Journal Title: Calculus of a single variable /

ISSN:

Volume:
Issue:
Month/Year:
Pages: 784-843

Article Author:

Article Title: Chapter 9: Parametric Equations and Polar Coordinates

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INTRODUCTION

THE NATURAL WORLD is filled with curves that excite both the imagination of the artist and the curiosity of the scientist: the outline of the moon against the evening sky, the delicate folds of a flower, the sinuous curve of a river, the graceful silhouette of a bird in flight, the curls and spirals of a cresting wave. In this chapter, we examine several different ways of representing curves in mathematical forms that will enable us both to understand the curves better and to gain new appreciation of their beauty.

For an equation of the form $y = f(x)$, where f is a function, the graph is a *curve* in the xy-plane. The concept of curve is more general, however, than that of the graph of a function, since a curve may cross itself in figure-eight style, be closed (as are circles and ellipses), or spiral around a fixed point. In fact, some curves studied in advanced mathematics pass through every point in a coordinate plane!

The curves discussed in this chapter lie in an xy-plane, and each has the property that the coordinates x and y of an arbitrary point P on the curve can be expressed as functions of a variable t, called a *parameter*. We choose the letter t because in many applications this variable denotes time and P represents a moving object that has position (x, y) at time t. In later chapters, we will use such representations to define velocity, acceleration, and other concepts associated with motion. In Section 9.1, we consider the definitions of a curve and parametric equations, and we discuss a number of examples and applications. The determination of tangent lines and arc length from a parametric representation of a curve is the topic of Section 9.2; we also discuss how to determine the area of a surface of revolution of a curve given its parametric equations.

In Sections 9.3 and 9.4, we discuss polar (or circular) coordinates and use definite integrals to find areas enclosed by graphs of polar equations. Our methods are analogous to those developed in Chapter 5. The principal difference is that we consider limits of sums of circular sectors instead of vertical or horizontal rectangles. Switching from an xy-coordinate system to a polar coordinate system often yields a much simpler equation for a plane curve. The circles and spirals evident in such natural phenomena as the curl of an ocean wave have much simpler representations in polar coordinates than in rectangular xy-coordinates. Thus, we may be able to describe and understand some facets of nature more easily by using polar coordinates. Equations for a curve in the xy-coordinate system can also be simplified by adopting a rectangular coordinate system obtained from the standard xy-coordinates by a *translation* or *rotation* of axes, which we discuss in Section 9.5.

CHAPTER 9

Parametric equations and polar coordinates provide useful frameworks for the analysis of many natural phenomena such as those exhibited by ocean waves.

Parametric Equations and Polar Coordinates

9.1 PARAMETRIC EQUATIONS

In this section, we introduce a new way to describe curves in the plane by using parametric equations. If f is a continuous function, the graph of the equation $y = f(x)$ is often called a *plane curve*. However, this definition is restrictive because it excludes many useful graphs. The following definition is more general.

Definition 9.1

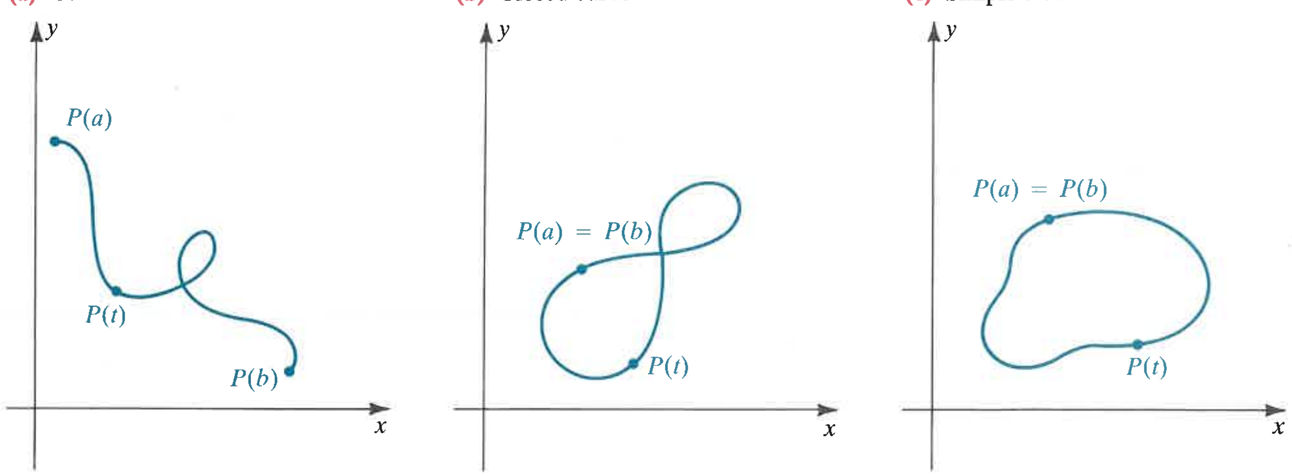
> A **plane curve** is a set C of ordered pairs $(f(t), g(t))$, where f and g are continuous functions on an interval I.

