned}$$

The domain of both f and g is $\mathbb{R}$. Since for each x in $\mathbb{R}$ (the domain of g) the function value $g(x)$ is in $\mathbb{R}$ (the domain of f), the domain of $f \circ g$ is also $\mathbb{R}$.

(b)
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{definition of } g \circ f\\ &= g(x^2 - 1) && \text{definition of } f\\ &= 3(x^2 - 1) + 5 && \text{definition of } g\\ &= 3x^2 + 2 && \text{simplifying}\end{aligned}$$

Since for each x in $\mathbb{R}$ (the domain of f) the function value $f(x)$ is in $\mathbb{R}$ (the domain of g), the domain of $g \circ f$ is $\mathbb{R}$.

Note that in Example 8, $f(g(x))$ and $g(f(x))$ are not always the same; that is, $f \circ g \neq g \circ f$.

If two functions f and g both have domain $\mathbb{R}$, then the domain of $f \circ g$ and $g \circ f$ is also $\mathbb{R}$, as was illustrated in Example 8. The next example shows that the domain of a composite function may differ from those of the two given functions.

EXAMPLE ■ 9 If $f(x) = x^2 - 16$ and $g(x) = \sqrt{x}$, find

(a) $(f \circ g)(x)$ and the domain of $f \circ g$

(b) $(g \circ f)(x)$ and the domain of $g \circ f$

SOLUTION We first note that the domain of f is $\mathbb{R}$ and the domain of g is the set of all nonnegative real numbers—that is, the interval $[0, \infty)$. We may proceed as follows.

(a)
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{definition of } f \circ g\\ &= f(\sqrt{x}) && \text{definition of } g\\ &= (\sqrt{x})^2 - 16 && \text{definition of } f\\ &= x - 16 && \text{simplifying}\end{aligned}$$

If we consider only the final expression $x - 16$, we might be led to believe that the domain of $f \circ g$ is $\mathbb{R}$, since $x - 16$ is defined for every real number x. However, this is not the case. By definition, the domain of $f \circ g$ is the set of all x in $[0, \infty)$ (the domain of g) such that $g(x)$ is in $\mathbb{R}$ (the domain of f). Since $g(x) = \sqrt{x}$ is in $\mathbb{R}$ for every x in $[0, \infty)$, it follows that the domain of $f \circ g$ is $[0, \infty)$.

(b)
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{definition of } g \circ f\\ &= g(x^2 - 16) && \text{definition of } f\\ &= \sqrt{x^2 - 16} && \text{definition of } g\end{aligned}$$

By definition, the domain of $g \circ f$ is the set of all x in $\mathbb{R}$ (the domain of f) such that $f(x) = x^2 - 16$ is in $[0, \infty)$ (the domain of g). The statement

$x^2 - 16$ is in $[0, \infty)$ is equivalent to each of the inequalities

$$x^2 - 16 \geq 0, \quad x^2 \geq 16, \quad \text{and} \quad |x| \geq 4.$$

Thus, the domain of $g \circ f$ is $(-\infty, -4] \cup [4, \infty)$. Note that this domain is different from the domains of both f and g.

If f and g are functions such that

$$y = f(u) \quad \text{and} \quad u = g(x),$$

then substituting for u in $y = f(u)$ yields

$$y = f(g(x)).$$

For certain problems in calculus, we *reverse* this procedure; that is, given $y = h(x)$ for some function h, we find a *composite function form* $y = f(u)$ and $u = g(x)$ such that $h(x) = f(g(x))$.

EXAMPLE ■ 10 Express $y = (2x + 5)^8$ in composite function form.

SOLUTION A simple method for solving this problem is to assume that we want to evaluate the expression $(2x + 5)^8$ by using a calculator. We might first calculate $2x + 5$ and then raise the result to the eighth power. This procedure suggests that we let

$$u = 2x + 5 \quad \text{and} \quad y = u^8,$$

which is a composite function form for $y = (2x + 5)^8$.

The method of the preceding example can be extended to other functions. In general, suppose we are given $y = h(x)$. To choose the *inside* expression $u = g(x)$ in a composite function form, ask the following question: If you were using a calculator, which part of the expression $h(x)$ would you evaluate first? The answer often leads to a suitable choice for $u = g(x)$. After choosing u, refer to $h(x)$ to determine $y = f(u)$. The following illustration provides some typical examples.

ILLUSTRATION

Function value	Choice for $u = g(x)$	Choice for $y = f(u)$
■ $y = (x^3 - 5x + 1)^4$	$u = x^3 - 5x + 1$	$y = u^4$
■ $y = \sqrt{x^2 - 4}$	$u = x^2 - 4$	$y = \sqrt{u}$
■ $y = \dfrac{2}{3x + 7}$	$u = 3x + 7$	$y = \dfrac{2}{u}$

The composite function form is never unique. For example, consider the first expression in the preceding illustration:

$$y = (x^3 - 5x + 1)^4$$

If n is any nonzero integer, we could choose

$$u = (x^3 - 5x + 1)^n \quad \text{and} \quad y = u^{4/n}.$$

Thus, there are an *unlimited* number of composite function forms. Generally, our goal is to choose a form such that the expression for y is simple, as we did in the illustration.

As a general rule, the composition $f \circ g$ of two functions will be more complex than either f or g. In some instances, however, the composition may turn out to be particularly simple, as the following example illustrates.

EXAMPLE ▪ 11 If $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$, find

(a) $(f \circ g)(x)$ and the domain of $f \circ g$

(b) $(g \circ f)(x)$ and the domain of $g \circ f$

SOLUTION

(a)

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{definition of } f \circ g \\ &= f(\sqrt[3]{x-1}) && \text{definition of } g \\ &= (\sqrt[3]{x-1})^3 + 1 && \text{definition of } f \\ &= x - 1 + 1 = x && \text{simplifying} \end{aligned}$$

Since for each x in $\mathbb{R}$ (the domain of g) the function value $g(x)$ is in $\mathbb{R}$ (the domain of f), the domain of $f \circ g$ is $\mathbb{R}$.

(b) A similar computation shows that $(g \circ f)(x) = x$ for all real numbers x and the domain of $g \circ f$ is also $\mathbb{R}$.

An **identity function** is a function h with the property that $h(x) = x$ for all x in the domain of h. The graph of an identity function lies along the line $y = x$. For the functions f and g of Example 11, both $f \circ g$ and $g \circ f$ are identity functions.

If the composition of two functions f and g is an identity function, then the functions are **inverses** of each other; that is, applying g to $f(x)$ returns x and applying f to $g(x)$ returns x. For inverse functions, it follows that if the point (a, b) lies on the graph of one of the functions, then the point (b, a) lies on the graph of the other. Thus, for inverse functions, the graph of either function is the reflection of the graph of the other across the line $y = x$.

EXERCISES B

1 If $f(x) = \sqrt{x-4} - 3x$, find $f(4)$, $f(8)$, and $f(13)$.

2 If $f(x) = \dfrac{x}{x-3}$, find $f(-2)$, $f(0)$, and $f(3.01)$.

Exer. 3–6: If a and h are real numbers, find and simplify (a) $f(a)$, (b) $f(-a)$, (c) $-f(a)$, (d) $f(a+h)$, (e) $f(a) + f(h)$, and (f) $\dfrac{f(a+h) - f(a)}{h}$, provided $h \neq 0$.

3 $f(x) = 5x - 2$ **4** $f(x) = 3 - 4x$

5 $f(x) = x^2 - x + 3$ **6** $f(x) = 2x^2 + 3x - 7$

Exer. 7–10: Find the domain of f.

7 $f(x) = \dfrac{x+1}{x^3 - 4x}$ **8** $f(x) = \dfrac{4x}{6x^2 + 13x - 5}$

9 $f(x) = \dfrac{\sqrt{2x-3}}{x^2 - 5x + 4}$ **10** $f(x) = \dfrac{\sqrt{4x-3}}{x^2 - 4}$

Exer. 11–12: Determine whether f is even, odd, or neither even nor odd.

11 **(a)** $f(x) = 5x^3 + 2x$ **(b)** $f(x) = |x| - 3$
(c) $f(x) = (8x^3 - 3x^2)^3$

12 **(a)** $f(x) = \sqrt{3x^4 + 2x^2 - 5}$
(b) $f(x) = 6x^5 - 4x^3 + 2x$
(c) $f(x) = x(x - 5)$

Exer. 13–18: Sketch the graph of f.

13 $f(x) = \begin{cases} x + 2 & \text{if } x \le -1 \\ x^3 & \text{if } |x| < 1 \\ -x + 3 & \text{if } x \ge 1 \end{cases}$

14 $f(x) = \begin{cases} x - 3 & \text{if } x \le -2 \\ -x^2 & \text{if } -2 < x < 1 \\ -x + 4 & \text{if } x \ge 1 \end{cases}$

15 $f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{if } x \neq -1 \\ 2 & \text{if } x = -1 \end{cases}$

16 $f(x) = \begin{cases} \dfrac{x^2 - 4}{2 - x} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

17 **(a)** $f(x) = [\![x - 3]\!]$ **(b)** $f(x) = [\![x]\!] - 3$
(c) $f(x) = 2[\![x]\!]$ **(d)** $f(x) = [\![2x]\!]$

18 **(a)** $f(x) = [\![x + 2]\!]$ **(b)** $f(x) = [\![x]\!] + 2$
(c) $f(x) = \frac{1}{2}[\![x]\!]$ **(d)** $f(x) = [\![\frac{1}{2}x]\!]$

Exer. 19–28: Sketch, on the same coordinate plane, the graphs of f for the given values of c. (Make use of symmetry, vertical shifts, horizontal shifts, stretching, or reflecting.)

19 $f(x) = |x| + c$; $c = 0, 1, -3$

20 $f(x) = |x - c|$; $c = 0, 2, -3$

21 $f(x) = 2\sqrt{x} + c$; $c = 0, 3, -2$

22 $f(x) = \sqrt{9 - x^2} + c$; $c = 0, 1, -3$

23 $f(x) = 2\sqrt{x - c}$; $c = 0, 1, -2$

24 $f(x) = -2(x - c)^2$; $c = 0, 1, -2$

25 $f(x) = c\sqrt{4 - x^2}$; $c = 1, 3, -2$

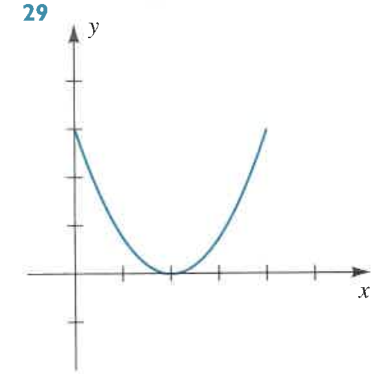
26 $f(x) = (x + c)^3$; $c = 0, 1, -2$

27 $f(x) = (x - c)^{2/3} + 2$; $c = 0, 4, -3$

28 $f(x) = (x - 1)^{1/3} - c$; $c = 0, 2, -1$

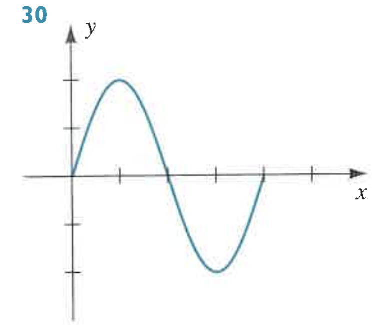
Exer. 29–30: The graph of a function f with domain $0 \le x \le 4$ is shown in the figure. Sketch the graph of the given equation.

29



(a) $y = f(x + 3)$
(b) $y = f(x - 3)$
(c) $y = f(x) + 3$
(d) $y = f(x) - 3$
(e) $y = -3f(x)$
(f) $y = -\frac{1}{3}f(x)$
(g) $y = -f(x + 2) - 3$
(h) $y = f(x - 2) + 3$

30



(a) $y = f(x - 2)$
(b) $y = f(x + 2)$
(c) $y = f(x) - 2$
(d) $y = f(x) + 2$
(e) $y = -2f(x)$
(f) $y = -\frac{1}{2}f(x)$
(g) $y = -f(x + 4) - 2$
(h) $y = f(x - 4) + 2$

c **Exer. 31–40: Use a graphing utility to examine several different views of the graph of the function f. Copy one that displays the important features of the function. Clearly indicate the scaling or the range of the viewing window selected.**

31 $f(x) = \dfrac{1}{\sqrt{x^4 + 1}}$

32 $f(x) = x|x^2 - 7|$

33 $f(x) = \sqrt[3]{x^3 - 2}$

34 $f(x) = \sqrt[3]{x^3 - 8x}$

35 $f(x) = \dfrac{\sqrt{1+x} - 1}{x}$

36 $f(x) = \dfrac{\sqrt{2+x} - \sqrt{5}}{x - 3}$

37 $f(x) = |x^3 - x + 1|$

38 $f(x) = \frac{1}{5}x^4 + \frac{2}{3}x^3 + 1$

39 $f(x) = x^4 + 5x^3 - 6x^2 - 7x - 8$

40 $f(x) = x^5 - 7x^3 + 8x + 5$

Exer. 41–44: (a) Find $(f+g)(x), (f-g)(x), (fg)(x)$, and $(f/g)(x)$. (b) Find the domain of $f+g, f-g$, and fg; and find the domain of f/g.

41 $f(x) = \sqrt{x+5}$; $\quad g(x) = \sqrt{x+5}$

42 $f(x) = \sqrt{3-2x}$; $\quad g(x) = \sqrt{x+4}$

43 $f(x) = \dfrac{2x}{x-4}$; $\quad g(x) = \dfrac{x}{x+5}$

44 $f(x) = \dfrac{x}{x-2}$; $\quad g(x) = \dfrac{3x}{x+4}$

Exer. 45–52: (a) Find $(f \circ g)(x)$ and the domain of $f \circ g$. (b) Find $(g \circ f)(x)$ and the domain of $g \circ f$.

45 $f(x) = x^2 - 3x$; $\quad g(x) = \sqrt{x+2}$

46 $f(x) = \sqrt{x - 15}$; $\quad g(x) = x^2 + 2x$

47 $f(x) = \sqrt{x-2}$; $\quad g(x) = \sqrt{x+5}$

48 $f(x) = \sqrt{3-x}$; $\quad g(x) = \sqrt{x+2}$

49 $f(x) = \sqrt{25 - x^2}$; $\quad g(x) = \sqrt{x-3}$

50 $f(x) = \sqrt{3-x}$; $\quad g(x) = \sqrt{x^2 - 16}$

51 $f(x) = \dfrac{x}{3x+2}$; $\quad g(x) = \dfrac{2}{x}$

52 $f(x) = \dfrac{x}{x-2}$; $\quad g(x) = \dfrac{3}{x}$

Exer. 53–60: Find a composite function form for y.

53 $y = (x^2 + 3x)^{1/3}$

54 $y = \sqrt[4]{x^4 - 16}$

55 $y = \dfrac{1}{(x-3)^4}$

56 $y = 4 + \sqrt{x^2 + 1}$

57 $y = (x^4 - 2x^2 + 5)^5$

58 $y = \dfrac{1}{(x^2 + 3x - 5)^3}$

59 $y = \dfrac{\sqrt{x+4} - 2}{\sqrt{x+4} + 2}$

60 $y = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}}$

c **61** If

$$f(x) = \sqrt{x^2 - 1.7} \quad \text{and} \quad g(x) = \frac{x^3 - x + 1}{\sqrt{x}},$$

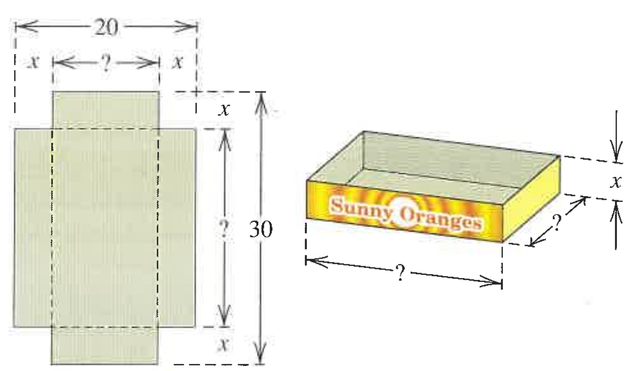
approximate $(f \circ g)(2.4)$ and $(g \circ f)(2.4)$.

c **62** If $f(x) = \sqrt{x^3 + 1} - 1$, approximate $f(0.0001)$. In order to avoid calculating a zero value for $f(0.0001)$, rewrite the formula for f as

$$f(x) = \frac{x^3}{\sqrt{x^3 + 1} + 1}.$$

63 An open box is to be made from a rectangular piece of cardboard 20 in. × 30 in. by cutting out identical squares of area x^2 from each corner and turning up the sides (see figure). Express the volume V of the box as a function of x.

Exercise 63

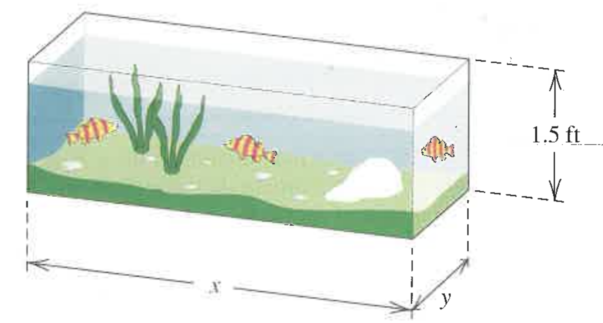


64 An open-top aquarium of height 1.5 ft is to have a volume of 6 ft^3. Let x denote the length of the base, and let y denote the width (see figure on the following page).

(a) Express y as a function of x.

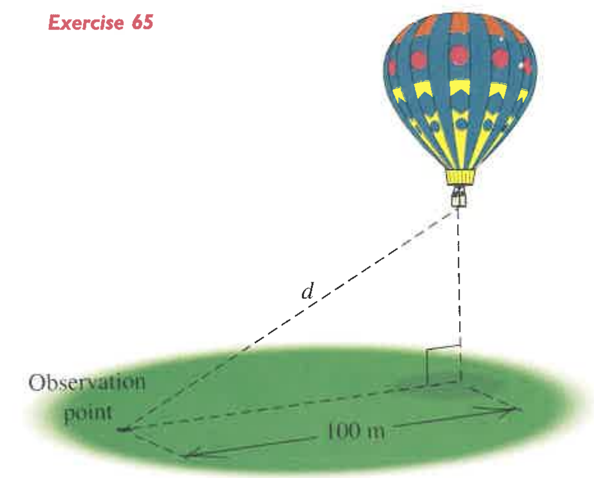
(b) Express the total number of square feet S of glass needed as a function of x.

Exercise 64



65 A hot-air balloon is released at 1:00 P.M. and rises vertically at a rate of 2 m/sec. An observation point is situated 100 m from a point on the ground directly below the balloon (see figure). If t denotes the time (in seconds) after 1:00 P.M., express the distance d between the balloon and the observation point as a function of t.

Exercise 65



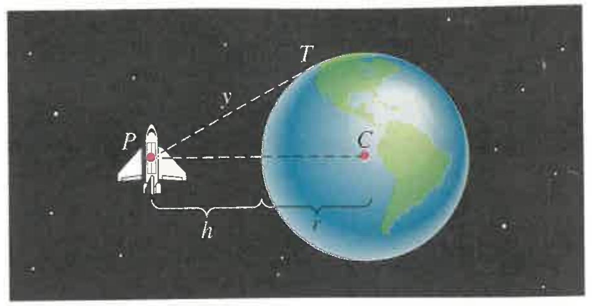
66 Refer to Example 3. A steel storage tank for propane gas is to be constructed in the shape of a right circular cylinder of altitude 10 ft with a hemisphere attached to each end. The radius r is yet to be determined. Express the surface area S of the tank as a function of r.