For simplicity, we often refer to a plane curve as a **curve.** The **graph** of C in Definition (9.1) consists of all points $P(t) = (f(t), g(t))$ in an xy-plane, for t in I. We shall use the term *curve* interchangeably with *graph of a curve*. We sometimes regard the point $P(t)$ as tracing the curve C as t varies through the values of the interval I.

The graphs of several curves are sketched in Figure 9.1, where I is a closed interval $[a, b]$. In Figure 9.1(a), $P(a) \neq P(b)$, and $P(a)$ and $P(b)$ are called the **endpoints** of C. The curve in (a) intersects itself; that is, two different values of t produce the same point. If $P(a) = P(b)$, as in Figure 9.1(b), then C is a **closed curve**. If $P(a) = P(b)$ and C does not intersect itself at any other point, as in Figure 9.1(c), then C is a **simple closed curve.**

Figure 9.1

(a) Curve (b) Closed curve (c) Simple closed curve



A convenient way to represent curves is given in the next definition.

Definition 9.2

> Let C be the curve consisting of all ordered pairs $(f(t), g(t))$, where f and g are continuous on an interval I. The equations
>
> $$x = f(t), \qquad y = g(t),$$
>
> for t in I, are **parametric equations** for C with **parameter** t.

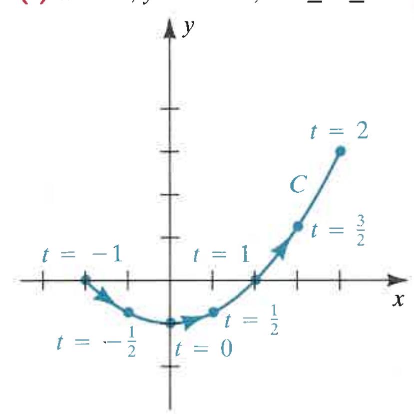
The curve C in this definition is referred to as a **parametrized curve,** and the parametric equations are a **parametrization** for C. We often use the notation

$$x = f(t), \quad y = g(t); \quad t \text{ in } I$$

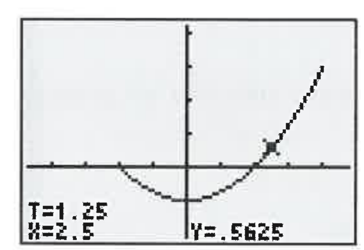
to indicate the domain I of f and g. Sometimes it may be possible to eliminate the parameter and obtain a familiar equation in x and y for C. In simple cases, we may sketch a graph of a parametrized curve by plotting points and connecting them in the order of increasing t, as illustrated in the next example.

Figure 9.2

(a) $x = 2t,\ y = t^2 - 1;\ -1 \le t \le 2$



(b) $-1 \le t \le 2,\ -4.7 \le x \le 4.8,$ $-2.1 \le y \le 4.2$



EXAMPLE ■ 1 Let C be the curve that has parametrization

$$x = 2t, \quad y = t^2 - 1; \quad -1 \le t \le 2.$$

(a) Sketch the graph of C by hand by plotting several points and joining them with a smooth curve.

(b) Obtain an equation for the curve in the form $y = f(x)$ for some function f.

(c) Use a graphing utility to plot a graph of C. Set the viewing window so that it contains the entire graph.

SOLUTION

(a) We use the parametric equations to tabulate coordinates of points $P(x, y)$ on C as follows.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x	-2	-1	0	1	2	3	4
y	0	$-\frac{3}{4}$	-1	$-\frac{3}{4}$	0	$\frac{5}{4}$	3

Plotting points leads to the sketch in Figure 9.2(a). The arrowheads on the graph indicate the direction in which $P(x, y)$ traces the curve as t *increases* from -1 to 2.

(b) We may obtain a clearer description of the graph by eliminating the parameter. Solving the first parametric equation for t, we obtain $t = \frac{1}{2}x$. Substituting this expression for t in the second equation gives us

$$y = (\tfrac{1}{2}x)^2 - 1.$$

The graph of this equation in x and y is a parabola symmetric with respect to the y-axis with vertex $(0, -1)$. However, since $x = 2t$ and it satisfies $-1 \le t \le 2$, we see that $-2 \le x \le 4$ for points (x, y) on C, and hence C is that part of the parabola between the points $(-2, 0)$ and $(4, 3)$ shown in Figure 9.2(a).

(c) We set the graphing utility to parametric mode and enter the parametric equations. We also specify the interval for the parameter t as $[-1, 2]$. To select a viewing window that will contain the entire graph, we first note that since $x = 2t$, x will range from -2 to 4 as t ranges from -1 to 2. Similarly, since $y = t^2 - 1$, y has a minimum value of -1 at $t = 0$ and a maximum value of 3 at $t = 2$. Thus, the smallest viewing window that will accommodate the entire graph is $-2 \le x \le 4, -1 \le y \le 3$. We will use the slightly larger viewing window shown in Figure 9.2(b).

The graphing utility may show the curve tracing out its path in the direction indicated by the arrowheads in Figure 9.2(a). If not, we can use the trace operation to verify that the graph begins at $(-2, 0)$, moves downward through the third quadrant to $(0, -1)$, and then moves upward through quadrants IV and I until it reaches $(4, 3)$.

As indicated by the arrowheads in Figure 9.2(a), the point $P(x, y)$ traces the curve C from *left to right* as t increases. The parametric equations

$$x = -2t, \quad y = t^2 - 1; \quad -2 \le t \le 1$$

give us the same graph; however, as t increases, $P(x, y)$ traces the curve from *right to left*. For other parametrizations, the point $P(x, y)$ may oscillate back and forth as t increases.

The **orientation** of a parametrized curve C is the direction determined by *increasing* values of the parameter. We often indicate an orientation by placing arrowheads on C as in Figure 9.2(a). If $P(x, y)$ moves back and forth as t increases, we may place arrows *alongside* of C. As we have observed, a curve may have different orientations, depending on the parametrization.

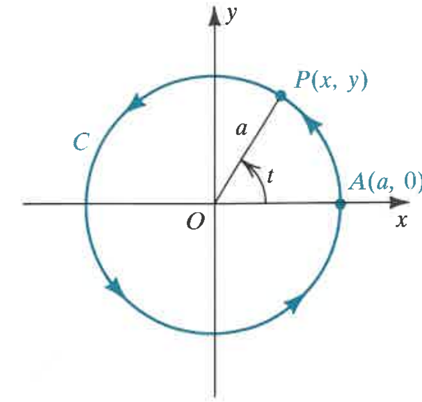
The next example demonstrates that it is sometimes useful to eliminate the parameter *before* plotting points.

EXAMPLE ■ 2 A point moves in a plane such that its position $P(x, y)$ at time t is given by

$$x = a\cos t, \quad y = a \sin t; \quad t \ge 0,$$

where $a > 0$. Describe the motion of the point.

Figure 9.3
$x = a\cos t,\ y = a\sin t;\ t \ge 0$



SOLUTION We may eliminate the parameter by rewriting the parametric equations as

$$\frac{x}{a} = \cos t, \qquad \frac{y}{a} = \sin t$$

and using the identity $\cos^2 t + \sin^2 t = 1$ to obtain

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1,$$

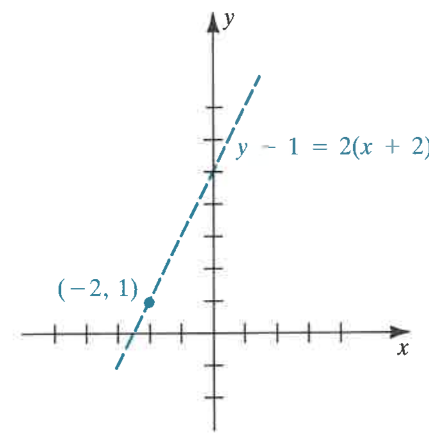
or

$$x^2 + y^2 = a^2.$$

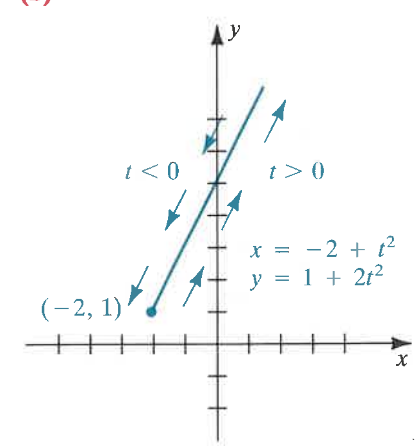
This result shows that the point $P(x, y)$ moves on the circle C of radius a with center at the origin (see Figure 9.3). The point is at $A(a, 0)$ when $t = 0$, at $(0, a)$ when $t = \pi/2$, at $(-a, 0)$ when $t = \pi$, at $(0, -a)$ when $t = 3\pi/2$, and back at $A(a, 0)$ when $t = 2\pi$. Thus, P moves around C in a counterclockwise direction, making one revolution every 2π units of time. The orientation of C is indicated by the arrowheads in the figure.

Note that in this example we may interpret t geometrically as the radian measure of the angle generated by the line segment OP.

Figure 9.4
(a)



(b)



EXAMPLE ■ 3 Sketch the graph of the curve C that has the parametrization

$$x = -2 + t^2, \quad y = 1 + 2t^2; \quad t \text{ in } \mathbb{R}$$

and indicate the orientation.