67 From an exterior point P that is h units from a circle of radius r, a tangent line is drawn to the circle (see figure). Let y denote the distance from the point P to the point of tangency T.

(a) Express y as a function of h. (*Hint:* If C is the center of the circle, then PT is perpendicular to CT.)

(b) If r is the radius of the earth and h is the altitude of a space shuttle, then we can derive a formula for the maximum distance (to the earth) that an astronaut can see from the shuttle. In particular, if $h = 200$ mi and $r \approx 4000$ mi, approximate y.

Exercise 67

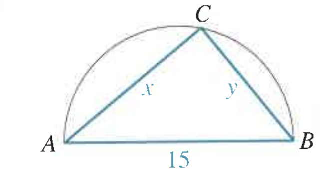


68 Triangle ABC is inscribed in a semicircle of diameter 15 (see figure).

(a) If x denotes the length of side AC, express the length y of side BC as a function of x, and state its domain. (*Hint:* Angle ACB is a right angle.)

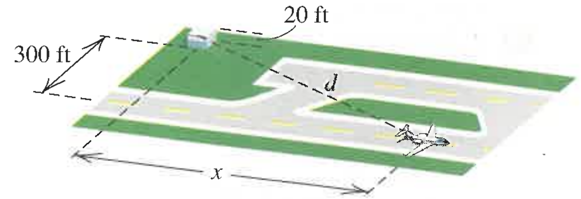
(b) Express the area of triangle ABC as a function of x.

Exercise 68



69 The relative positions of an airport runway and a 20-ft-tall control tower are shown in the figure. The beginning of the runway is at a perpendicular distance of 300 ft from the base of the tower. If x denotes the distance that an airplane has moved down the runway, express the distance d between the airplane and the control booth as a function of x.

Exercise 69

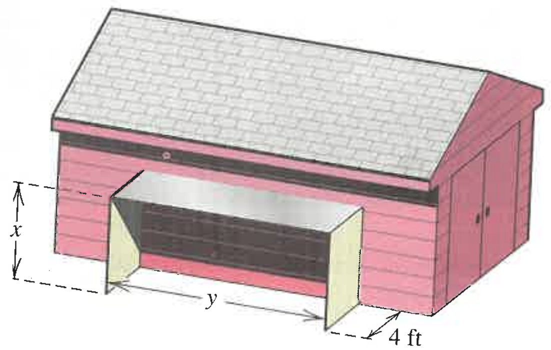


70 An open rectangular storage shelter consisting of two vertical sides, 4 ft wide, and a flat roof is to be attached to an existing structure as illustrated in the figure on the following page. The flat roof is made of tin that costs \$5 per square foot, and the other two sides are made of plywood that costs \$2 per square foot.

(a) If \$400 is to be spent on construction, express the length y as a function of the height x.

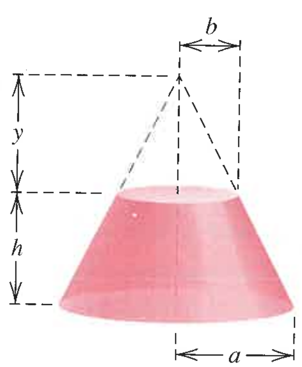
(b) Express the volume V inside the shelter as a function of x.

Exercise 70



71 The shape of the first spacecraft in the Apollo program was a frustum of a right circular cone, a solid formed by truncating a cone by a plane parallel to its base. For the frustum shown in the figure, the radii a and b have already been determined.

Exercise 71



(a) Use similar triangles to express y as a function of h.

(b) Express the volume of the frustum as a function of h.

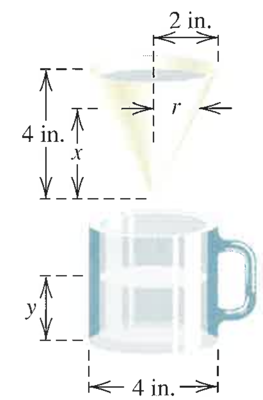
(c) If $a = 6$ ft and $b = 3$ ft, for what value of h is the volume of the frustum 600 ft^3?

72 Suppose 5 in^3 of water is poured into a conical filter and subsequently drips into a cup, as shown in the figure. Let x denote the height of the water in the filter, and let y denote the height of the water in the cup.

(a) Express the radius r shown in the figure as a function of x. (*Hint:* Use similar triangles.)

(b) Express the height y of the water in the cup as a function of x. (*Hint:* What is the sum of the two volumes shown in the figure?)

Exercise 72



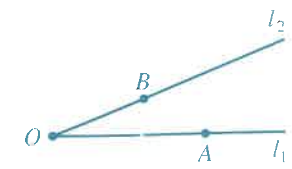
C TRIGONOMETRY

Trigonometry helps us understand angles, triangles, and circles through the use of six special *trigonometric functions.* In this section, we review some of the basic ideas and formulas of trigonometry that are especially important for calculus.

ANGLES

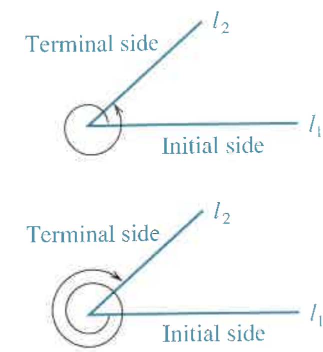
An angle is determined by two rays, or line segments, having the same initial point O (the **vertex** of the angle). If A and B are points on the rays l_1 and l_2 in Figure 30, we refer to **angle** AOB, or $\angle AOB$.

Figure 30



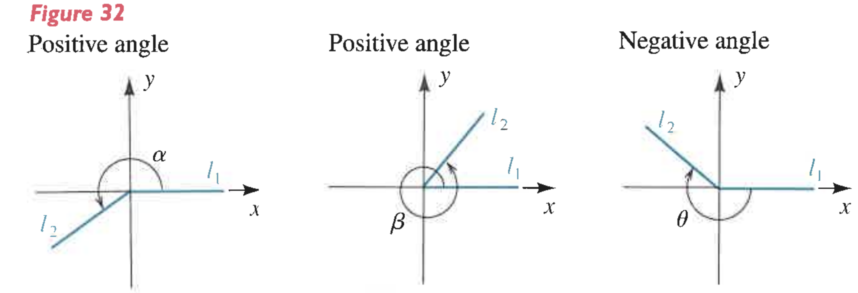
We may also interpret $\angle AOB$ as a rotation about O of the ray l_1 (the **initial side** of the angle) to a position specified by l_2 (the **terminal side**). There is no restriction on the amount or direction of rotation. We can let l_1 make several full revolutions in either direction about O before stopping at l_2, as shown by the curved arrows in Figure 31. Thus, many different angles have the same initial and terminal sides.

Figure 31



In a rectangular coordinate system, the **standard position** of an angle has the vertex at the origin and the initial side along the positive x-axis (see Figure 32). A counterclockwise rotation of the initial side produces a **positive angle**, whereas a clockwise rotation gives a **negative angle**. Lower-case Greek letters such as α, β, and θ are often used to denote angles.

Figure 32



The magnitude of an angle is expressed in either degrees or radians. An angle of **degree measure** 1° corresponds to 1/360 of a complete counterclockwise revolution. An angle of **radian measure** 1 corresponds to $1/(2\pi)$ of a complete counterclockwise revolution. In calculus, the radian is a more important unit of angular measure. To visualize radian measure, consider a circle of radius 1 with center at the vertex of the angle. The radian measure of an angle is the length of the arc on the circle that lies between the initial and the terminal sides. If the length of arc AP (sometimes denoted $\widehat{AP}$) is 1 unit, as in Figure 33, then θ is an angle of 1 radian. Figure 34 shows a more general case in which the radian measure of angle θ is the length t of arc AP. For convenience, we show the angle θ in Figures 33 and 34 in standard position.

Figure 33

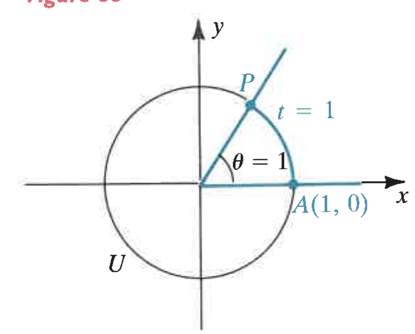
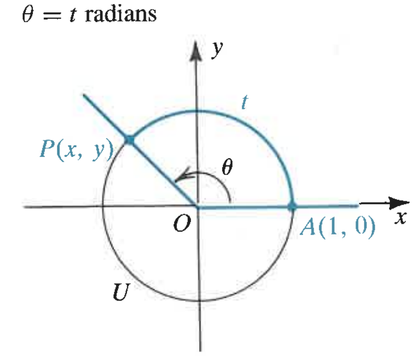


Figure 34
$\theta = t$ radians



Since the circumference of the unit circle is 2π, it follows that

$$2\pi \text{ radians} = 360°.$$

From this relationship between degrees and radians, we find that

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \approx 57.29578° \quad \text{and} \quad 1° \approx 0.01745 \text{ radian}.$$

The following rules are a more general consequence of these relationships.

Conversion Rules for Radians and Degrees 12

(i) To change radian measure to degrees, multiply by $180/\pi$.
(ii) To change degree measure to radians, multiply by $\pi/180$.

This table displays the relationship between the radian and the degree measures of several common angles.

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°

CAUTION *When radian measure of an angle is used, no units are indicated.* Thus, if an angle θ has radian measure 5, we write $\theta = 5$ instead of $\theta = 5$ radians. There should be no confusion as to whether radian or degree measure is intended, since if θ has degree measure 5°, we write $\theta = 5°$, *not* $\theta = 5$.

EXAMPLE ■ 1

(a) Express $7\pi/9$ radians in degrees.

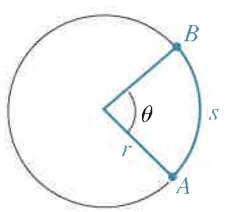
(b) Express 105° in radians.

SOLUTION

(a) By (12)(i), to convert radians to degrees, we multiply $7\pi/9$ by $180/\pi$ to obtain 140°.

(b) By (12)(ii), to convert degrees to radians, we multiply 105° by $\pi/180$ to obtain $7\pi/12$ radians.

Figure 35



A **central angle** of a circle is an angle θ whose vertex is at the center of the circle, as illustrated in Figure 35. We say that arc AB *subtends* θ or that θ *is subtended by arc AB.* A central angle also determines a circular sector AOB. The length of the arc AB and the area of the sector AOB are functions of the radian measure of the angle and the radius of the circle.

Length of a Circular Arc and Area of a Circular Sector 13

If an arc of length s on a circle of radius r subtends a central angle of radian measure θ, and if A is the area of the circular sector determined by θ, then

(i) $$s = r\theta$$

and

(ii) $$A = \tfrac{1}{2}r^2\theta.$$

EXAMPLE ■ 2 An arc of length 6 cm on a circle of radius 3 cm subtends a central angle θ.

(a) Find the radian measure of θ.

(b) Find the area of the circular sector determined by θ.

SOLUTION

(a) From (13)(i), $s = r\theta$, so

$$\theta = \frac{s}{r} = \frac{6}{3} = 2 \text{ radians.}$$

(b) From (13)(ii), the area A of the circular sector is

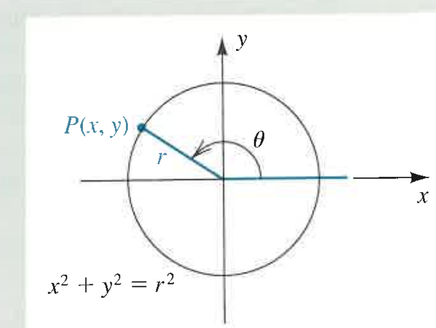
$$\tfrac{1}{2}r^2\theta = \tfrac{1}{2}(3^2)(2) = 9 \text{ cm}^2.$$

TRIGONOMETRIC FUNCTIONS

The six trigonometric functions are the **sine, cosine, tangent, cosecant, secant,** and **cotangent**. We denote them by **sin, cos, tan, csc, sec,** and **cot,** respectively.

We may define the trigonometric functions in terms of either an angle θ or a real number x. We begin with the angle approach. Let θ be any angle in standard position, and let C be a circle of radius r with center at the origin. Let P be a point on the circle that lies on the terminal side of the angle and has coordinates (x, y). The trigonometric functions are defined as ratios involving the values x, y, and r.

Trigonometric Functions of Any Angle 14



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

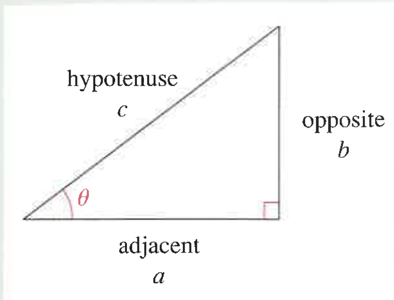
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

In the special case where θ is an acute angle (between 0 and $\pi/2$), the vertical segment PQ from P to a point Q on the x-axis determines a right triangle POQ. The value of the x-coordinate of P is equal to the length of the segment OQ, and the value of the y-coordinate of P is equal to the length of the segment PQ.

The trigonometric functions of θ can also be expressed as ratios involving the hypotenuse c, the adjacent side a, and the opposite side b of a right triangle.

Trigonometric Functions of an Acute Angle **15**



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$$

Now that we have a definition of the trigonometric functions using the angle approach, it is easy to define these functions for an arbitrary real number x.

Trigonometric Functions of a Real Number **16**

The **value of a trigonometric function at a real number x** is its value at an angle of x radians.

From this definition, we see that there is no difference between trigonometric functions of angles measured in radians and trigonometric functions of real numbers. We can interpret sin 2, for example, as *either* the sine of an angle of 2 radians *or* the sine of the real number 2.

The sign of the value of a trigonometric function of an angle depends on the quadrant containing the terminal side of θ. For example, if θ is in quadrant IV (as in Figure 36), then the point $P(x, y)$ has $x > 0$ and $y < 0$,

Figure 36

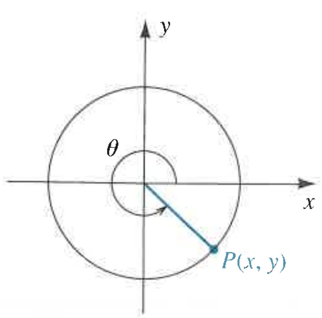
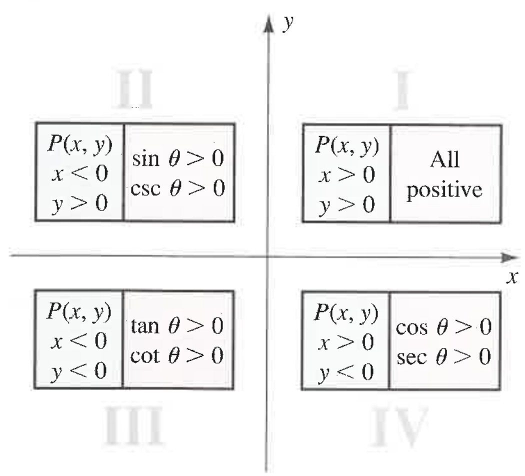


Figure 37 Positive functions

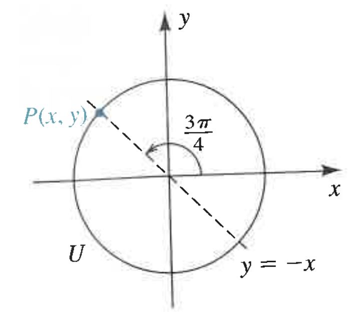


so $\cos\theta = x/r$ and $\sec\theta = r/x$ are positive while the other four functions are negative. Figure 37 indicates the *positive* trigonometric functions for each quadrant.

EXAMPLE ■ 3 Find the values of the trigonometric functions for $\theta = 3\pi/4$.

SOLUTION For $\theta = 3\pi/4$, the point $P(x, y)$ is in quadrant II on the unit circle U and on the line $y = -x$, as illustrated in Figure 38. Since $x^2 + y^2 = 1$ and $y = -x$, we have $x^2 + (-x)^2 = 1$, so $2x^2 = 1$. Thus,

Figure 38



$$\cos\frac{3\pi}{4} = x = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{3\pi}{4} = y = \frac{\sqrt{2}}{2}.$$

The other trigonometric function values are

$$\tan\frac{3\pi}{4} = -1, \qquad \cot\frac{3\pi}{4} = -1,$$

$$\sec\frac{3\pi}{4} = -\sqrt{2}, \quad \text{and} \quad \csc\frac{3\pi}{4} = \sqrt{2}.$$

Let us now consider the domain of the trigonometric functions. Since these functions are ratios, there is the possibility of an undefined fraction occurring because a denominator is 0. Since r, the radius of a circle, is always positive, $\sin\theta = y/r$ and $\cos\theta = x/r$ are defined for all angles. Hence, the domain of the sine and the cosine functions consists of all real numbers. The cosecant and the cotangent functions are undefined when $y = 0$, which occurs when the terminal side of the angle lies along the x-axis; that is, when θ is an integer multiple of π. Similarly, the secant and the tangent functions are undefined when $x = 0$, which occurs when the terminal side lies along the y-axis; that is, when $\theta = \pi/2$ plus an integer multiple of π.

From the definition of the trigonometric functions of any angle, $|x| \le r$ and $|y| \le r$ or, equivalently, $|x/r| \le 1$ and $|y/r| \le 1$. Thus,

$$|\sin\theta| \le 1, \qquad |\cos\theta| \le 1, \qquad |\csc\theta| \ge 1, \quad \text{and} \quad |\sec\theta| \ge 1$$

for every θ in the domains of these functions.

TRIGONOMETRIC IDENTITIES

We next examine some important relationships or identities that exist among the trigonometric functions. Trigonometric identities provide us with ways in which to rewrite expressions in forms that may be simpler to work with.

Several **fundamental identities** follow directly from the definition of the trigonometric functions of any angle.

Reciprocal and Ratio Identities 17

$$\csc\theta = \frac{1}{\sin\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

A second set of identities, called the **Pythagorean identities,** can be formulated from the observation that if $P(x, y)$ is a point on the unit circle centered at the origin, then $\sin\theta = y$ and $\cos\theta = x$. Thus, the equation of the circle, $x^2 + y^2 = 1$, or

$$y^2 + x^2 = 1, \quad \text{is equivalent to} \quad \sin^2\theta + \cos^2\theta = 1.$$

CAUTION The notation $\sin^2\theta$ represents the square of the sine of θ; that is, $\sin^2\theta = (\sin\theta)(\sin\theta)$. To indicate the sine of the square of θ, we write $\sin(\theta^2)$.

Dividing both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ or $\cos^2\theta$ yields two more useful identities.

Pythagorean Identities 18

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

EXAMPLE ■ 4 Express $\sqrt{16 - x^2}$ in terms of a trigonometric function of θ without radicals by making the trigonometric substitution

$$x = 4\sin\theta \quad \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

SOLUTION We let $x = 4\sin\theta$. Then

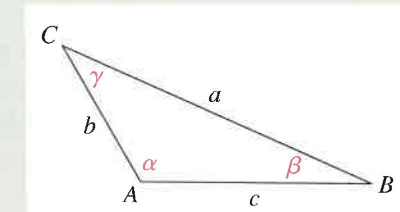
$$\begin{aligned}\sqrt{16 - x^2} &= \sqrt{16 - (4\sin\theta)^2}\\ &= \sqrt{16 - 16\sin^2\theta}\\ &= \sqrt{16(1 - \sin^2\theta)}\\ &= \sqrt{16\cos^2\theta}\\ &= 4\cos\theta.\end{aligned}$$

The last equality is true because if $-\pi/2 \le \theta \le \pi/2$, then $\cos\theta \ge 0$ and so $\sqrt{\cos^2\theta} = \cos\theta$.