SOLUTION To eliminate the parameter, we use the first equation to obtain $t^2 = x + 2$ and then substitute for t^2 in the second equation. Thus,

$$y = 1 + 2(x + 2).$$

This result is an equation of the line of slope 2 through the point $(-2, 1)$, as indicated by the dashes in Figure 9.4(a). However, since $t^2 \ge 0$, we see from the parametric equations for C that

$$x = -2 + t^2 \ge -2 \quad \text{and} \quad y = 1 + 2t^2 \ge 1.$$

Thus, the graph of C is that part of the line to the right of $(-2, 1)$ (the point corresponding to $t = 0$), as shown in Figure 9.4(b). The orientation is indicated by the arrows alongside of C. As t increases in the interval $(-\infty, 0]$, $P(x, y)$ moves down the curve toward the point $(-2, 1)$. As t increases in $[0, \infty)$, $P(x, y)$ moves up the curve away from $(-2, 1)$.

If a curve C is described by an equation $y = f(x)$ for a continuous function f, then an easy way to obtain parametric equations for C is to let

$$x = t, \qquad y = f(t),$$

where t is in the domain of f. For example, if $y = x^3$, then parametric equations are

$$x = t, \quad y = t^3; \quad t \text{ in } \mathbb{R}.$$

We can use many different substitutions for x, provided that as t varies through some interval, x takes on every value in the domain of f. Thus, the graph of $y = x^3$ is also given by

$$x = t^{1/3}, \quad y = t; \quad t \text{ in } \mathbb{R}.$$

Note, however, that the parametric equations

$$x = \sin t, \quad y = \sin^3 t; \quad t \text{ in } \mathbb{R}$$

give only that part of the graph of $y = x^3$ between the points $(-1, -1)$ and $(1, 1)$.

EXAMPLE ■ 4 Find three parametrizations for the line of slope m through the point (x_1, y_1).

SOLUTION By the point–slope form, an equation for the line is

$$y - y_1 = m(x - x_1).$$

If we let $x = t$, then $y - y_1 = m(t - x_1)$ and we obtain the parametrization

$$x = t, \quad y = y_1 + m(t - x_1); \quad t \text{ in } \mathbb{R}.$$

We obtain another parametrization for the line if we let $x - x_1 = t$. In this case, $y - y_1 = mt$, and we have

$$x = x_1 + t, \quad y = y_1 + mt; \quad t \text{ in } \mathbb{R}.$$

As a third illustration, if we let $x - x_1 = \tan t$, then

$$x = x_1 + \tan t, \quad y = y_1 + m \tan t; \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

There are many other parametrizations for the line.

Parametric equations of the form

$$x = a \sin \omega_1 t, \quad y = b \cos \omega_2 t; \quad t \geq 0,$$

where a, b, ω_1, and ω_2 are constants, occur in electrical theory. The variables x and y usually represent voltages or currents at time t. The resulting curve is often difficult to sketch; however, using an oscilloscope and imposing voltages or currents on the input terminals, we can represent the graph, a **Lissajous figure**,* on the screen of the oscilloscope. Computers are also useful in obtaining these complicated graphs.

*Jules Antoine Lissajous (1822–1890) was a French physicist known for his research in acoustics and optics. He invented a system of optical telegraphy used during the 1871 siege of Paris.

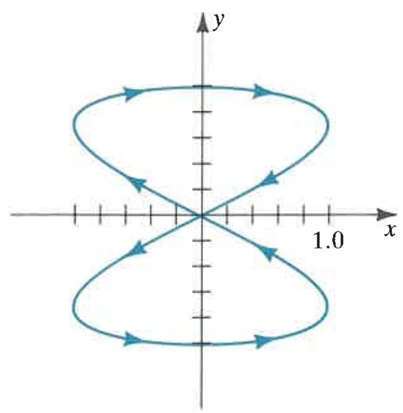


Figure 9.5
$x = \sin 2t,\ y = \cos t;\ 0 \leq t \leq 2\pi$

EXAMPLE ■ 5 A graph of the Lissajous figure

$$x = \sin 2t, \quad y = \cos t; \quad 0 \leq t \leq 2\pi$$

is shown in Figure 9.5, with the arrowheads indicating the orientation. Verify the orientation and find an equation in x and y for the curve.

SOLUTION Referring to the parametric equations, we see that as t increases from 0 to $\pi/2$, the point $P(x, y)$ starts at $(0, 1)$ and traces the part of the curve in quadrant I (in a generally clockwise direction). As t increases from $\pi/2$ to π, $P(x, y)$ traces the part in quadrant III (in a counterclockwise direction). For $\pi \leq t \leq 3\pi/2$, we obtain the part in quadrant IV; and $3\pi/2 \leq t \leq 2\pi$ gives us the part in quadrant II.

We may find an equation in x and y for the curve by using trigonometric identities and algebraic manipulations. Writing $x = 2 \sin t \cos t$ and squaring, we have

$$x^2 = 4 \sin^2 t \cos^2 t,$$

or

$$x^2 = 4(1 - \cos^2 t) \cos^2 t.$$

Using $y = \cos t$ gives us

$$x^2 = 4(1 - y^2)y^2.$$

To express y in terms of x, let us rewrite the last equation as

$$4y^4 - 4y^2 + x^2 = 0$$

and use the quadratic formula to solve for y^2 as follows:

$$y^2 = \frac{4 \pm \sqrt{16 - 16x^2}}{8} = \frac{1 \pm \sqrt{1 - x^2}}{2}$$

Taking square roots, we obtain

$$y = \pm\sqrt{\frac{1 \pm \sqrt{1 - x^2}}{2}}.$$

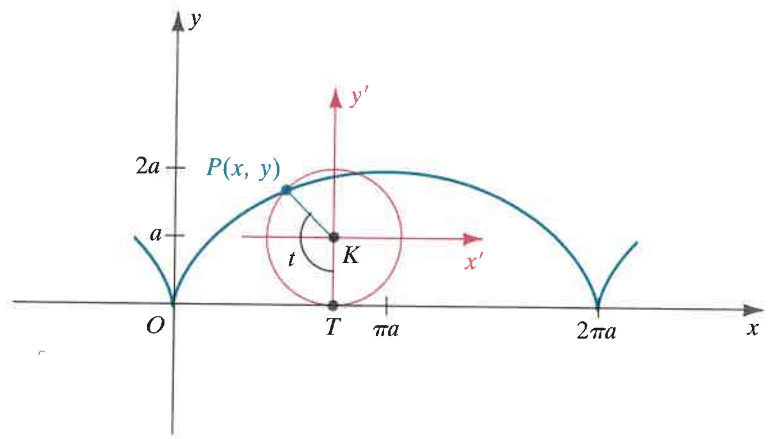
These complicated equations should indicate the advantage of expressing the curve in parametric form.

A curve C is **smooth** if it has a parametrization $x = f(t)$, $y = g(t)$ on an interval I such that the derivatives f' and g' are continuous and not simultaneously zero, except possibly at endpoints of I. A curve C is **piecewise smooth** if the interval I can be partitioned into closed subintervals with C smooth on each subinterval. The graph of a smooth curve has no corners or cusps. The curves given in Examples 1–5 are smooth. The curve in the next example is piecewise smooth.

EXAMPLE ■ 6 The curve traced by a fixed point P on the circumference of a circle as the circle rolls along a line in a plane is called a **cycloid**. Find parametric equations for a cycloid and determine the intervals on which it is smooth.

SOLUTION Suppose the circle has radius a and that it rolls along (and above) the x-axis in the positive direction. If one position of P is the origin, then Figure 9.6 displays part of the curve and a possible position of the circle.

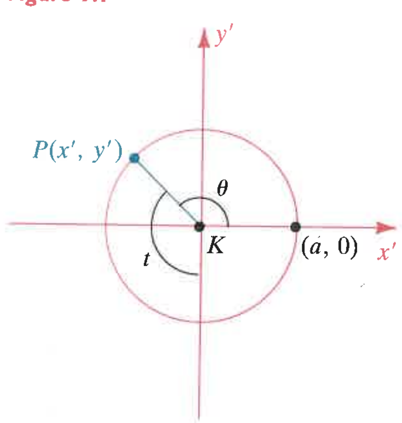
Figure 9.6



Let K denote the center of the circle and T the point of tangency with the x-axis. We introduce, as a parameter t, the radian measure of angle TKP. The distance that the circle has rolled is $d(O, T) = at$. Consequently, the coordinates of K are (at, a). We set up a new rectangular coordinate system centered at $K(at, a)$ with the horizontal and vertical axes designated by x' and y', respectively. In this $x'y'$-coordinate system, if $P(x', y')$ denotes the point P, then we have

$$x = at + x', \qquad y = a + y'.$$

Figure 9.7



If, as in Figure 9.7, θ denotes an angle in standard position on the $x'y'$-plane, then $\theta = (3\pi/2) - t$. Hence,

$$x' = a\cos\theta = a\cos\left(\frac{3\pi}{2} - t\right) = -a\sin t$$

$$y' = a\sin\theta = a\sin\left(\frac{3\pi}{2} - t\right) = -a\cos t,$$

and substitution in $x = at + x'$, $y = a + y'$ gives us parametric equations for the cycloid:

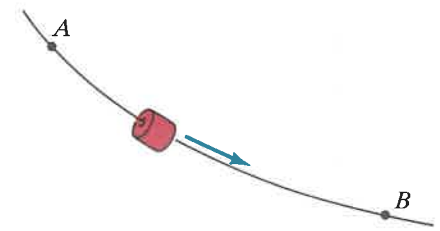
$$x = a(t - \sin t), \quad y = a(1 - \cos t); \quad t \text{ in } \mathbb{R}$$

Differentiating the parametric equations of the cycloid yields

$$\frac{dx}{dt} = a(1 - \cos t), \qquad \frac{dy}{dt} = a\sin t.$$

These derivatives are continuous for every t, but are simultaneously 0 at $t = 2\pi n$ for every integer n. The points corresponding to $t = 2\pi n$ are the x-intercepts of the graph, and the cycloid has a cusp at each such point (see Figure 9.6). The graph is piecewise smooth, since it is smooth on the t-interval $[2\pi n, 2\pi(n + 1)]$ for every integer n.