Some trigonometric identities state relationships that hold among the lengths of sides of a triangle and the sine and cosine of the angles of the triangle. Of particular usefulness in applications are the *law of sines* and the *law of cosines.*

Law of Sines and Law of Cosines 19

If ABC is a triangle labeled as shown, then the following relationships are true.



The law of sines:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

The law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Note that if ABC is a right triangle with $\gamma = \pi/2$, then the third equation in the law of cosines becomes the familiar Pythagorean theorem, $c^2 = a^2 + b^2$, since $\cos\pi/2 = 0$. Thus, we may regard the law of cosines as a generalization of the Pythagorean theorem.

Many other important relationships exist among the trigonometric functions.

Additional Trigonometric Identities 20

Formulas for negatives:

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$	$\cot(-\theta) = -\cot\theta$

for any real number θ.

Addition and subtraction formulas for the sine and cosine:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

for any real numbers α and β.

Double-angle formulas for the sine and cosine:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

(continued)

Half-angle formulas for the sine and cosine:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

for any real number θ.

The negative formulas show that the sine, tangent, cosecant, and cotangent functions are odd and the cosine and secant functions are even. Other trigonometric identities useful in calculus are listed on the inside back cover of this text.

EXAMPLE ■ 5 Verify the following addition formula for the tangent function.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

SOLUTION

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \qquad \text{tangent identity}$$

$$= \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \qquad \text{addition formulas for sine and cosine}$$

If $\cos\alpha\cos\beta \neq 0$, then we may divide the numerator and the denominator by $\cos\alpha\cos\beta$, thereby obtaining 1 as the first term in the denominator.

$$\tan(\alpha+\beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \qquad \text{divide by } \cos\alpha\cos\beta$$

$$= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \qquad \text{simplify}$$

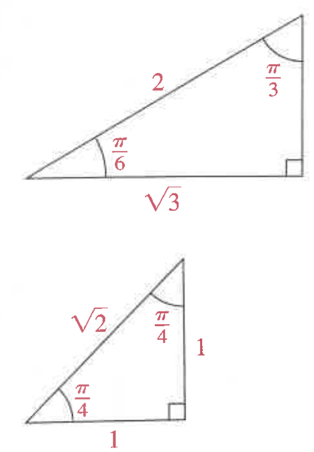
EVALUATING TRIGONOMETRIC FUNCTIONS

There are a variety of ways to find the values of a trigonometric function, including the use of scientific calculators. For certain important special cases, we can obtain them from familiar right triangles. Figure 39 shows a right triangle with acute angles of $\pi/6$ and $\pi/3$ and an isosceles right triangle with acute angles of $\pi/4$. From these triangles, the following values can be determined.

Special Values of the Trigonometric Functions 21

θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

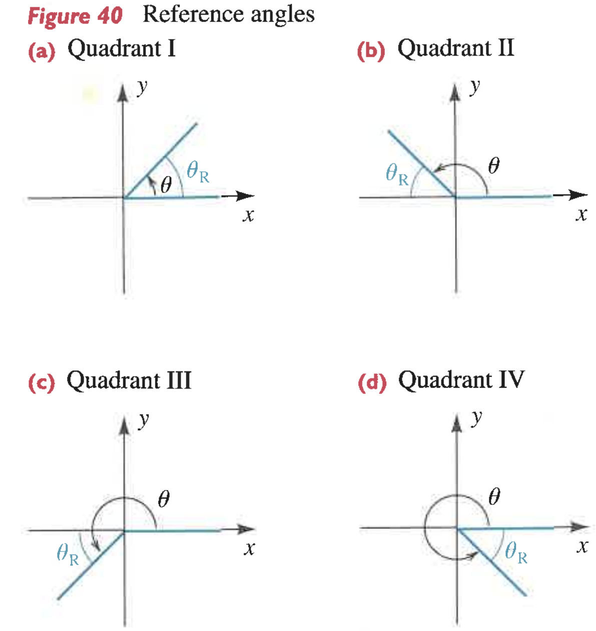
Figure 39



Since these particular values occur frequently in work involving trigonometry, it is a good idea either to memorize the table or to be able to find the values quickly by using the triangles in Figure 39.

Another method for finding values of trigonometric functions for an angle θ uses the **reference angle** of θ, which is the acute angle θ_R that the terminal side of θ makes with the x-axis when θ is in standard position. Figure 40 illustrates the reference angle θ_R for an angle in each of the four quadrants. To find the value of a trigonometric function at angle θ, we first determine the value for the reference angle θ_R of θ and then prefix with the appropriate sign.

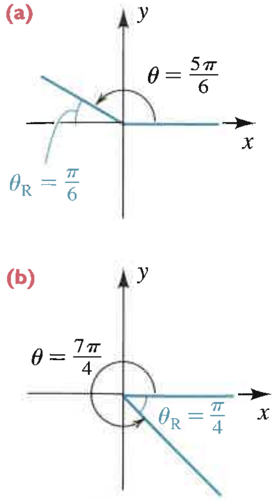
Figure 40 Reference angles



EXAMPLE ■ 6 Find $\sin\theta$ and $\cos\theta$ for the following:

(a) $\theta = \frac{5\pi}{6}$ **(b)** $\theta = \frac{7\pi}{4}$

Figure 41



SOLUTION We sketch the angles and their reference angles in Figure 41. Using the table of special values (20) gives the following:

(a) $\sin \dfrac{5\pi}{6} = \sin \dfrac{\pi}{6} = \dfrac{1}{2}$

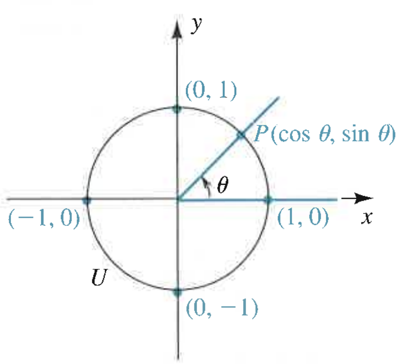
$\cos \dfrac{5\pi}{6} = -\cos \dfrac{\pi}{6} = -\dfrac{\sqrt{3}}{2}$

(b) $\sin \dfrac{7\pi}{4} = -\sin \dfrac{\pi}{4} = -\dfrac{\sqrt{2}}{2}$

$\cos \dfrac{7\pi}{4} = \cos \dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

Figure 42



To graph the sine function, we first study the variation of $\sin\theta$ as θ increases. For convenience, consider arcs along the unit circle U in Figure 42. Since $r = 1$, the formulas $\sin\theta = y/r$ and $\cos\theta = x/r$ take on the simpler forms $\sin\theta = y$ and $\cos\theta = x$. Thus, the coordinates (x, y) of the point P corresponding to θ can be written as $(\cos\theta, \sin\theta)$. At $\theta = 0$, P is the point $(1, 0)$. As θ increases from 0 to 2π, the point $P(\cos\theta, \sin\theta)$ travels around the unit circle once in a counterclockwise direction. Observation of the y-coordinate leads to the following facts, where arrows are used to indicate the variations of θ and $\sin\theta$. (For example, $0 \to \pi/2$ means that θ increases from 0 to $\pi/2$, and $0 \to 1$ means that $\sin\theta$ increases from 0 to 1.)

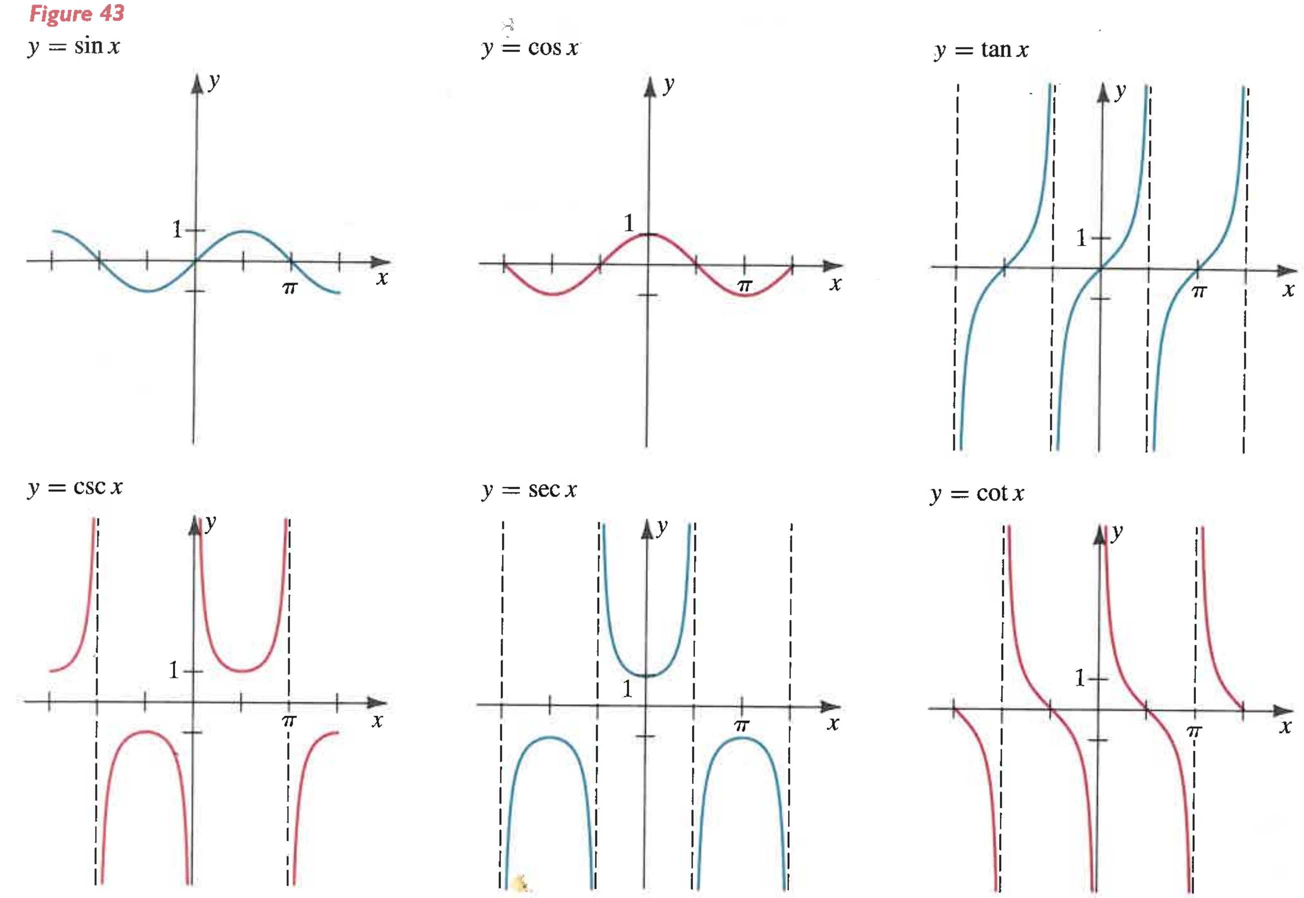
$$\theta:\quad 0 \;\to \frac{\pi}{2} \;\to \pi \;\to \frac{3\pi}{2} \;\to 2\pi$$

$$\sin\theta:\quad 0 \;\to 1 \;\to 0 \;\to -1 \;\to 0$$

If we let P continue to travel around U, the same pattern repeats in θ-intervals $[2\pi, 4\pi]$ and $[4\pi, 6\pi]$. In general, the values of $\sin\theta$ repeat in all successive intervals of length 2π. A function f with domain D is **periodic** if there is a positive real number k such that $x + k$ is in D and $f(x + k) = f(x)$ for every x in D. If a smallest such positive number k exists, it is called the **period** of f. We have seen that the sine function is periodic with period 2π. Using these facts and plotting several points corresponding to the special values of θ gives the graph of the sine function, shown in Figure 43, where we have used $\theta = x$ for the independent variable (measured in radians or real numbers).

The graph of the cosine function can be found in a similar fashion by studying the behavior of the horizontal component of P as θ increases. The graphs of all the trigonometric functions are given in Figure 43. Note that the period of the tangent and the cotangent functions is π.

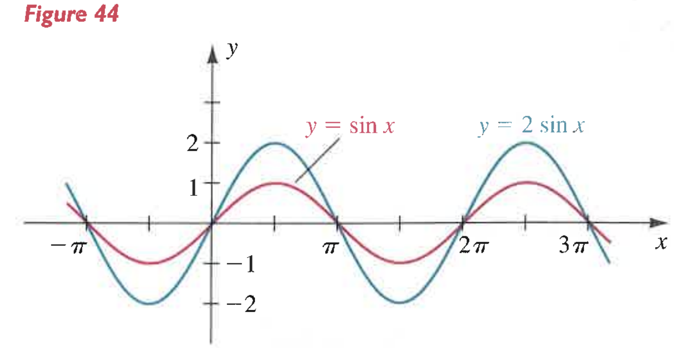
Figure 43



EXAMPLE 7 Sketch the graph of the function $f(x) = 2\sin x$.

SOLUTION We begin by sketching the graph of $\sin x$, as in Figure 43. We can then stretch this graph by multiplying each of the y-coordinates by a factor of 2 to obtain the graph of $y = 2\sin x$, shown in Figure 44.

Figure 44



TRIGONOMETRIC EQUATIONS

A **trigonometric equation** is an equation that contains trigonometric expressions. Each fundamental identity is an example of a trigonometric equation where every number (or angle) in the domain of the variable is a solution of the equation. If a trigonometric equation is not an identity, we often find solutions by using techniques similar to those used for algebraic equations. The main difference is that we first solve the trigonometric equation for $\sin x$, $\cos\theta$, and so on, and then find values of x or θ that satisfy the equation. *If degree measure is not specified, then solutions of a trigonometric equation should be expressed in radian measure (or as real numbers).*

EXAMPLE ■ 8 Find the solutions of the equation $\sin\theta = \frac{1}{2}$ if

(a) θ is in the interval $[0, 2\pi)$

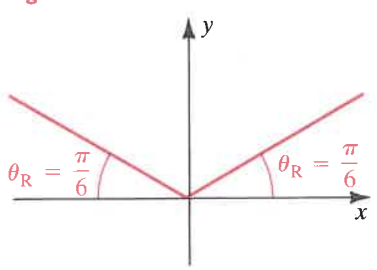
(b) θ is any real number

SOLUTION

(a) If $\sin\theta = \frac{1}{2}$, then the reference angle for θ is $\theta_R = \pi/6$. If we regard θ as an angle in standard position, then, since $\sin\theta > 0$, the terminal side is in either quadrant I or quadrant II, as illustrated in Figure 45. Thus there are two solutions for $0 \le \theta < 2\pi$:

$$\theta = \frac{\pi}{6} \quad \text{and} \quad \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Figure 45

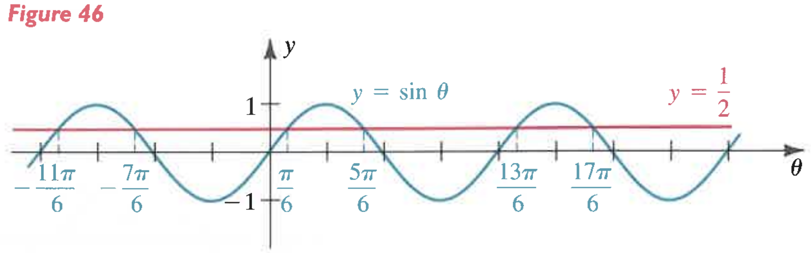


(b) Since the sine function has period 2π, we may obtain all solutions by adding multiples of 2π to $\pi/6$ and $5\pi/6$. This procedure gives us

$$\theta = \frac{\pi}{6} + 2\pi n \quad \text{and} \quad \theta = \frac{5\pi}{6} + 2\pi n \quad \text{for every integer } n.$$

NOTE An alternative (graphical) solution involves determining where the graph of $y = \sin\theta$ intersects the horizontal line $y = \frac{1}{2}$, as shown in Figure 46.

Figure 46



Most calculators have [SIN], [COS], and [TAN] keys to approximate values for these trigonometric functions. The values of the cosecant, secant, and cotangent functions can also be found using a calculator (by using the [1/x] key) and the formulas in (17).

CAUTION Calculators have both a radian and a degree mode. Choosing the wrong mode on a calculator is a very common error made when evaluating trigonometric functions. For example, in radian mode, the calculator gives the approximate value

$$\sin 1.3 \approx 0.963558185417,$$

but in degree mode, it yields a different value:

$$\sin 1.3 \approx 0.022687333573$$

The next example illustrates the use of a calculator in solving a trigonometric equation.

EXAMPLE ■ 9 Approximate, to the accuracy of your calculator (in radian mode), the solutions of the following equation in the interval $[0, 2\pi)$:

$$5\sin\theta\tan\theta - 10\tan\theta + 3\sin\theta - 6 = 0$$

SOLUTION

$5\sin\theta\tan\theta - 10\tan\theta + 3\sin\theta - 6 = 0$	given
$5\tan\theta(\sin\theta - 2) + 3(\sin\theta - 2) = 0$	factor groups
$(5\tan\theta + 3)(\sin\theta - 2) = 0$	factor out $\sin\theta - 2$
$5\tan\theta + 3 = 0, \quad \sin\theta - 2 = 0$	set each factor equal to 0
$\tan\theta = -\frac{3}{5}, \quad \sin\theta = 2$	solve for $\tan\theta$ and $\sin\theta$

The equation $\sin\theta = 2$ has no solution, since $\sin\theta \le 1$ for every θ. To solve $\tan\theta = -\frac{3}{5}$, we need to find the number θ whose tangent is $-\frac{3}{5}$. Many scientific calculators have a key labeled [TAN^{-1}] that can be used to find such a number. In this case, the calculator gives

$$\theta = \tan^{-1} \approx -0.540419500271.$$

(We will discuss inverse trigonometric functions more in Chapter 6.) Hence, the reference angle is $\theta_R \approx 0.540419500271$, which we store temporarily in a calculator memory. Then, without re-entering any numbers, we obtain the following solutions in quadrants II and IV:

$$\theta = \pi - \theta_R \approx 2.60117315332,$$

$$\theta = 2\pi - \theta_R \approx 5.74276580691$$

We may not always report all the digits shown on the final calculator screen, but we try not to round intermediate results and not to re-enter numbers.

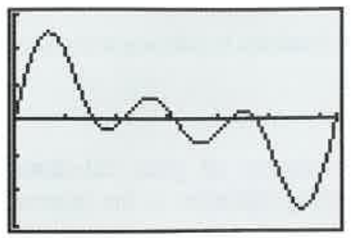
The next example illustrates how a graphing utility can aid in solving trigonometric equations.