Figure 9.8



If $a < 0$, then the graph of $x = a(t - \sin t)$, $y = a(1 - \cos t)$ is the inverted cycloid that results if the circle of Example 6 rolls *below* the x-axis. This curve has a number of important physical properties. To illustrate, suppose that a thin wire passes through two fixed points A and B, as shown in Figure 9.8, and that the shape of the wire can be changed by bending it in any manner. Suppose further that a bead is allowed to slide along the wire and the only force acting on the bead is gravity. We now ask which of all the possible paths will allow the bead to slide from A to B in the least amount of time. This question is the brachistochrone problem discussed in Section 7.5. It is natural to believe that the desired path is the straight line segment from A to B; however, this is not the correct answer. In Section 7.5, we saw that the path requiring the least amount of time coincides with the graph of an inverted cycloid with A at the origin. Because the velocity of the bead increases more rapidly along the cycloid than along the line through A and B, the bead reaches B more rapidly, even though the distance is greater.

There is another interesting property of this **curve of least descent.** Suppose that A is the origin and B is the point with x-coordinate $\pi|a|$—that is, the lowest point on the cycloid in the first arc to the right of A. If the bead is released at *any* point between A and B, it can be shown that the time required for the bead to reach B is always the same.

Variations of the cycloid occur in applications. For example, if a motorcycle wheel rolls along a straight road, then the curve traced by a fixed point on one of the spokes is a cycloidlike curve. In this case, the curve does not have corners or cusps, nor does it intersect the road (the x-axis) as does the graph of a cycloid. If the wheel of a train rolls along a railroad track, then the curve traced by a fixed point on the circumference of the wheel (which extends below the track) contains loops at regular intervals. Other cycloids are defined in Exercises 33 and 34.

As another application of parametric equations, we consider how they can be used to study projectile motion. Suppose that an object is projected into the air with an initial horizontal velocity of h ft/sec and an initial vertical velocity of v ft/sec. If an xy-coordinate system is set up with the origin at the object's initial position at time $t = 0$, then we can find parametric equations that describe the position of the object at subsequent times if we assume that the only force acting on the object is the gravitational attraction of the earth. (We ignore air resistance, for example.) Then since there is no force to accelerate or decelerate the horizontal motion, the horizontal velocity remains constant, and we have $x(t) = ht$. For the vertical motion, we have, from our discussion of free fall in Section 3.7, $y(t) = -16t^2 + vt$.

In Chapter 11, we will see that if the object is projected into the air at an angle of θ with an initial speed of s ft/sec, then the initial horizontal and vertical velocities are given by

$$h = s\cos\theta \quad \text{and} \quad v = s\sin\theta.$$

We summarize our discussion with a slightly generalized statement of parametric equations for the motion of a projectile acted on only by a constant gravitational force.

Parametric Equations for Projectile Motion 9.3

The equations of motion of a projectile in a plane launched from an initial position (x_0, y_0) at time $t = 0$ with an initial horizontal velocity h_0 and an initial vertical velocity v_0 are

$$\text{(i)}\quad x(t) = x_0 + h_0 t, \quad y(t) = -\tfrac{1}{2}gt^2 + v_0 t + y_0,$$

where g is the magnitude of the assumed constant acceleration of gravity. If the projectile is launched at an angle of elevation θ with an initial speed s_0, then

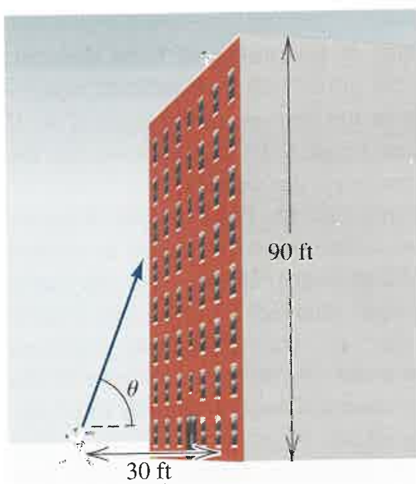
$$\text{(ii)}\quad h_0 = s_0 \cos\theta, \quad v_0 = s_0 \sin\theta.$$

EXAMPLE ■ 7 A pitcher on a baseball team throws a ball to a friend who is standing on the roof of a building 90 ft high. The pitcher stands 30 ft from the base of the building and releases the ball from a height of 8 ft with an initial horizontal velocity of 23.5 ft/sec and an initial vertical velocity of 84.8 ft/sec. (See Figure 9.9.)

(a) Determine whether the ball will reach the top of the building.

(b) Estimate the initial speed of the ball and the angle of release.