EXAMPLE ▪ 10 Find the solutions of the following equation that are in the interval $[0, 2\pi)$:

$$\sin x + \sin 2x + \sin 3x = 0$$

SOLUTION Since $2\pi \approx 6.3$ and $|\sin\theta| \le 1$ for $\theta = x, 2x$, and $3x$, we choose the viewing window [0, 6.3] by [−3, 3] and obtain a sketch similar to Figure 47. Using the zoom and tracing features, we obtain the following approximations for the x-intercepts—that is, the *approximate* solutions of the given equation in $[0, 2\pi)$:

Figure 47
$0 \le x \le 6.3, -3 \le y \le 3$



$$0, \quad 1.57, \quad 2.09, \quad 3.14, \quad 4.19, \quad 4.71$$

The approximate solution 3.14 might lead us to guess that π is a solution. Checking $x = \pi$ in the given equation confirms that π is an exact solution.

We will now apply algebraic methods to find the *exact* solutions, *knowing that there should be six solutions in this interval.* We use the addition and double-angle formulas (19) to change the form of the given equation:

$$\sin x + \sin 2x + \sin 3x = 2\sin x\cos x(2\cos x + 1) = 0$$

Setting the factors equal to 0 gives us

$$\sin x = 0 \quad \text{or} \quad x = 0, \pi$$

$$\cos x = 0 \quad \text{or} \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\cos x = -\frac{1}{2} \quad \text{or} \quad x = \frac{2\pi}{3}, \frac{4\pi}{3}.$$

Frequently, exact solutions are "lost" when careless algebraic work is performed. By comparing the exact solutions

$$0, \quad \frac{\pi}{2}, \quad \frac{2\pi}{3}, \quad \pi, \quad \frac{4\pi}{3}, \quad \frac{3\pi}{2}$$

with the numerical estimates obtained from the graph, we confirm that the number of solutions and their approximate values agree.

In the preceding example, we were able to use a graphing utility to help us find the exact solutions of the equation. For many equations that occur in applications, however, it is possible only to approximate the solutions.

EXAMPLE ▪ 11 In Boston, the number of hours of daylight $D(t)$ at a particular time of the year may be approximated by

$$D(t) = 3\sin\left[\frac{2\pi}{365}(t - 79)\right] + 12,$$

with t in days and $t = 0$ corresponding to January 1. How many days of the year have more than 10.5 hr of daylight?

SOLUTION The graph of D is shown in Figure 48. If we can find two numbers a and b with $D(a) = 10.5$, $D(b) = 10.5$, and $0 < a < b < 365$, then there will be more than 10.5 hr of daylight in the tth day of the year if $a < t < b$.

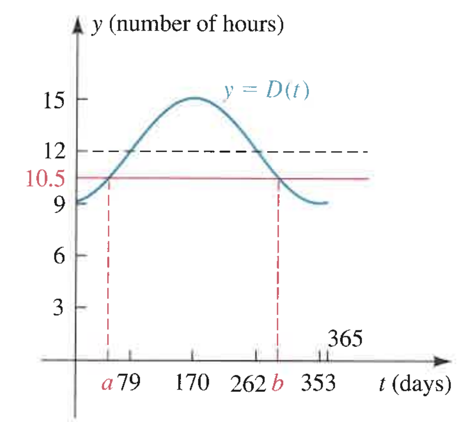
Let us solve the equation $D(t) = 10.5$ as follows:

$$3\sin\left[\frac{2\pi}{365}(t - 79)\right] + 12 = 10.5 \qquad \text{let } D(t) = 10.5$$

$$3\sin\left[\frac{2\pi}{365}(t - 79)\right] = -1.5 \qquad \text{subtract 12}$$

$$\sin\left[\frac{2\pi}{365}(t - 79)\right] = -0.5 = -\frac{1}{2} \qquad \text{divide by 3}$$

Figure 48



If $\sin\theta = -\frac{1}{2}$, then the reference angle is $\pi/6$ and the angle θ is in either quadrant III or quadrant IV. Thus, we can find the numbers a and b by solving the equations

$$\frac{2\pi}{365}(t - 79) = \frac{7\pi}{6} \quad \text{and} \quad \frac{2\pi}{365}(t - 79) = \frac{11\pi}{6}.$$

From the first of these equations, we obtain

$$t - 79 = \frac{7\pi}{6} \cdot \frac{365}{2\pi} = \frac{2555}{12} \approx 213,$$

and hence, $\quad t \approx 213 + 79, \quad$ or $\quad t \approx 292.$

Similarly, the second equation gives us $t \approx 414$. Since the period of the function D is 365 days (see Figure 48), we obtain

$$t \approx 414 - 365, \quad \text{or} \quad t \approx 49.$$

Thus, there will be at least 10.5 hr of daylight from $t = 49$ to $t = 292$—that is, for 242 days of the year.

EXERCISES C

Exer. 1–2: Find the exact radian measure of the angle.

1 (a) 150° (b) 120° (c) 450° (d) −60°

2 (a) 225° (b) 210° (c) 630° (d) −135°

Exer. 3–4: Find the exact degree measure of the angle.

3 (a) $\frac{2\pi}{3}$ (b) $\frac{5\pi}{6}$ (c) $\frac{3\pi}{4}$ (d) $-\frac{7\pi}{2}$

4 (a) $\frac{11\pi}{6}$ (b) $\frac{4\pi}{3}$ (c) $\frac{11\pi}{4}$ (d) $-\frac{5\pi}{2}$

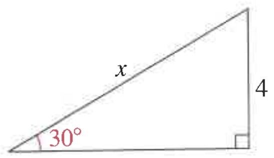
Exer. 5–6: Find the length of arc that subtends a central angle θ on a circle of diameter d and the area of the circular sector that θ determines.

5 $\theta = 50°$; $d = 16$

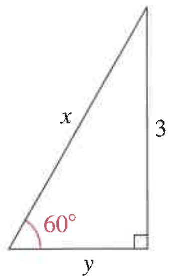
6 $\theta = 2.2$; $d = 120$

Exer. 7–8: Find the values of x and y in the figure.

7



8



Exer. 9–12: Find the values of the trigonometric functions if θ is an acute angle.

9 $\sin\theta = \frac{3}{5}$ 10 $\cos\theta = \frac{8}{17}$

11 $\tan\theta = \frac{5}{12}$ 12 $\cot\theta = 1$

Exer. 13–14: If θ is in standard position and Q is on the terminal side of θ, find the values of the trigonometric functions of θ.

13 $Q(4, -3)$ 14 $Q(-8, -15)$

Exer. 15–20: Refer to Example 4. Make the indicated trigonometric substitution and use fundamental identities to obtain a simplified trigonometric expression that contains no radicals.

15 $\sqrt{16 - x^2}$; $x = 4\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

16 $\dfrac{x^2}{\sqrt{9 - x^2}}$; $x = 3\sin\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

17 $\dfrac{x}{\sqrt{25 + x^2}}$; $x = 5\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

18 $\dfrac{\sqrt{x^2 + 4}}{x^2}$; $x = 2\tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

19 $\dfrac{\sqrt{x^2 - 9}}{x}$; $x = 3\sec\theta$ for $0 < \theta < \frac{\pi}{2}$

20 $x^3\sqrt{x^2 - 25}$; $x = 5\sec\theta$ for $0 < \theta < \frac{\pi}{2}$

Exer. 21–26: Find the exact value.

21 (a) $\sin(2\pi/3)$ (b) $\sin(-5\pi/4)$

22 (a) $\cos 150°$ (b) $\cos(-60°)$

23 (a) $\tan(5\pi/6)$ (b) $\tan(-\pi/3)$

24 (a) $\cot 120°$ (b) $\cot(-150°)$

25 (a) $\sec(2\pi/3)$ (b) $\sec(-\pi/6)$

26 (a) $\csc 240°$ (b) $\csc(-330°)$

c **Exer. 27–32: Find approximate values using a calculator.**

27 (a) $\sin 67°$ (b) $\csc 25°$

28 (a) $\sin(-2.743)$ (b) $\csc 51.314$

29 (a) $\cos(-12°)$ (b) $\sec 39°$

30 (a) $\cos(-4.2)$ (b) $\sec 15.9$

31 (a) $\tan 15$ (b) $\cot 5$

32 (a) $\tan 1.8$ (b) $\cot(-3)$

Exer. 33–38: Sketch the graph of f, making use of stretching, reflecting, or shifting.

33 (a) $f(x) = \frac{1}{4}\sin x$ (b) $f(x) = -4\sin x$

34 (a) $f(x) = \sin(x - \pi/2)$ (b) $f(x) = \sin x - \pi/2$

35 (a) $f(x) = 2\cos(x + \pi)$ (b) $f(x) = 2\cos x + \pi$

36 (a) $f(x) = \frac{1}{3}\cos x$ (b) $f(x) = -3\cos x$

37 (a) $f(x) = 4\tan x$ (b) $f(x) = \tan(x - \pi/4)$

38 (a) $f(x) = \frac{1}{4}\tan x$ (b) $f(x) = \tan(x + 3\pi/4)$

Exer. 39–42: Find a composite function form for y.

39 $y = \sqrt{\tan^2 x + 4}$ 40 $y = \cot^3(2x)$

41 $y = \sec(x + \pi/4)$ 42 $y = \csc\sqrt{x - \pi}$

43 If $f(x) = \cos x$, show that

$$\frac{f(x+h) - f(x)}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right).$$

44 If $f(x) = \sin x$, show that

$$\frac{f(x+h) - f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right).$$

Exer. 45–54: Verify the identity.

45 $(1 - \sin^2 t)(1 + \tan^2 t) = 1$

46 $\sec\beta - \cos\beta = \tan\beta\sin\beta$

47 $\dfrac{\csc^2\theta}{1 + \tan^2\theta} = \cot^2\theta$

48 $\cot t + \tan t = \csc t \sec t$

49 $\dfrac{1 + \csc\beta}{\sec\beta} - \cot\beta = \cos\beta$

50 $\dfrac{1}{\csc z - \cot z} = \csc z + \cot z$

51 $\sin 3u = \sin u(3 - 4\sin^2 u)$

52 $2\sin^2 2t + \cos 4t = 1$

53 $\cos^4(\theta/2) = \frac{3}{8} + \frac{1}{2}\cos\theta + \frac{1}{8}\cos 2\theta$

54 $\sin^4 2x = \frac{3}{8} - \frac{1}{2}\cos 4x + \frac{1}{8}\cos 8x$

Exer. 55–56: Find all solutions of the equation.

55 $2\cos 2\theta - \sqrt{3} = 0$ 56 $2\sin 3\theta + \sqrt{2} = 0$

Exer. 57–64: Find the solutions of the equation in $[0, 2\pi)$.

57 $2\sin^2 u = 1 - \sin u$ 58 $\cos\theta - \sin\theta = 1$

59 $2\tan t - \sec^2 t = 0$

60 $\sin x + \cos x \cot x = \csc x$

61 $\sin 2t + \sin t = 0$ 62 $\cos u + \cos 2u = 0$

63 $\tan 2x = \tan x$ 64 $\sin\frac{1}{2}u + \cos u = 1$

c **Exer. 65–70: Approximate, to the accuracy of your calculator or computer in radians, the solutions of the equation that are in the interval $[0, 2\pi)$.**

65 $\sin\theta = -0.5640$ 66 $\cos\theta = 0.7490$

67 $\tan\theta = 2.798$ 68 $\cot\theta = -0.9601$

69 $\sec\theta = -1.116$ 70 $\csc\theta = 1.485$

c 71 Use a graphing utility to graph $f(x) = (\sin x)/(x - \pi)$. Zoom in several times near $x = \pi$ and investigate the behavior of f.

c 72 Approximate the solution of the equation $x = \frac{1}{2}\cos x$ by using the following procedure.

(1) Graph $y = x$ and $y = \frac{1}{2}\cos x$ on the same coordinate axes.

(2) Use the graphs in (1) to find a first approximation x_1 to the solution.

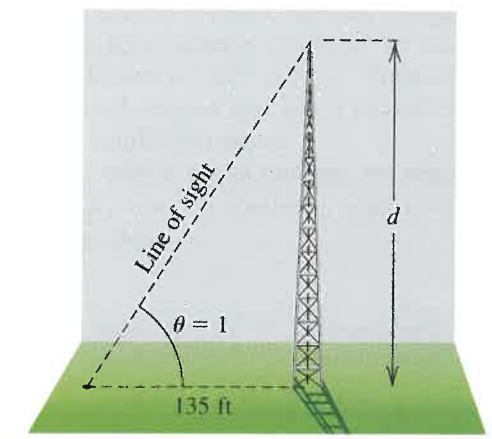
(3) Find successive approximations $x_2, x_3, \ldots$ by using the formulas $x_2 = \frac{1}{2}\cos x_1$, $x_3 = \frac{1}{2}\cos x_2, \ldots$ until accuracy to six decimal places is obtained.

c 73 Graph $y = (\sin x - \cos\pi x)/\cos x$ for $-1 \le x \le 1$ and estimate the x-intercepts.

Exer. 74–75: The *angle of elevation* of an object is the angle between a horizontal line at an observer's position and the line of sight from the observer to the object. Use the angle of elevation to estimate the heights specified.

74 From a point on level ground 135 ft from the base of a tower, the angle of elevation θ of the top of the tower is 1 radian. Approximate the height of the tower.

Exercise 74



75 A motorist, traveling along a level highway at a speed of 60 km/hr directly toward a mountain, observes that between 1:00 P.M. and 1:10 P.M. the angle of elevation changes from $\theta = 0.17$ to $\theta = 1.2$. Approximate the height of the mountain.

D EXPONENTIALS AND LOGARITHMS

Exponential and logarithmic functions play a major role in calculus. They are examples of *transcendental functions*. We defer a complete rigorous definition of exponential and logarithmic functions until we have developed the necessary tools of calculus. We review some of their properties in this section.

EXPONENTIAL FUNCTIONS

Exponential functions involve raising a constant base to a variable exponent. Two simple examples are $f(x) = 10^x$ and $g(x) = (\frac{1}{3})^x$.

Exponential Functions 22

> The **exponential function with base a** is defined by
> $$f(x) = a^x,$$
> where $a > 0$, $a \neq 1$, and x is any real number.

From algebra, we know how to evaluate a^x if x is a positive or a negative integer or if x is a rational number. If x is a positive integer, then a^x is the product of x factors of a.

$$a^x = \underbrace{(a)(a)\cdots(a)}_{x \text{ times}} \quad \text{if } x \text{ is a positive integer}$$

If x is a negative integer, then $x = -n$ for some positive integer n and

$$a^x = a^{-n} = \frac{1}{a^n}.$$

If x is a rational number of the form $x = m/n$, where m and n are integers with $n > 0$, then a^x is well-defined as

$$a^x = a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

ILLUSTRATION

Exponential notation a^x	**Exponential notation a^{-n}**
$2^4 = (2)(2)(2)(2) = 16$	$3^{-5} = \frac{1}{3^5} = \frac{1}{243}$
$(\frac{1}{5})^3 = (\frac{1}{5})(\frac{1}{5})(\frac{1}{5}) = \frac{1}{125}$	$\frac{1}{4^{-3}} = \frac{1}{1/4^3} = 4^3 = 64$

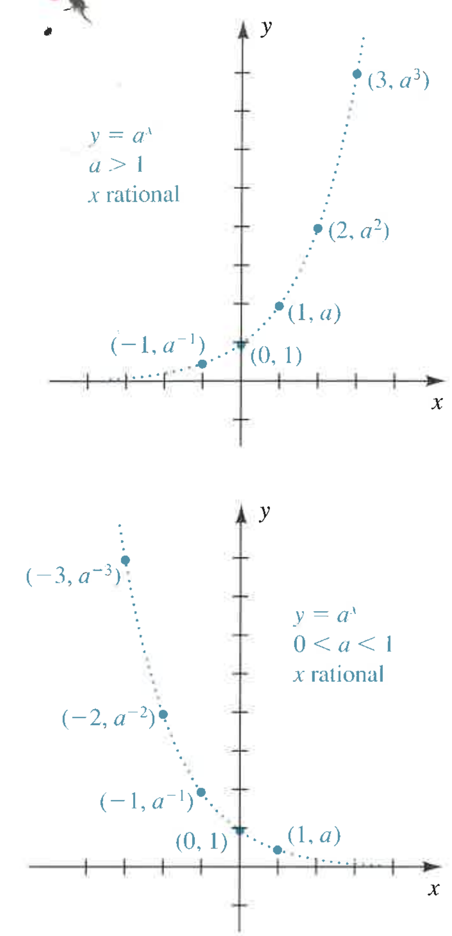
Exponential notation $a^{m/n}$

- $2^{3/5} = \sqrt[5]{2^3} = \sqrt[5]{8} \approx 1.5157$
- $\left(\frac{1}{9}\right)^{-5/2} = \frac{1}{(1/9)^{5/2}} = \frac{1}{\left(\sqrt{1/9}\right)^5} = \frac{1}{(1/3)^5} = 3^5 = 243$

We can make use of the $\boxed{x^y}$ or $\boxed{\wedge}$ key on a calculator for computation of a positive number raised to a rational power.

It can be shown algebraically that if x_1 and x_2 are any two rational numbers with $x_1 < x_2$, then $a^{x_1} < a^{x_2}$ if $a > 1$ and $a^{x_1} > a^{x_2}$ if $0 < a < 1$. Thus, if $a > 1$, then $f(x) = a^x$ is an increasing function, sometimes called an **exponential growth function,** whose graph rises. If $0 < a < 1$, then $f(x) = a^x$ is a decreasing function, sometimes called an **exponential decay function,** whose graph falls. In the graphs of $y = a^x$ shown in Figure 49, the dots indicate that only the points with *rational* x-coordinates are on the graphs. There is a *hole* in the graph whenever the x-coordinate of a point is irrational.

Figure 49

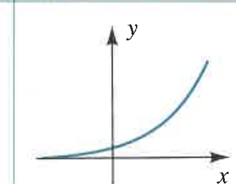
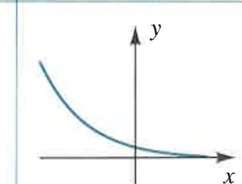


To extend the domain of the exponential function a^x to all real numbers, we must define a^x for *irrational* values of the exponent x. For example, in order to rigorously define 2^π, we need some knowledge of *limits* (the subject of Chapter 1), but for now, we use the nonterminating decimal representing 3.1415926... for π and consider the following *rational* powers of 2:

$$2^3, \quad 2^{3.1}, \quad 2^{3.14}, \quad 2^{3.141}, \quad 2^{3.1415}, \quad 2^{3.14159}, \ldots$$

We will show, in Chapter 6, that each successive power gets closer to a unique real number, which is designated as 2^π. The numerical value of 2^π is the nonterminating decimal 8.824977827... . We use the same technique for any other irrational value of x; that is, we find a sequence of rational numbers $x_1, x_2, x_3, \ldots$ that approaches x and let a^x be the unique real number approached by the numbers $a^{x_1}, a^{x_2}, a^{x_3}, \ldots$. Note that each of the values $a^{x_1}, a^{x_2}, a^{x_3}, \ldots$ is well-defined and has a numerical value that is easily approximated on a scientific calculator.

To sketch the graph of $y = a^x$ with x a real number, we replace any hole in the graph in Figure 49 with a point. The following chart summarizes this discussion and shows typical graphs.

Definition	Graph of f for $a > 1$	Graph of f for $0 < a < 1$
$f(x) = a^x$ for every x in $\mathbb{R}$, where $a > 0$ and $a \neq 1$		

The graphs merely indicate the *general* appearance; the *exact* shape of each depends on the value of a. Since a^x is either strictly increasing or strictly decreasing, it never takes on the same value twice. Thus, exponential functions are one-to-one functions.

One-to-One Property of Exponential Functions 23

> The exponential function f given by
> $$f(x) = a^x \quad \text{for } 0 < a < 1 \text{ or } a > 1$$
> is one-to-one; that is, for any real numbers x_1 and x_2:
>
> **(i)** If $x_1 \neq x_2$, then $a^{x_1} \neq a^{x_2}$.
>
> **(ii)** If $a^{x_1} = a^{x_2}$, then $x_1 = x_2$.

CAUTION We do not define exponential functions for $a = 1$, $a = 0$, or negative values of a. For such choices of a, the values of a^x do not give a one-to-one function whose domain is the set of all real numbers. If $a = 1$, then $a^x = 1$ for all values of x and we have a constant function. If $a = 0$, then a^x is undefined if $x < 0$ and has the constant value 0 for $x > 0$. If $a < 0$, then a^x is undefined for many values of x. For example, $(-2)^{1/2} = \sqrt{-2}$ is not a real number.

Exponential functions also satisfy the familiar laws of exponents.

Laws of Exponents 24

If u and v are any two real numbers, then

(i) $a^u a^v = a^{u+v}$

(ii) $\dfrac{a^u}{a^v} = a^{u-v}$

(iii) $(a^u)^v = a^{uv}$

Note too that the domain of an exponential function is the set of all real numbers and the range is the set of positive numbers ($a^x > 0$ for all x).

We frequently use the base 10 for exponential functions because of our familiarity with the decimal representation of numbers. In Chapter 6, we will study another important base, the irrational number e, which has a nonterminating decimal expansion that begins 2.7182818284. . . . Computer scientists often use exponentials with base 2 because computers store numbers internally in base 2 format. Most calculators provide special keys to compute 10^x and e^x.

An **exponential equation** is an equation involving exponential functions. We can often solve exponential equations by using the one-to-one property of exponential functions.

EXAMPLE ■ 1 Solve the exponential equation $5^{5x} = 5^{4x+7}$.

SOLUTION From (23)(ii),

$$5^{5x} = 5^{4x+7} \quad \text{implies} \quad 5x = 4x + 7.$$

Subtracting $4x$ from each side of the equation gives the solution $x = 7$. Checking the answer by substituting 7 for x in the original equation, we obtain the identity $5^{35} = 5^{35}$.

EXAMPLE ■ 2 Solve the exponential function $2^{5x-8} = 4^{x+2}$.

SOLUTION We first express 4^{x+2} with the base 2:

$$4^{x+2} = (2^2)^{x+2} = 2^{2(x+2)} = 2^{2x+4}$$

By (23)(ii),

$$2^{5x-8} = 2^{2x+4} \quad \text{implies} \quad 5x - 8 = 2x + 4,$$

which simplifies to $3x = 12$, so $x = 4$. Checking the answer by substituting 4 for x in the original equation yields the identity $2^{12} = 4^6$.

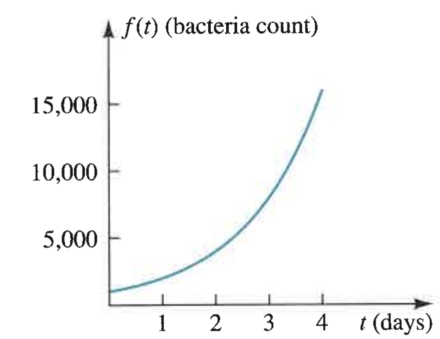
EXAMPLE ■ 3 Suppose it is observed experimentally that the number of bacteria in a given culture doubles every day. If 1000 bacteria are present at the start, then we obtain the following table, where t is the time in days and $f(t)$ is the bacteria count at time t.

t (time in days)	0	1	2	3	4
$f(t)$ (bacteria count)	1000	2000	4000	8000	16,000

(a) Determine a function of the form $f(t) = ba^t$ that can be used to predict the number of bacteria present at any time $t \geq 0$.

(b) Sketch the graph of f from part (a) and approximate the number of bacteria present after $1\frac{1}{2}$ days.

Figure 50



SOLUTION

(a) Since $f(t) = 1000$ when $t = 0$, we have $1000 = ba^0$ or, equivalently, $b = 1000$. Because the number of bacteria are doubling every day, $a = 2$. Hence,

$$f(t) = (1000)2^t.$$

(b) The graph of f is sketched in Figure 50. The number of bacteria present after $1\frac{1}{2}$ days is

$$f(\tfrac{3}{2}) = (1000)2^{3/2} \approx 2828.$$

LOGARITHMIC FUNCTIONS

If a is a positive number (other than 1), then the exponential function with base a is a one-to-one function whose range is the set of positive real numbers. Thus, given a positive number x, there will be a unique number y such that $x = a^y$. The number y is called the *logarithm of x with base a*. We denote this number as $\log_a(x)$ or as $\log_a x$ (read "the logarithm of x with base a").

Logarithmic Function 25

If a is a positive real number other than 1, then the **logarithm of x with base a** is defined by

$$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

for every $x > 0$ and every real number y.

Note that the domain of a logarithmic function is the set of positive real numbers ($\log_a x$ is defined only if $x > 0$), and the range is the set of all real numbers. The two equations in (25) are equivalent; they assert the same relationship between the variables x and y. We call the first equation the **logarithmic form** and the second the **exponential form**. Consider the following equivalent forms.

ILLUSTRATION

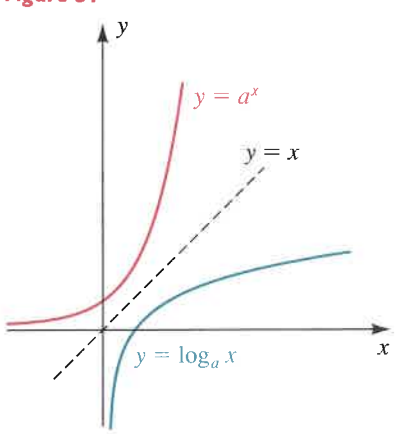
Logarithmic form	Exponential form
$\log_5 u = 2$	$5^2 = u$
$\log_b 8 = 3$	$b^3 = 8$
$r = \log_p q$	$p^r = q$
$w = \log_4(2t+3)$	$4^w = 2t+3$
$\log_3 x = 5 + 2z$	$3^{5+2z} = x$

We can use equivalent forms to verify a number of general properties of logarithmic functions.

***Properties of Logarithms and Equivalent Exponential Forms* 26**

Property of $\log_a x$	Exponential form
(i) $\log_a 1 = 0$	$a^0 = 1$
(ii) $\log_a a = 1$	$a^1 = a$
(iii) $\log_a a^x = x$	$a^x = a^x$
(iv) $\log_a x = \log_a x$	$a^{\log_a x} = x$

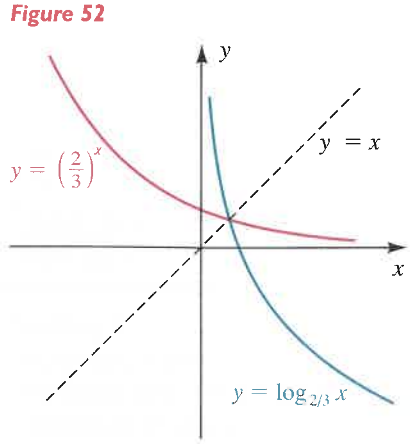
Figure 51



To obtain graphs of logarithmic functions, we first show that $\log_a x$ and a^x are inverses of each other. If $f(x) = \log_a x$ and $g(x) = a^x$, then the composite function $f \circ g$ is computed by

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{definition of } f \circ g\\ &= f(a^x) && \text{definition of } g\\ &= \log_a a^x && \text{definition of } f\\ &= x. && \text{by (25)(iii)}\end{aligned}$$

Figure 52



A similar computation using the exponential form of (26)(iv) shows that $(g \circ f)(x) = x$ for all positive real numbers x. The functions $\log_a x$ and a^x are inverses since both $f \circ g$ and $g \circ f$ are identity functions. Because $\log_a x$ and a^x are inverses of each other, the graph of either function is the reflection of the graph of the other across the line $y = x$. Figure 51 shows typical graphs of these functions for $a > 1$.

EXAMPLE ■ 4 Sketch the graphs of $y = (2/3)^x$ and $y = \log_{2/3} x$.

SOLUTION We begin with the graph of $y = (2/3)^x$. Since we have $0 < 2/3 < 1$, the graph will decrease as x increases, with positive values for all real numbers x. Reflecting this graph across the line $y = x$ yields the graph of $y = \log_{2/3} x$. Figure 52 shows both graphs.

Logarithmic functions either strictly increase or strictly decrease in their domains and hence are one-to-one functions.

***One-to-One Property of Logarithmic Functions* 27**

The logarithmic function f given by

$$f(x) = \log_a x \quad \text{for } 0 < a < 1 \text{ or } a > 1$$

is one-to-one; that is, for any two positive real numbers x_1 and x_2:

(i) If $x_1 \neq x_2$, then $\log_a x_1 \neq \log_a x_2$.
(ii) If $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$.

Other properties of logarithms may be stated as laws, which correspond to the laws of exponents.

***Laws of Logarithms* 28**

If u and v are any two positive real numbers, then

(i) $\log_a(uv) = \log_a u + \log_a v$
(ii) $\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$
(iii) $\log_a(u^c) = c\log_a u$ for every real number c

We will prove (28)(i) here; the others have similar proofs.

PROOF Let $x = \log_a u$ and $y = \log_a v$.

Then $a^x = u$ and $a^y = v$. by definition of the logarithm

Now $uv = a^x a^y = a^{x+y}$. by the properties of exponents

The exponential equation

$$uv = a^{x+y}$$

has the equivalent logarithmic form

$$\log_a uv = x + y.$$

But since $x = \log_a u$ and $y = \log_a v$, the last equation can be written as

$$\log_a uv = \log_a u + \log_a v. \quad ■$$

Logarithms with base 10 are called **common logarithms,** and the symbol **log *x*** is an abbreviation for $\log_{10} x$. A second widely used logarithm is the **natural logarithm,** denoted **ln *x*,** which has the irrational number e for its base.

Most calculators have a [LOG] key for the calculation of common logarithms and an [LN] key for natural logarithms. To numerically calculate logarithms with bases other than 10 and e, we need to use the following change-of-base formula.

Change-of-Base Formula for Logarithms 29

If $x > 0$ and if a and b are positive real numbers other than 1, then

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

PROOF Let $u = \log_b x$.

Then
$$b^u = x.$$

If we take the logarithm with base a of both sides of this equation, we obtain

$$\log_a x = \log_a(b^u) = u \log_a b,$$

which we can write as

$$\log_a x = (\log_b x)(\log_a b).$$

Dividing each side by $\log_a b$ gives the formula. ■

EXAMPLE ■ 5 Approximate $\log_7 32$ using common logarithms.

SOLUTION Using the change-of-base formula with $a = 10$, we have

$$\log_7 32 = \frac{\log_{10} 32}{\log_{10} 7} = \frac{\log 32}{\log 7}.$$

We can now use the [LOG] key to obtain

$$\frac{\log 32}{\log 7} \approx \frac{1.5051}{0.8451} \approx 1.7810.$$

Note that we could have also used (ln 32)/(ln 7) to obtain the approximation. We can check our approximation by using the [x^y] key to evaluate $7^{1.7810}$.

A **logarithmic equation** is an equation involving logarithmic functions. We can often solve logarithmic equations by using the one-to-one property of logarithmic functions.

EXAMPLE ■ 6 Solve the logarithmic equation

$$\log_3(4x - 5) = \log_3(2x + 1).$$

SOLUTION Since logarithmic functions are one-to-one, if there is a solution, then $4x - 5$ must equal $2x + 1$ or, equivalently, $x = 3$. We must check that $x = 3$ does not make $4x - 5$ or $2x + 1$ zero or negative, because then the logarithms in the given equation would be undefined. In this case, $4x - 5$ and $2x + 1$ both equal 7, so $x = 3$ is a valid solution.

EXAMPLE ■ 7 If the number N of bacteria in a culture after t days is given by $N = (1000)2^t$,

(a) express t as a logarithmic function of N with base 2

(b) determine the time when the number of bacteria is 8000

SOLUTION

(a) From $N = (1000)2^t$, we have

$$2^t = \frac{N}{1000} \quad \text{or, equivalently,} \quad t = \log_2 \frac{N}{1000}.$$

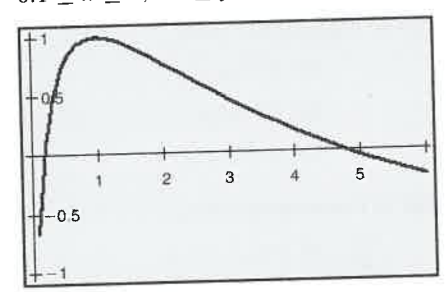
(b) Using the result of part (a) with $N = 8000$,

$$t = \log_2 \frac{8000}{1000} = \log_2 8 = \log_2 2^3 = 3.$$

EXAMPLE ■ 8 Use a graphing utility to estimate the x-intercepts of $f(x) = \cos(\ln x)$ for $0.1 \le x \le 6$.

SOLUTION Since the range of the cosine function is the interval $[-1, 1]$, we set the viewing window so that $-1 \le y \le 1$. Using a graphing utility, we obtain the graph of the function, shown in Figure 53. We estimate the x-intercepts to be 0.21 and 4.81. An interesting problem arises if you investigate the x-intercepts on the interval $0 < x \le 0.1$.

Figure 53
$0.1 \le x \le 6, -1 \le y \le 1$



The next example is a good illustration of the power of a graphing utility, since it is impossible to find the exact solution using only algebraic methods.

EXAMPLE ■ 9 Estimate the point of intersection of the graphs of

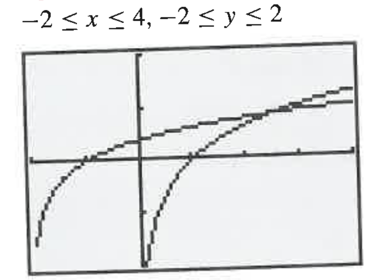
$$f(x) = \log_3 x \quad \text{and} \quad g(x) = \log_6(x + 2).$$

SOLUTION Most graphing utilities work directly only with common and natural logarithmic functions. Thus, we first use the change-of-base formula to rewrite f and g as

$$f(x) = \frac{\ln x}{\ln 3} \quad \text{and} \quad g(x) = \frac{\ln(x + 2)}{\ln 6}.$$

We then use a graphing utility with a viewing window of $-2 \le x \le 4$ and $-2 \le y \le 2$ to obtain graphs like those in Figure 54. We see that there is a point of intersection in the first quadrant with $2 < x < 3$. Using the tracing and zoom features, we find that the point of intersection is approximately (2.52, 0.84).

Figure 54
$-2 \le x \le 4, -2 \le y \le 2$



EXERCISES D

Exer. 1–6: Sketch the graph of f.

1 $f(x) = 2^x$
2 $f(x) = -3^x$
3 $f(x) = 2(5)^x$
4 $f(x) = 7^x + 3$
5 $f(x) = 4^{-x}$
6 $f(x) = (\frac{1}{2})^x$

Exer. 7–14: Solve the equation.

7 $5^{x+8} = 5^{3x-2}$
8 $8^{7-x} = 8^{2x+1}$
9 $5^{(x^2)} = 5^{2x+3}$
10 $25^{(x^2)} = 5^{3x+2}$
11 $(\frac{1}{2})^{5-x} = 2$
12 $2^{-100x} = (0.5)^{x-4}$
13 $27^{4-x} = 9^{x-3}$
14 $8^{x-1} = 4^{2x-3}$

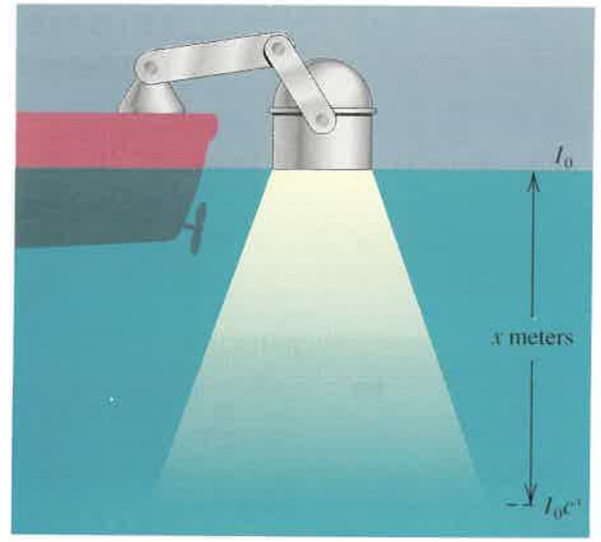
15 A colony of an endangered species originally numbering 1000 was predicted to have a population N after t years given by the equation $N(t) = 1000(0.9)^t$. Estimate the population after
(a) 1 year
(b) 5 years
(c) 10 years

16 The number of bacteria in a certain culture increased from 600 to 1800 between 8 A.M. and 10 A.M. Assuming the growth is exponential, the number $f(t)$ of bacteria t hours after 8 A.M. is given by $f(t) = 600(3)^{t/2}$.
(a) Estimate the number of bacteria at 9 A.M., 11 A.M., and noon.
(b) Sketch the graph of f.

17 Prescription drugs that enter the body are eventually eliminated through excretion. For an initial dose of 20 mg, suppose that the amount $A(t)$ remaining in the body t hours later is given by $A(t) = 20(0.7)^t$.
(a) Estimate the amount of the drug in the body 8 hr after the initial dose.
(b) What percentage of the drug still in the body is eliminated each hour?

18 An important problem in oceanography is to determine the amount of light that can penetrate to various ocean depths. The Beer–Lambert law asserts that the exponential function given by $I(x) = I_0c^x$ is a model for this phenomenon (see figure). For a certain location, $I(x) = 10(0.4)^x$ is the amount of light (in calories/cm²/sec) reaching a depth of x meters.
(a) Find the amount of light at a depth of 2 m.
(b) Sketch the graph of I for $0 \le x \le 5$.

Exercise 18



Exer. 19–24: Change to logarithmic form.

19 $5^3 = 125$
20 $5^{-3} = \frac{1}{125}$
21 $3^x = 7 + t$
22 $m^n = p$
23 $(0.7)^t = \frac{2}{3}$
24 $3^{-2x} = P/F$

Exer. 25–30: Change to exponential form.

25 $\log_2 32 = 5$
26 $\log_3 27 = 3$
27 $\log_{10} 1000 = 3$
28 $\log_2 \frac{1}{64} = -6$
29 $\log_7 m = 5x + 3$
30 $\log_a 1994 = 7$

Exer. 31–34: Solve for t using logarithms with base a.

31 $2a^{t/5} = 5$
32 $5a^{3t} = 63$
33 $A = Ba^{Ct} + D$
34 $C = Ba^{t/D} - Q$

Exer. 35–40: Find the number, if possible.

35 $\log_9 1$
36 $\log_6 6$
37 $\log_9(-3)$
38 $\log_3 3^2$
39 $17^{\log_{17} 8}$
40 $\log_2 1024$

Exer. 41–44: Solve the logarithmic equation.

41 $\log_4 x = \log_4(8 - x)$
42 $\log_3(x + 4) = \log_3(1 - x)$
43 $\log x^2 = \log(-3x - 2)$
44 $\log x^2 = -4$

Exer. 45–50: Sketch the graph of f.