Figure 9.9



SOLUTION

(a) The parametric equations for the motion of the ball are

$$x(t) = 23.5t \quad \text{and} \quad y(t) = -16t^2 + 84.8t + 8.$$

When the ball reaches the wall, $x = 30$ and the corresponding time T satisfies $23.5T = 30$, so

$$T = \frac{30}{23.5} \approx 1.2765957 \text{ sec.}$$

At this time T, the y-coordinate is given by

$$y(T) \approx -16(1.2765957)^2 + 84.8(1.2765957) + 8 \approx 90.18 \text{ ft.}$$

Thus, since the ball will be 90.18 ft high when it reaches the building, it will just clear the top of the building.

(b) If s is the initial speed and the ball is released at an angle of θ, then

$$s\sin\theta = 84.8 \quad \text{and} \quad s\cos\theta = 23.5.$$

If we square each equation and then add them, we obtain

$$s^2\sin^2\theta + s^2\cos^2\theta = (84.8)^2 + (23.5)^2$$
$$s^2(\sin^2\theta + \cos^2\theta) = 7191.04 + 552.25$$
$$s^2 = 7743.29$$
$$s = \sqrt{7743.29} \approx 87.99596582.$$

Thus, the original speed was approximately 88 ft/sec (about 60 mi/hr). To determine the angle θ of release, we note that

$$\frac{s\sin\theta}{s\cos\theta} = \frac{84.8}{23.5} \quad \text{or, equivalently,} \quad \tan\theta = \frac{84.8}{23.5}.$$

Thus, the angle of release is

$$\theta = \tan^{-1}\left(\frac{84.8}{23.5}\right) \approx 1.3 \text{ radians} \quad \text{or about } 74.5°.$$

We next examine an important application of parametric curves that was first introduced by the French scientist Pierre Bézier.* **Bézier curves** are special parametric curves commonly used in computer-aided design, microcomputer drawing applications, and the mathematical representation of different fonts for laser printers. Bézier was trying to solve a problem plaguing the designers of stamped parts such as car-body panels: The curves they created at the drawing board did not coincide exactly with what was produced. He wanted to devise a method to represent in a computer "an accurate, complete and indisputable definition of freeform shapes." He discovered that this could be done by piecing together particular types of cubic polynomials.

A cubic Bézier curve is specified by four control points in the plane, $P_0(p_0, q_0)$, $P_1(p_1, q_1)$, $P_2(p_2, q_2)$, and $P_3(p_3, q_3)$. The curve starts at the first point for the parameter $t = 0$, ends at the last point for $t = 1$, and roughly "heads toward" the middle points for parameter values between 0 and 1. Artists and engineering designers can move the control points to adjust the end locations and the shape of the parametric curve until what appears on the computer screen is the shape they want. The cubic Bézier curve for the four control points has the following parametric equations:

$$x(t) = p_0(1-t)^3 + 3p_1(1-t)^2 t + 3p_2(1-t)t^2 + p_3 t^3,$$
$$y(t) = q_0(1-t)^3 + 3q_1(1-t)^2 t + 3q_2(1-t)t^2 + q_3 t^3; \quad 0 \le t \le 1.$$

Note that

$$x(0) = p_0(1-0)^3 + 3p_1(1-0)^2 \cdot 0 + 3p_2(1-0) \cdot 0^2 + p_3 \cdot 0^3 = p_0.$$

Similarly, we have $y(0) = q_0$, so the curve passes through the control point P_0 at $t = 0$. We may also compute that $x(1) = p_3$ and $y(1) = q_3$ so the curve passes through P_3 at $t = 1$. For values of t between 0 and 1, $x(t)$ is a "weighted average" of the x-coordinates of all four control points and $y(t)$ is a weighted average of their y-coordinates. For example, at $t = \frac{1}{2}$, we have

$$x\left(\tfrac{1}{2}\right) = \frac{p_0 + 3p_1 + 3p_2 + p_3}{8} \quad \text{and} \quad y\left(\tfrac{1}{2}\right) = \frac{q_0 + 3q_1 + 3q_2 + q_3}{8}.$$

*Pierre E. Bézier is a contemporary French scientist whose mathematical, engineering, and design work for the Rénault automobile company beginning in the mid-1960s led to the development of the field of computer-aided geometric design.

In general, the cubic Bézier curve will *not* pass through the control points P_1 and P_2. The relative locations of these points, however, will determine the shape of the Bézier curve.

EXAMPLE ■ 8 Use a graphing utility to obtain the graph of the cubic Bézier curve with the control points $P_0(32, 6)$, $P_1(85, 30)$, $P_2(6, 35)$, and $P_3(45, 8)$. Set the viewing window so that the control points appear within it and plot the control points.