45 $f(x) = \log_6 x$
46 $f(x) = -\log_6 x$
47 $f(x) = 2\log_6 x$
48 $f(x) = 3 + \log_6 x$
49 $f(x) = \log_6(x - 2)$
50 $f(x) = \log_6 |x|$

51 The loudness of a sound, as experienced by the human ear, is based on its intensity level. The intensity level α (in decibels) that corresponds to a sound intensity I is $\alpha = 10\log(I/I_0)$, where I_0 is a special value of I agreed to be the weakest sound that can be detected by the human ear under certain conditions. Find α if
(a) I is 10 times as great as I_0
(b) I is 1000 times as great as I_0

52 A sound intensity level of 140 decibels produces pain in the average human ear (refer to Exercise 51). Approximately how many times greater than I_0 must I be in order for α to reach this level?

c 53 The population $N(t)$ of the United States (in millions) t years after 1990 may be approximated by the formula $N(t) = 253(2.72)^{0.007t}$.
(a) Estimate the population in 1990 and 2000.
(b) Approximately when will the population be twice what it was in 1990?

54 Find the error in the following "solution" to the problem: Solve

$$\log_3(5x - 17) = \log_3(4x - 14).$$

"*Solution:* Since logarithmic functions are one-to-one, we must have $5x - 17 = 4x - 14$, which implies that $5x - 4x = 17 - 14$, so $x = 3$."

c **Exer. 55–62: Use a graphing utility to obtain a graph of f on the indicated x-interval. Adjust the y-interval so that the viewing window contains the entire graph.**

55 $f(x) = 2^x + 2^{-x}$; $-2 \le x \le 2$
56 $f(x) = \dfrac{2}{\sqrt{\pi}}\, 2^{-x^2}$; $-3 \le x \le 3$
57 $f(x) = \log(2^x + 5)$; $-2 \le x \le 10$
58 $f(x) = \dfrac{5(3)^x}{3^x + 1}$; $-8 \le x \le 15$
59 $f(x) = \log(\sin x)$; $0.1 \le x \le 3$
60 $f(x) = \sin(\log x)$; $0.1 \le x \le 800$
61 $f(x) = \log(x(1.2 + \sin x))$; $0.1 \le x \le 20$
62 $f(x) = 2.56(3)^{-0.22x}\cos 4.9x$; $0 \le x \le 12$

E CONIC SECTIONS

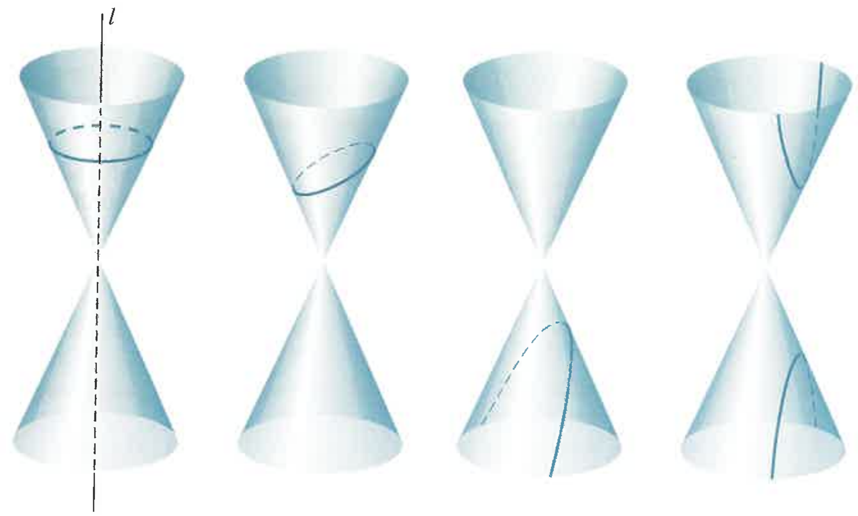
We now review some of the elementary geometric properties of the conic sections. In later chapters, we will use the methods of calculus to solve problems that include finding equations of tangent lines to conics and calculating areas and volumes of regions determined by conics.

Intersecting a double-napped right circular cone with a plane produces curves known as the *conic sections*. By varying the position of the plane, we can obtain a *circle*, an *ellipse*, a *parabola*, or a *hyperbola*, as Figure 55 illustrates on the following page. *Degenerate conics* occur if the plane intersects the cone in a single point or along either one or two lines that lie on the cone.

PARABOLAS

We first define a *parabola* and present equations for parabolas that have a vertical or a horizontal axis.

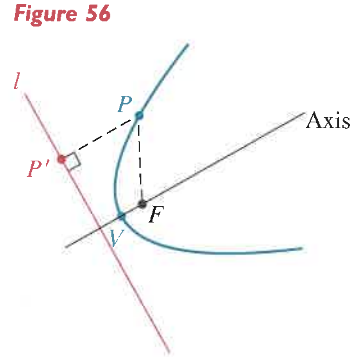
Figure 55
(a) Circle (b) Ellipse (c) Parabola (d) Hyperbola



Definition 30

> A **parabola** is the set of all points in a plane equidistant from a fixed point F (the **focus**) and a fixed line l (the **directrix**) that lie in the plane.

Figure 56

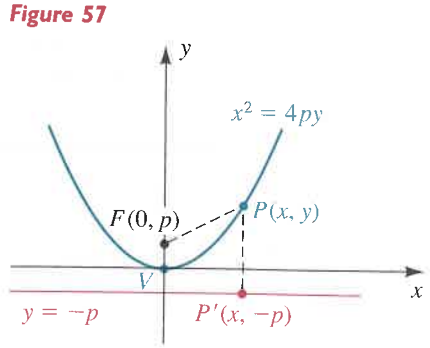


If the focus F lies on the directrix l, then we have a degenerate parabola, the line through F perpendicular to l. Thus, we assume that F does not lie on l. If P is a point in the plane, then the distance from P to the line l is the distance $d(P, P')$, where P' is the point determined by the line through P that is perpendicular to l (see Figure 56). The point P is on the parabola if and only if $d(P, P') = d(P, F)$. The **axis** of the parabola is the line through F that is perpendicular to the directrix. The **vertex** of the parabola is the point V on the axis halfway from F to l.

If the axis of the parabola is the y-axis and the vertex is at the origin with focus F at $(0, p)$, then the equation of the parabola is

$$x^2 = 4py.$$

Figure 57



If $p > 0$, the parabola opens upward, as in Figure 57. If $p < 0$, the parabola opens downward. The graph is symmetric with respect to the y-axis; substitution of $-x$ for x does not change the equation $x^2 = 4py$.

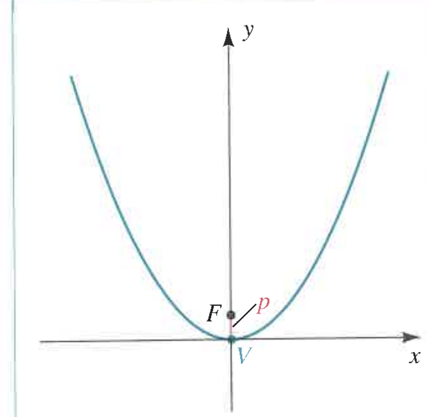
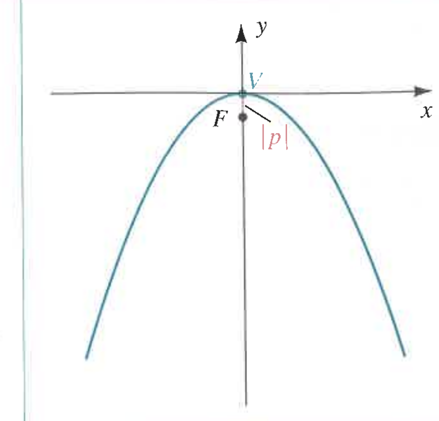
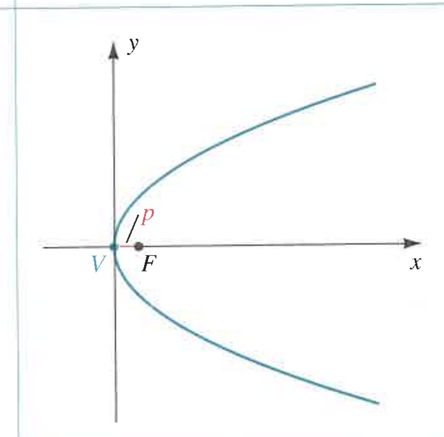
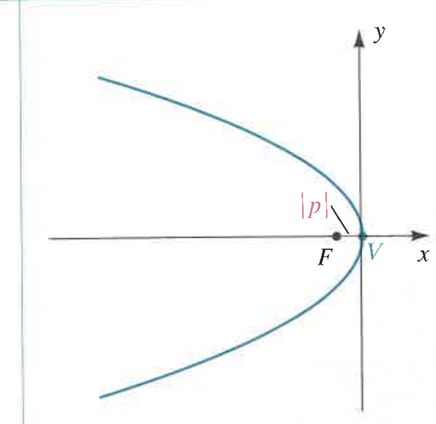
Interchanging the roles of x and y yields the similar equations

$$x^2 = 4py \quad \text{and} \quad y^2 = 4px,$$

which are the equations of a parabola with vertex at the origin and focus $F(0, p)$ and $F(p, 0)$, respectively. If $p > 0$, the parabola opens upward or to the right, and if $p < 0$, it opens downward or to the left.

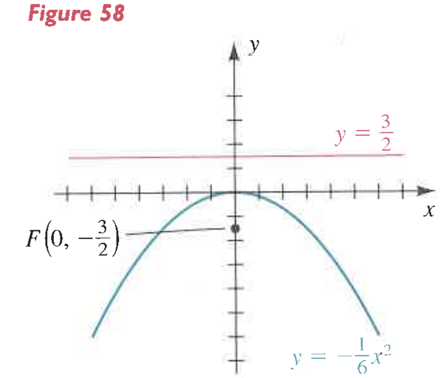
We see from (31) that for any nonzero real number a, the graph of $y = ax^2$ or $x = ay^2$ is a parabola with vertex $V(0, 0)$. Moreover, $a = 1/(4p)$,

Parabolas with Vertex $V(0,0)$ 31

Equation	Graph for $p > 0$	Graph for $p < 0$
$x^2 = 4py$, or $y = \frac{1}{4p}x^2$		
$y^2 = 4px$, or $x = \frac{1}{4p}y^2$		

or, equivalently, $p = 1/(4a)$, where $|p|$ is the distance between the focus F and the vertex V. To find the directrix l, recall that l is also a distance $|p|$ from V.

Figure 58



EXAMPLE ■ 1 Find the focus and the directrix of the parabola having equation $y = -\frac{1}{6}x^2$, and sketch its graph.

SOLUTION The equation has the form $y = ax^2$ with $a = -\frac{1}{6}$. As in (31), $a = 1/(4p)$, or

$$p = \frac{1}{4a} = \frac{1}{4(-\frac{1}{6})} = -\frac{3}{2}.$$

The parabola opens downward and has focus $F(0, -\frac{3}{2})$, as illustrated in Figure 58. The directrix is the horizontal line $y = \frac{3}{2}$, which is a distance $\frac{3}{2}$ above V, as shown in the figure.

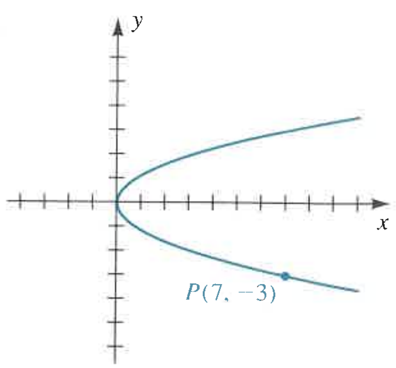
EXAMPLE ■ 2

(a) Find an equation of a parabola that has vertex at the origin, opens right, and passes through the point $P(7, -3)$.

(b) Find the focus.

SOLUTION

Figure 59



(a) The parabola is sketched in Figure 59. By (31), the equation of the parabola has the form $x = ay^2$ for some number a. If $P(7, -3)$ is on the graph, then

$$7 = a(-3)^2, \quad \text{or} \quad a = \tfrac{7}{9}.$$

Hence an equation of the parabola is $x = \frac{7}{9}y^2$.

(b) The focus is a distance p to the right of the vertex, where

$$p = \frac{1}{4a} = \frac{1}{4(\frac{7}{9})} = \frac{9}{28}.$$

Thus, the focus has coordinates $(\frac{9}{28}, 0)$.

If the vertex V of a parabola lies at any point (h, k) in the xy-plane, then we can also find equations for the parabola if the directrix is horizontal or vertical. If the focus is $F(h, k+p)$ and the directrix is the horizontal line $y = k - p$, an equation of the parabola is

$$(x-h)^2 = 4p(y-k).$$

Expanding the left-hand side of $(x-h)^2 = 4p(y-k)$ and simplifying leads to an equation of the form

$$y = ax^2 + bx + c,$$

where a, b, and c are real numbers. Conversely, if $a \neq 0$, then the graph of $y = ax^2 + bx + c$ is a parabola with a vertical axis. As with $y = ax^2$, we can show that $a = 1/(4p)$. The parabola opens upward if $p > 0$ and downward if $p < 0$.

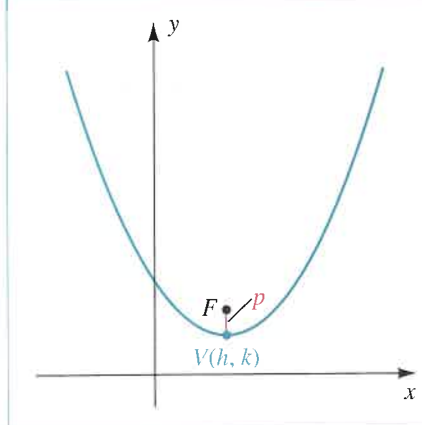
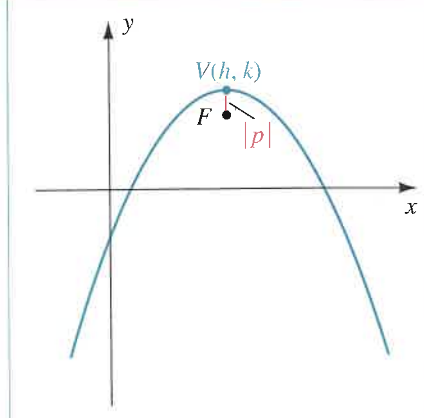
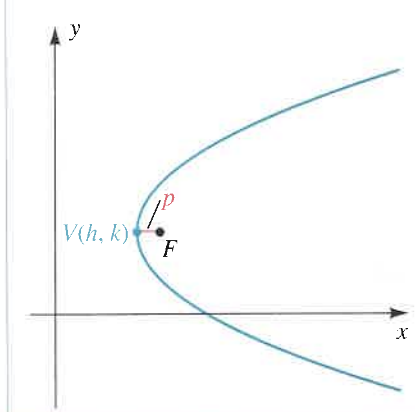
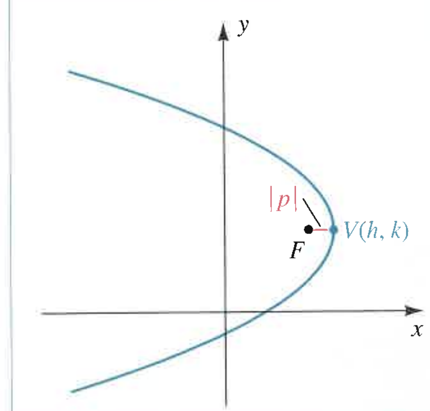
Similarly, if the directrix is a vertical line $x = h - p$ and the vertex is at $V(h, k)$ with focus $F(h+p, k)$, then an equation is

$$(y-k)^2 = 4p(x-h),$$

with the parabola opening to the right if $p > 0$ and to the left if $p < 0$. We may write this equation in the form $x = ay^2 + by + c$, where $a = 1/(4p)$.

For convenience, in the following summary of this discussion, we have taken $V(h, k)$ in the first quadrant of the figures.

Parabolas with Vertex $V(h, k)$ **32**

Equation	Graph for $p > 0$	Graph for $p < 0$
$(x-h)^2 = 4p(y-k)$, or $y = ax^2 + bx + c$, where $p = \dfrac{1}{4a}$		
$(y-k)^2 = 4p(x-h)$, or $x = ay^2 + by + c$, where $p = \dfrac{1}{4a}$		

If the equation of a parabola is in the form $y = ax^2 + bx + c$, then we can find the vertex $V(h, k)$ algebraically by completing the square in x and changing the equation to the form $(x-h)^2 = 4p(y-k)$.

EXAMPLE ■ 3 Discuss and sketch the graph of $y = 2x^2 - 6x + 4$.

SOLUTION By (32), the graph is a parabola with vertical axis. We rewrite the equation as $y/2 = x^2 - 3x + 2$ and complete the square in x:

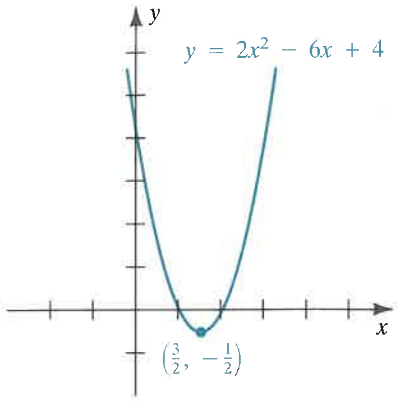
$$x^2 - 3x + 2 = x^2 - 2(\tfrac{3}{2})x + \tfrac{9}{4} - \tfrac{9}{4} + 2 = (x - \tfrac{3}{2})^2 - \tfrac{1}{4}$$

so we have

$$\left(x - \frac{3}{2}\right)^2 = \frac{y}{2} + \frac{1}{4} = 4\left(\frac{1}{8}\right)\left(y - \left(-\frac{1}{2}\right)\right).$$

Hence, the vertex is $V(\frac{3}{2}, -\frac{1}{2})$ and $p = \frac{1}{8}$.

Figure 60



Since the parabola opens upward, the focus F is a distance $p = \frac{1}{8}$ above V, which gives us

$$F(\tfrac{3}{2}, -\tfrac{1}{2} + \tfrac{1}{8}) = F(\tfrac{3}{2}, -\tfrac{3}{8}).$$

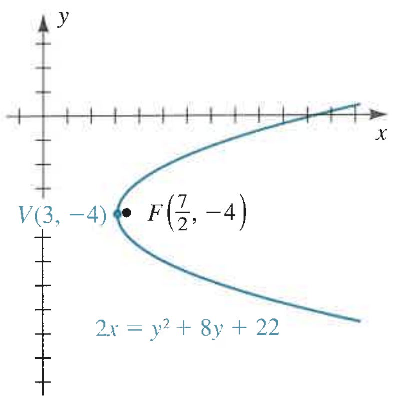
The directrix is the horizontal line l that is a distance $p = \frac{1}{8}$ below V. Therefore, an equation for l is

$$y = -\tfrac{1}{2} - \tfrac{1}{8}, \quad \text{or, equivalently,} \quad y = -\tfrac{5}{8}.$$

The graph is sketched in Figure 60. Note that the y-intercept is 4 and the x-intercepts (the solutions of $2x^2 - 6x + 4 = 0$) are 1 and 2.

For equations of the form $x = ay^2 + by + c$, we complete the square in y and write the equation in the form $(y - k)^2 = 4p(x - h)$.

Figure 61



EXAMPLE ■ 4 Discuss and sketch the graph of $2x = y^2 + 8y + 22$.

SOLUTION By (32), the graph is a parabola with horizontal axis. Note that

$$x = \tfrac{1}{2}(y^2 + 8y + 22) = \tfrac{1}{2}(y^2 + 8y + 16 + 6) = \tfrac{1}{2}(y + 4)^2 + 3,$$

which we may rewrite as

$$\tfrac{1}{2}(y + 4)^2 = x - 3$$

so that

$$(y + 4)^2 = 2x - 6 = 4(\tfrac{1}{2})(x - 3).$$

Hence, the vertex is $V(3, -4)$ and $p = \frac{1}{2}$. Since the parabola opens to the right, the focus F is a distance $p = \frac{1}{2}$ to the right of V, which gives us

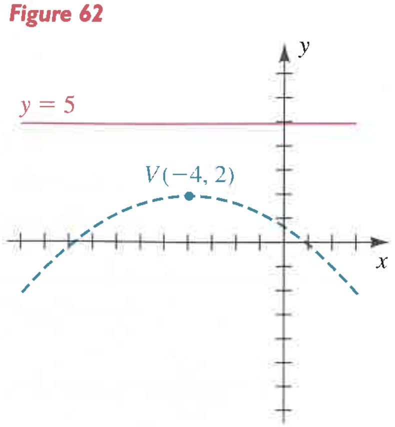
$$F(3 + \tfrac{1}{2}, -4) = F(\tfrac{7}{2}, -4).$$

The directrix is the vertical line l that is a distance $p = \frac{1}{2}$ to the left of V. Therefore, an equation for l is

$$x = 3 - \tfrac{1}{2}, \quad \text{or} \quad x = \tfrac{5}{2}.$$

The parabola is sketched in Figure 61.

Figure 62



EXAMPLE ■ 5 Find an equation of the parabola with vertex $V(-4, 2)$ and directrix $y = 5$.

SOLUTION The vertex and the directrix are shown in Figure 62. The dashes indicate a possible shape for the parabola. From (32), we have the following equation of the parabola:

$$(x - h)^2 = 4p(y - k),$$

with $h = -4$, $k = 2$, and $p = -3$, since V is 3 units *below* the directrix. Substituting gives us

$$(x + 4)^2 = -12(y - 2).$$

This equation can be expressed in the form $y = ax^2 + bx + c$, as follows:

$$x^2 + 8x + 16 = -12y + 24$$
$$12y = -x^2 - 8x + 8$$
$$y = -\tfrac{1}{12}x^2 - \tfrac{2}{3}x + \tfrac{2}{3}$$

ELLIPSES

We may define an ellipse as follows. (Note that *foci* is the plural of *focus*.)

Definition 33

An **ellipse** is the set of all points in a plane, the sum of whose distances from two fixed points (the **foci**) in the plane is constant.

Figure 63

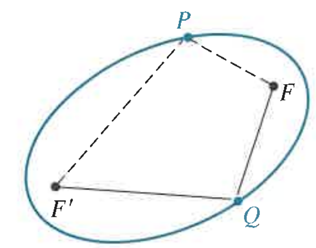


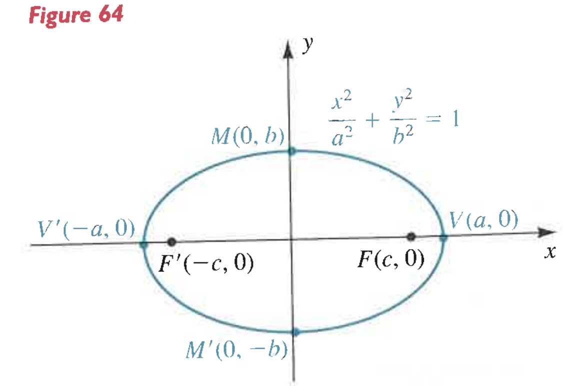
Figure 63 shows an ellipse with foci F and F'. If P and Q are any two points on the ellipse, then $d(P, F) + d(P, F') = d(Q, F) + d(Q, F')$. When F and F' are close to each other, the ellipse is almost circular. If $F = F'$, then we obtain a circle with center F. The midpoint of the segment FF' is the **center** of the ellipse.

If the foci lie along the x-axis and the center is at the origin, then the ellipse has a simple equation. Suppose F has coordinates $(c, 0)$ so that F' has coordinates $(-c, 0)$. Let $2a$ denote the constant sum of the distances of P from F and F', and let $b = \sqrt{a^2 - c^2}$. Note that $b < a$. It can be shown that the coordinates (x, y) of every point on the ellipse satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Figure 64 illustrates this case. We may find the x-intercepts of the ellipse by letting $y = 0$ in the equation. Doing so gives $x^2 = a^2$, so there are two

Figure 64



x-intercepts, a and $-a$. The corresponding points $V(a, 0)$ and $V'(-a, 0)$ on the graph are the **vertices** of the ellipse. The line segment $V'V$ is the **major axis**. Similarly, letting $x = 0$ in the equation, we obtain $y^2/b^2 = 1$, or $y^2 = b^2$. Hence, the y-intercepts are b and $-b$. The segment between $M'(0, -b)$ and $M(0, b)$ is the **minor axis** of the ellipse. The major axis is always longer than the minor axis, since $a > b$. The foci are always on the major axis.

Applying tests for symmetry, we see that the ellipse is symmetric with respect to the x-axis, the y-axis, and the origin.

The preceding discussion may be summarized as follows.

Theorem 34

The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for $a^2 > b^2$ is an ellipse with vertices $(\pm a, 0)$. The endpoints of the minor axis are $(0, \pm b)$. The foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$.

EXAMPLE 6 Sketch the graph of $2x^2 + 9y^2 = 18$, and find the foci.

SOLUTION To obtain the form in Theorem (34), we divide both sides of the equation by 18 and simplify to get

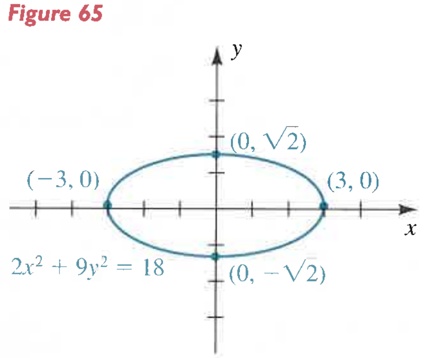
$$\frac{x^2}{9} + \frac{y^2}{2} = 1,$$

which is in the proper form, with $a^2 = 9$ and $b^2 = 2$. Thus, $a = 3$ and $b = \sqrt{2}$; hence the endpoints of the major axis are $(\pm 3, 0)$, and the endpoints of the minor axis are $(0, \pm\sqrt{2})$. Since

$$c^2 = a^2 - b^2 = 9 - 2 = 7, \quad \text{or} \quad c = \sqrt{7},$$

the foci are $(\pm\sqrt{7}, 0)$. The graph is sketched in Figure 65.

Figure 65



EXAMPLE 7 Find an equation of the ellipse with vertices $(\pm 4, 0)$ and foci $(\pm 2, 0)$.

SOLUTION Using the notation of Theorem (34), we conclude that $a = 4$ and $c = 2$. Since $c^2 = a^2 - b^2$, we have $b^2 = a^2 - c^2 = 16 - 4 = 12$. Hence, an equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1.$$

We sometimes choose the major axis of the ellipse along the y-axis. If the foci are $(0, \pm c)$, then by the same type of argument used previously, we obtain the following.

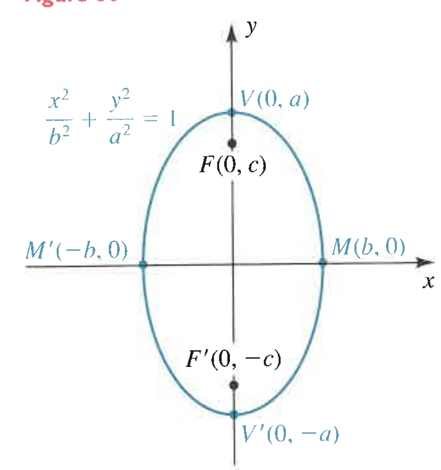
Theorem 35

The graph of the equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

for $a^2 > b^2$ is an ellipse with vertices $(0, \pm a)$. The endpoints of the minor axis are $(\pm b, 0)$. The foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$.

Figure 66



A typical graph is sketched in Figure 66.

The preceding discussion shows that an equation of an ellipse with center at the origin and foci on a coordinate axis can always be written in the form

$$\frac{x^2}{p} + \frac{y^2}{q} = 1, \quad \text{or} \quad qx^2 + py^2 = pq,$$

with p and q positive and $p \neq q$. If $p > q$, the major axis is on the x-axis, and if $q > p$, the major axis is on the y-axis. It is unnecessary to memorize these facts, because in any given problem the major axis can be determined by examining the x- and y-intercepts.

EXAMPLE 8 Sketch the graph of $9x^2 + 4y^2 = 25$, and find the foci.

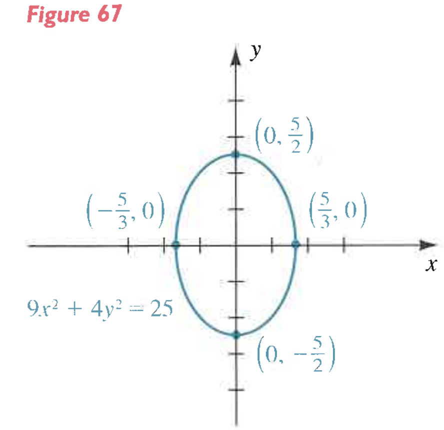
SOLUTION The graph is an ellipse with center at the origin and foci on one of the coordinate axes. To find x-intercepts, we let $y = 0$, obtaining

$$9x^2 = 25, \quad \text{or} \quad x = \pm\tfrac{5}{3}.$$

Similarly, to find the y-intercepts, we let $x = 0$, obtaining

$$4y^2 = 25, \quad \text{or} \quad y = \pm\tfrac{5}{2}.$$

Figure 67



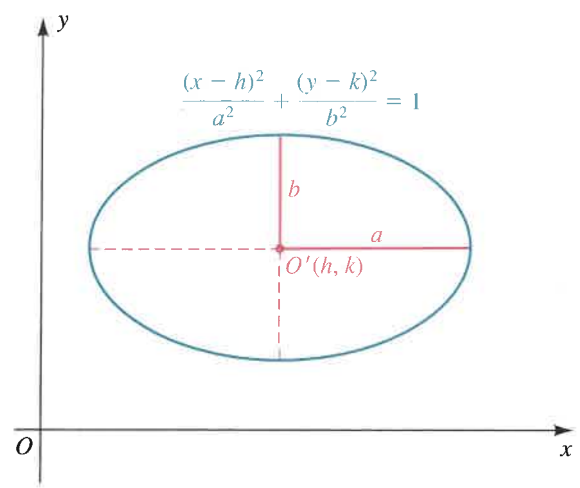
These results enable us to sketch the ellipse (see Figure 67). Since $\frac{5}{3} < \frac{5}{2}$, the major axis is on the y-axis.

To find the foci, we first calculate

$$c^2 = a^2 - b^2 = (\tfrac{5}{2})^2 - (\tfrac{5}{3})^2 = \tfrac{125}{36}.$$

Thus, $c = 5\sqrt{5}/6$ and the foci are $(0, \pm 5\sqrt{5}/6)$.

Figure 68



For an ellipse with center (h, k) at any point in the xy-plane and with major and minor axes that are horizontal or vertical (see Figure 68), the equation takes the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Squaring the indicated terms and simplifying gives us an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where the coefficients are real numbers and both A and C are positive. Conversely, if we start with such an equation, then by completing squares, we can obtain a form that displays the center of the ellipse and the lengths of the major and minor axes. This technique is illustrated in the next example.

EXAMPLE ■ 9 Discuss and sketch the graph of the equation

$$16x^2 + 9y^2 + 64x - 18y - 71 = 0.$$

SOLUTION We begin by writing the equation in the form

$$16(x^2 + 4x \quad) + 9(y^2 - 2y \quad) = 71.$$

Completing the squares gives us

$$16(x^2 + 4x + 4) + 9(y^2 - 2y + 1) = 71 + 64 + 9,$$

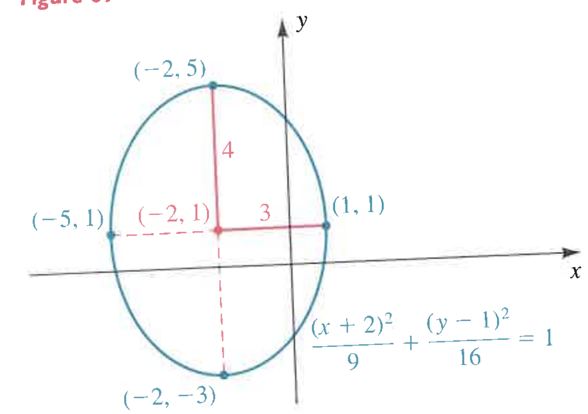
which may be written as

$$16(x+2)^2 + 9(y-1)^2 = 144.$$

Dividing by 144, we obtain

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{16} = 1,$$

Figure 69



which is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

with $h = -2$ and $k = 1$. The graph of the equation is an ellipse with center $(-2, 1)$ (see Figure 69). Since $16 > 9$, the major axis is on the vertical line $x = -2$.

To find the foci, note that $c^2 = 16 - 9 = 7$. Thus, the distance from the center of the ellipse to either focus is $c = \sqrt{7}$. Since the center is $(-2, 1)$, the foci are $(-2, 1 \pm \sqrt{7})$.

Ellipses can be very flat or almost circular. To obtain information about the *roundness* of an ellipse, we sometimes use the term *eccentricity*.

Definition 36

The **eccentricity** e of an ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

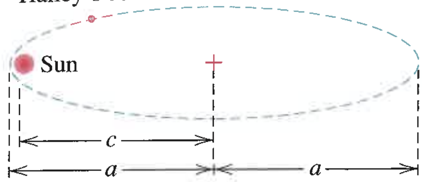
Consider the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, and suppose that the length $2a$ of the major axis is fixed and the length $2b$ of the minor axis is variable. Since $\sqrt{a^2 - b^2} < a$, we see that $0 < e < 1$. If $e \approx 1$, then $\sqrt{a^2 - b^2} \approx a$ and $b \approx 0$. Thus, the ellipse is very flat. If $e \approx 0$, then $\sqrt{a^2 - b^2} \approx 0$ and $a \approx b$. Thus, the ellipse is almost circular.

Many comets have elliptical orbits with the sun at a focus. In this case the eccentricity e is close to 1, and the ellipse is very flat. In the next example, we use the **astronomical unit** (AU)—that is, the average distance from the earth to the sun—to specify large distances (1 AU ≈ 93,000,000 miles).

EXAMPLE ■ 10 Halley's comet has an elliptical orbit with eccentricity $e = 0.967$. The closest that Halley's comet comes to the sun is

0.587 AU. Approximate the maximum distance of the comet from the sun, to the nearest 0.1 AU.

Figure 70
Halley's comet



SOLUTION Figure 70 illustrates the orbit of the comet, where c is the distance from the center of the ellipse to a focus (the sun) and $2a$ is the length of the major axis.

Since $a - c$ is the minimum distance between the sun and the comet, we have (in AU)

$$a - c = 0.587, \quad \text{or} \quad a = c + 0.587.$$

Since $e = c/a = 0.967$,

$$c = 0.967a = 0.967(c + 0.587)$$
$$c \approx 0.967c + 0.568.$$

Thus,

$$0.033c \approx 0.568 \quad \text{and} \quad c \approx \frac{0.568}{0.033} \approx 17.2.$$

Consequently,

$$a = c + 0.587$$
$$a \approx 17.2 + 0.587 \approx 17.8,$$

and the maximum distance between the sun and the comet is

$$a + c \approx 17.8 + 17.2, \quad \text{or} \quad a + c \approx 35.0 \text{ AU}.$$

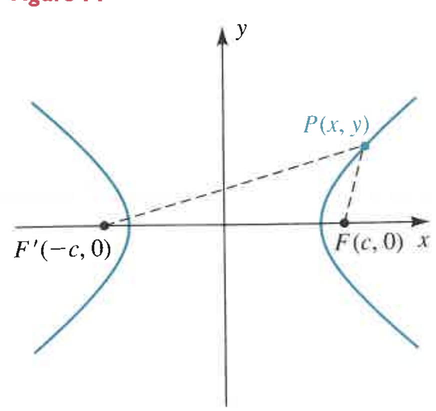
HYPERBOLAS

The definition of a hyperbola is similar to that of an ellipse. The only change is that instead of using the *sum* of distances from two fixed points, we use the *difference*.

Definition 37

A **hyperbola** is the set of all points in the plane, the difference of whose distances from two fixed points (the **foci**) in the plane is a positive constant.

Figure 71



The **center** of a hyperbola is the midpoint of the segment FF'. If the foci lie along the x-axis and the center is at the origin, then the hyperbola has a simple equation. Let P be a point on the hyperbola. Suppose F has coordinates $(c, 0)$ so that F' has coordinates $(-c, 0)$ (see Figure 71). Let $2a$ denote the constant difference of the distances of P from F and F', and let $b^2 = c^2 - a^2$. It can be shown that P is on the hyperbola if and only if its coordinates (x, y) satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

By tests for symmetry, the hyperbola is symmetric with respect to both axes and the origin. The x-intercepts are a and $-a$. The corresponding

points $V(a, 0)$ and $V'(-a, 0)$ are the **vertices**, and the line segment $V'V$ is the **transverse axis** of the hyperbola.

The preceding discussion may be summarized as follows.

Theorem 38

The graph of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

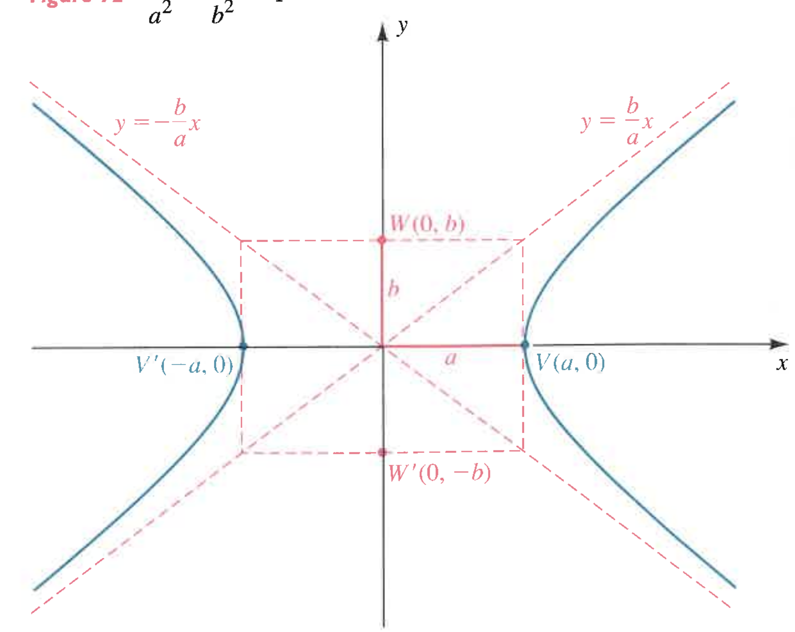
is a hyperbola with vertices $(\pm a, 0)$. The foci are $(\pm c, 0)$, where $c^2 = a^2 + b^2$.

If we solve the equation $(x^2/a^2) - (y^2/b^2) = 1$ for y, we obtain

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}.$$

If a graph approaches a line as the absolute value of x gets increasingly large, then the line is called an **asymptote** for the graph. It can be shown that the lines $y = (b/a)x$ and $y = -(b/a)x$ are asymptotes for the hyperbola. These asymptotes serve as excellent guides for sketching the graph. A convenient way to sketch the asymptotes is to first plot the vertices $V(a, 0)$, $V'(-a, 0)$ and the points $W(0, b)$, $W'(0, -b)$ (see Figure 72). The line segment $W'W$ of length $2b$ is the **conjugate axis** of the hyperbola. If horizontal and vertical lines are drawn through the endpoints of the conjugate and transverse axes, respectively, then the diagonals of the resulting rectangle have slopes b/a and $-b/a$. Hence, by extending these diagonals, we obtain lines with equations $y = (\pm b/a)x$, which are

Figure 72 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



the asymptotes. The hyperbola is then sketched as in Figure 72, using the asymptotes as guides. The two curves that make up the hyperbola are the **branches** of the hyperbola.

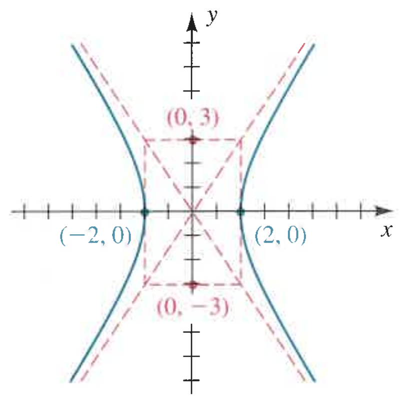
EXAMPLE ▪ 11 Discuss and sketch the graph of $9x^2 - 4y^2 = 36$. Then find the foci and the equations of the asymptotes.

SOLUTION The graph is a hyperbola with the center at the origin. Dividing both sides of the given equation by 36 and simplifying gives

$$\frac{x^2}{4} - \frac{y^2}{9} = 1,$$

which is of the form stated in Theorem (38), with $a^2 = 4$ and $b^2 = 9$. Hence, $a = 2$ and $b = 3$. The vertices $(\pm 2, 0)$ and the endpoints $(0, \pm 3)$ of the conjugate axis determine a rectangle whose diagonals (extended) give us the asymptotes. The graph of the equation is sketched in Figure 73.

Figure 73
$9x^2 - 4y^2 = 36$



To find the foci, we calculate

$$c^2 = a^2 + b^2 = 4 + 9 = 13.$$

Thus, $c = \sqrt{13}$ and the foci are $(\pm\sqrt{13}, 0)$.

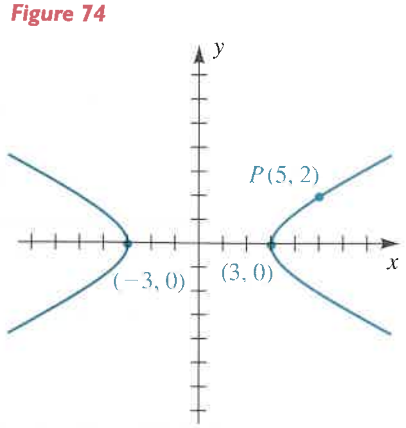
The equations of the asymptotes, $y = \pm\frac{3}{2}x$, can be found by referring to the graph or to the equations $y = \pm(b/a)x$.

The preceding example indicates that for hyperbolas it is not always true that $a > b$, as is the case for ellipses. Indeed, we may have $a < b$, $a > b$, or $a = b$.

EXAMPLE ▪ 12 A hyperbola has vertices $(\pm 3, 0)$ and passes through the point $P(5, 2)$. Find its equation, the foci, and the equations of the asymptotes.

SOLUTION We begin by sketching a hyperbola with vertices $(\pm 3, 0)$ that passes through the point $P(5, 2)$, as in Figure 74.

Figure 74



An equation of the hyperbola has the form

$$\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1.$$

Since $P(5, 2)$ is on the hyperbola, the x- and y-coordinates satisfy the last equation; that is,

$$\frac{25}{9} - \frac{4}{b^2} = 1.$$

Solving for b^2 gives us $b^2 = \frac{9}{4}$, so the desired equation is

$$\frac{x^2}{9} - \frac{y^2}{\frac{9}{4}} = 1,$$

or, equivalently,

$$x^2 - 4y^2 = 9.$$

To find the foci, we first calculate

$$c^2 = a^2 + b^2 = 9 + \tfrac{9}{4} = \tfrac{45}{4}.$$

Hence, $c = \sqrt{\frac{45}{4}} = \frac{3}{2}\sqrt{5}$, and the foci are $(\pm\frac{3}{2}\sqrt{5}, 0)$.

The general equations of the asymptotes are $y = \pm(b/a)x$. Substituting $a = 3$ and $b = \frac{3}{2}$ gives us $y = \pm\frac{1}{2}x$.

If the foci of a hyperbola are the points $(0, \pm c)$ on the y-axis, then by the same type of argument used previously, we obtain the following theorem.

Theorem 39

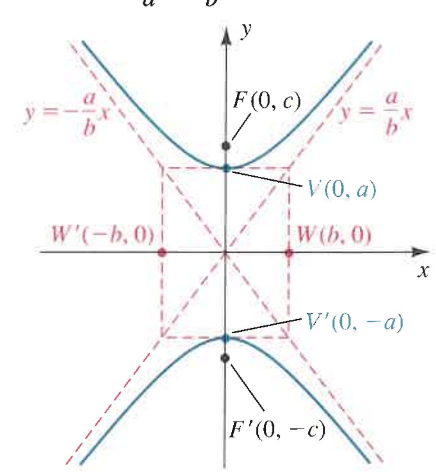
The graph of the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

is a hyperbola with vertices $(0, \pm a)$. The foci are $(0, \pm c)$, where $c^2 = a^2 + b^2$.

For the hyperbola in the preceding theorem, the endpoints of the conjugate axis are $W(b, 0)$ and $W'(-b, 0)$. We find the asymptotes as before, by using the diagonals of the rectangle determined by these points, the vertices, and lines parallel to the coordinate axes. The graph is sketched in Figure 75. The equations of the asymptotes are $y = \pm(a/b)x$. Note the difference between these equations and the equations $y = \pm(b/a)x$ for the asymptotes of the hyperbola considered first in this section.

Figure 75 $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$



EXAMPLE ■ 13 Discuss and sketch the graph of $4y^2 - 2x^2 = 1$. Then find the foci and the equations of the asymptotes.

SOLUTION We may obtain the form in Theorem (39) by writing the equation as

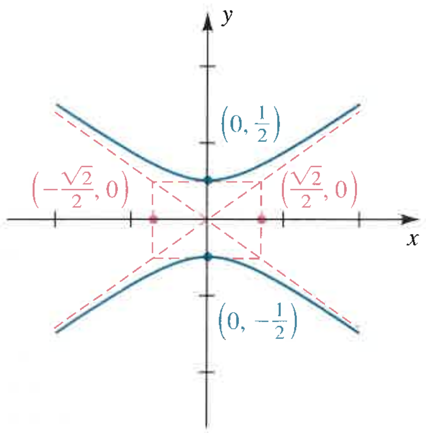
$$\frac{y^2}{\frac{1}{4}} - \frac{x^2}{\frac{1}{2}} = 1.$$

Thus, $\quad a^2 = \frac{1}{4}, \quad b^2 = \frac{1}{2}, \quad c^2 = a^2 + b^2 = \frac{3}{4},$

and hence, $\quad a = \frac{1}{2}, \quad b = \frac{\sqrt{2}}{2}, \quad c = \frac{\sqrt{3}}{2}.$

The vertices are $(0, \pm\frac{1}{2})$, the foci are $(0, \pm\sqrt{3}/2)$, and the endpoints of the conjugate axes are $(\pm\sqrt{2}/2, 0)$. The graph is sketched in Figure 76.

Figure 76 $4y^2 - 2x^2 = 1$



To find equations of the asymptotes, we can use $y = \pm(a/b)x$, obtaining $y = \pm(\sqrt{2}/2)x$.

If the center of a hyperbola is at any point (h, k) in the xy-plane, then it has the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

if the foci lie on a horizontal line, or the form of the equation is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

if the foci lie on a vertical line.

Squaring the indicated terms in these equations and simplifying allows us to write the equation for the hyperbola in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where the coefficients are real numbers and A and C have opposite signs (one is positive and the other is negative). Conversely, if we begin with such an equation, then by completing squares, we can obtain a form that displays the center of the hyperbola and the transverse and conjugate axes. The next example illustrates this technique.

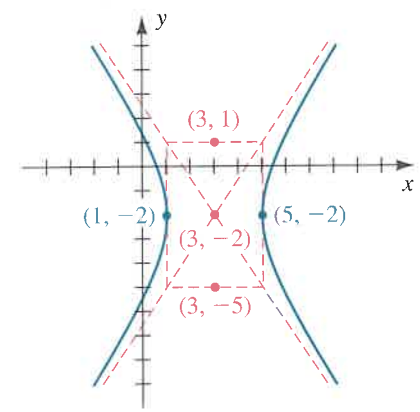
EXAMPLE ■ 14 Discuss and sketch the graph of the equation

$$9x^2 - 4y^2 - 54x - 16y + 29 = 0.$$

SOLUTION We arrange our work as follows:

$$9(x^2 - 6x \qquad) - 4(y^2 + 4y \qquad) = -29$$
$$9(x^2 - 6x + 9) - 4(y^2 + 4y + 4) = -29 + 81 - 16$$
$$9(x-3)^2 - 4(y+2)^2 = 36$$
$$\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$$

Figure 77 $\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$



Note that $h = 3$ and $k = -2$. The graph of the equation is a hyperbola with center $(3, -2)$. The foci lie on the horizontal line $y = -2$ through the center. We see that

$$a^2 = 4, \qquad b^2 = 9, \qquad c^2 = a^2 + b^2 = 13.$$

Hence, $\quad a = 2, \quad b = 3, \quad c = \sqrt{13}.$

As illustrated in Figure 77, the vertices are $(3 \pm 2, -2)$—that is, $(5, -2)$ and $(1, -2)$. The endpoints of the conjugate axis are $(3, -2 \pm 3)$—that is, $(3, 1)$ and $(3, -5)$. The foci are $(3 \pm \sqrt{13}, -2)$, and the equations of the asymptotes are

$$y + 2 = \pm\tfrac{3}{2}(x - 3).$$

The results of the last three sections indicate that the graph of every equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is a conic, except for certain degenerate cases in which a point, one or two lines, or no graph is obtained. Although we have considered only special examples, our methods are perfectly general. If A and C are equal and not 0, then the graph, when it exists, is a circle or, in exceptional cases, a point. If A and C are unequal but have the same sign, then by completing squares and properly translating axes, we obtain an equation whose graph, when it exists, is an ellipse (or a point). If A and C have opposite signs, an equation of a hyperbola is obtained or possibly, in the degenerate case, two intersecting straight lines. If either A or C (but not both) is 0, the graph is a parabola or, in certain cases, a pair of parallel lines.

EXERCISES E

Exer. 1–6: Find the vertex, the focus, and the directrix of the parabola. Sketch its graph, showing the focus and the directrix.

1 $y = -\frac{1}{12}x^2$ **2** $x = 2y^2$

3 $2y^2 = -3x$ **4** $x^2 = -3y$

5 $y = 8x^2$ **6** $y^2 = -100x$

Exer. 7–16: Find the vertex and the focus of the parabola. Sketch its graph, showing the focus.

7 $y = x^2 - 4x + 2$ **8** $y = 8x^2 + 16x + 10$

9 $y^2 - 12 = 12x$ **10** $y^2 - 20y + 100 = 6x$

11 $y^2 - 4y - 2x - 4 = 0$

12 $y^2 + 14y + 4x + 45 = 0$

13 $4x^2 + 40x + y + 106 = 0$

14 $y = 40x - 97 - 4x^2$

15 $x^2 + 20y = 10$

16 $4x^2 + 4x + 4y + 1 = 0$

Exer. 17–24: Find an equation of the parabola that satisfies the given conditions.

17 focus $F(2, 0)$; directrix $x = -2$

18 focus $F(0, -4)$; directrix $y = 4$

19 vertex $V(3, -5)$; directrix $x = 2$

20 vertex $V(-2, 3)$; directrix $y = 5$

21 vertex $V(-1, 0)$; focus $F(-4, 0)$

22 vertex $V(1, -2)$; focus $F(1, 0)$

23 vertex at the origin; symmetric to the y-axis; and passing through the point $(2, -3)$

24 vertex $V(-3, 5)$; axis parallel to the x-axis; and passing through the point $(5, 9)$

Exer. 25–38: Find the vertices and the foci of the ellipse. Sketch its graph, showing the foci.

25 $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **26** $\frac{x^2}{25} + \frac{y^2}{16} = 1$

27 $4x^2 + y^2 = 16$ **28** $y^2 + 9x^2 = 9$

29 $5x^2 + 2y^2 = 10$ **30** $\frac{1}{2}x^2 + 2y^2 = 8$

31 $4x^2 + 25y^2 = 1$ **32** $10y^2 + x^2 = 5$

33 $4x^2 + 9y^2 - 32x - 36y + 64 = 0$

34 $x^2 + 2y^2 + 2x - 20y + 43 = 0$

35 $9x^2 + 16y^2 + 54x - 32y - 47 = 0$

36 $4x^2 + 9y^2 + 24x + 18y + 9 = 0$

37 $25x^2 + 4y^2 - 250x - 16y + 541 = 0$

38 $4x^2 + y^2 = 2y$

Exer. 39–48: Find an equation for the ellipse that has its center at the origin and satisfies the given conditions.

39 vertices $V(\pm 8, 0)$; foci $F(\pm 5, 0)$

40 vertices $V(0, \pm 7)$; foci $F(0, \pm 2)$

41 vertices $V(0, \pm 5)$; minor axis of length 3

42 foci $F(\pm 3, 0)$; minor axis of length 2

43 vertices $V(0, \pm 6)$; passing through $(3, 2)$

44 passing through $(2, 3)$ and $(6, 1)$

45 eccentricity $\frac{3}{4}$; vertices $V(0, \pm 4)$

46 eccentricity $\frac{1}{2}$; vertices on the x-axis; passing through $(1, 3)$

47 x-intercepts ± 2; y-intercepts $\pm\frac{1}{3}$

48 x-intercepts $\pm\frac{1}{2}$; y-intercepts ± 4

Exer. 49–66: Find the vertices and the foci of the hyperbola. Sketch its graph, showing the asymptotes and the foci.

49 $\frac{x^2}{9} - \frac{y^2}{4} = 1$ **50** $\frac{y^2}{49} - \frac{x^2}{16} = 1$

51 $\frac{y^2}{9} - \frac{x^2}{4} = 1$ **52** $\frac{x^2}{49} - \frac{y^2}{16} = 1$

53 $y^2 - 4x^2 = 16$ **54** $x^2 - 2y^2 = 8$

55 $x^2 - y^2 = 1$ **56** $y^2 - 16x^2 = 1$

57 $x^2 - 5y^2 = 25$ **58** $4y^2 - 4x^2 = 1$

59 $3x^2 - y^2 = -3$ **60** $16x^2 - 36y^2 = 1$

61 $25x^2 - 16y^2 + 250x + 32y + 109 = 0$

62 $y^2 - 4x^2 - 12y - 16x + 16 = 0$

63 $4y^2 - x^2 + 40y - 4x + 60 = 0$

64 $25x^2 - 9y^2 + 100x - 54y + 10 = 0$

65 $9y^2 - x^2 - 36y + 12x - 36 = 0$

66 $4x^2 - y^2 + 32x - 8y + 49 = 0$

Exer. 67–76: Find an equation for the hyperbola that has its center at the origin and satisfies the given conditions.

67 foci $F(0, \pm 4)$; vertices $V(0, \pm 1)$

68 foci $F(\pm 8, 0)$; vertices $V(\pm 5, 0)$

69 foci $F(\pm 5, 0)$; vertices $V(\pm 3, 0)$

70 foci $F(0, \pm 3)$; vertices $V(0, \pm 2)$

71 foci $F(0, \pm 5)$; conjugate axis of length 4

72 vertices $V(\pm 4, 0)$; passing through $(8, 2)$

73 vertices $V(\pm 3, 0)$; asymptotes $y = \pm 2x$

74 foci $F(0, \pm 10)$; asymptotes $y = \pm\frac{1}{3}x$

75 x-intercepts ± 5; asymptotes $y = \pm 2x$

76 y-intercepts ± 2; asymptotes $y = \pm\frac{1}{4}x$

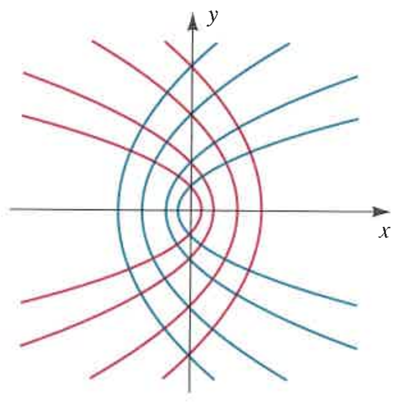
77 The graphs of the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

are called *conjugate hyperbolas*. Sketch the graphs of both equations on the same coordinate plane, with $a = 5$ and $b = 3$. Describe the relationship between the two graphs.

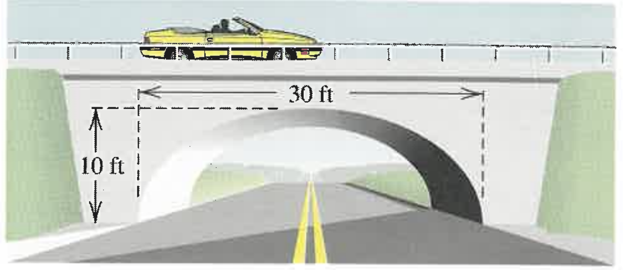
78 The parabola $y^2 = 4p(x + p)$ has its focus at the origin and its axis along the x-axis. By assigning different values to p, we obtain a family of *confocal parabolas*, as shown in the figure. Families of this type occur in the study of electricity and magnetism. Show that there are exactly two parabolas in the family that pass through a given point $P(x_1, y_1)$ if $y_1 \neq 0$.

Exercise 78



79 The arch of a bridge is semielliptical, with major axis horizontal. The base of the arch is 30 ft across, and the highest part is 10 ft above the horizontal roadway, as shown in the figure. Find the height of the arch 6 ft from the center of the base.

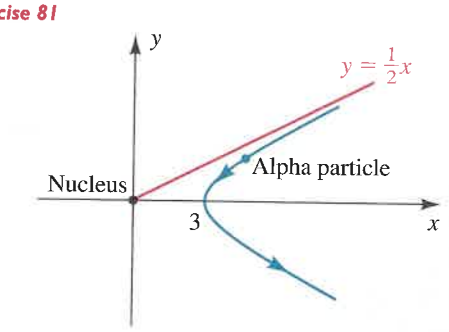
Exercise 79



80 Assume that the length of the major axis of the earth's orbit is 186,000,000 mi and the eccentricity is 0.017. Find, to the nearest 1000 mi, the maximum and minimum distances between the earth and the sun.

81 In 1911, the physicist Ernest Rutherford (1871–1937) discovered that when alpha particles are shot toward the nucleus of an atom, they are eventually repulsed away from the nucleus along hyperbolic paths. The figure illustrates the path of a particle that starts toward the origin along the line $y = \frac{1}{2}x$ and comes within 3 units of the nucleus. Find an equation of the path.

Exercise 81



82 A cruise ship is traveling a course that is 100 mi from, and parallel to, a straight shoreline. The ship sends out a distress signal, which is received by two Coast Guard stations A and B, located 200 mi apart, as shown in the figure. By measuring the difference in signal reception times, officials determine that the ship is 160 mi closer to B than to A. Where is the ship?

Exercise 82