SOLUTION Using the general form for the parametric equations for the cubic Bézier curve with

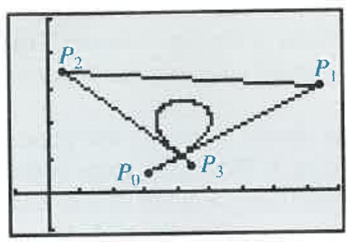
$$\begin{array}{llll} p_0 = 32 & p_1 = 85 & p_2 = 6 & p_3 = 45 \\ q_0 = 6 & q_1 = 30 & q_2 = 35 & q_3 = 8, \end{array}$$

we obtain the parametric equations for this Bézier curve:

$$x(t) = 32(1-t)^3 + 255(1-t)^2 t + 18(1-t)t^2 + 45t^3,$$
$$y(t) = 6(1-t)^3 + 90(1-t)^2 t + 105(1-t)t^2 + 8t^3; \quad 0 \le t \le 1$$

The x-coordinates of the control points range from 6 to 85, and the y-coordinates of the control points range from 6 to 35. To ensure that all four control points and the coordinate axes will appear on the screen, we set the viewing window to $-10 \le x \le 90$, $-10 \le y \le 50$, and plot the equations and the control points to obtain the curve shown in Figure 9.10.

Figure 9.10
$-10 \le x \le 90$, $-10 \le y \le 50$



Several Bézier curves can be pieced together continuously by making the last control point on one curve the first control point on the next curve. Piecewise parametric equations can be constructed in a similar manner. For simplicity, we treat each piece as a separate parametric curve so that we may use the same form (repeatedly) in a graphing utility. If we use the four control points P_0, P_1, P_2, and P_3 to determine the first piece of the curve with parametric equations for $0 \le t \le 1$, then control points P_3, P_4, P_5, and P_6 are used for the next piece, again with $0 \le t \le 1$. Because the fourth control point for the first piece is the first control point for the next piece, the two pieces fit together continuously. The equations for the second piece of the curve are

$$x(t) = p_3(1-t)^3 + 3p_4(1-t)^2 t + 3p_5(1-t)t^2 + p_6 t^3,$$
$$y(t) = q_3(1-t)^3 + 3q_4(1-t)^2 t + 3q_5(1-t)t^2 + q_6 t^3; \quad 0 \le t \le 1.$$

EXAMPLE ■ 9 Use a graphing utility to obtain the graph of the continuous piecewise Bézier curve with the control points $P_0(10, 15)$, $P_1(16, 14)$, $P_2(25, 38)$, $P_3(30, 40)$ (repeated), $P_4(18, 5)$, $P_5(50, 20)$, and $P_6(16, 30)$. Set the viewing window so that the control points appear within it and plot the control points.

SOLUTION Since seven control points have been specified, the curve will have two pieces. We use the first four control points (P_0, P_1, P_2, P_3) for the first piece and the last four (P_3, P_4, P_5, P_6) for the second piece. Using the general form for the parametric equations of a cubic Bézier curve, we obtain the parametric equations:

$$x_1(t) = 10(1-t)^3 + 48(1-t)^2 t + 75(1-t)t^2 + 30t^3,$$
$$y_1(t) = 15(1-t)^3 + 42(1-t)^2 t + 114(1-t)t^2 + 40t^3,$$

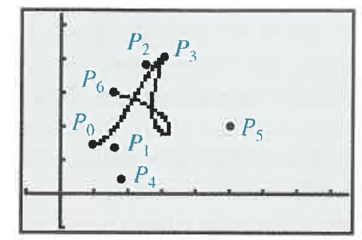
and

$$x_2(t) = 30(1-t)^3 + 54(1-t)^2 t + 150(1-t)t^2 + 16t^3,$$
$$y_2(t) = 40(1-t)^3 + 15(1-t)^2 t + 60(1-t)t^2 + 30t^3,$$

both for $0 \le t \le 1$.

Figure 9.11 shows a plot of the equations and the control points.

Figure 9.11
$-10 \le x \le 85$, $-10 \le y \le 53$



Cubic Bézier curves are often used in the design of characters (letters, numerals, and punctuation marks) that will be printed in different font sizes on a laser printer. Stored in the memory of the laser printer are the control points for each character. When it needs to print a letter "A," for example, the printer recalls the control points for "A" and then directs a graphing utility to plot the cubic Bézier curve with those points. Since there are a relatively small number of control points for each character, the memory requirements for storing the information about all the characters on the keyboard is not large. Another advantage is that only one size for the font need be stored in the memory of the laser printer. A simple scaling or rotation of the control points will yield characters of different size or rotational orientation. The general form of the cubic equations that give the parametrization of the Bézier curves and the specific control points for a particular character form a *mathematical representation* of that character.

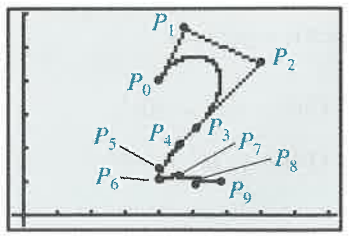
The mathematical representation of other designs besides characters is also given by specifying particular collections of control points. To obtain such a mathematical representation for a given figure, we begin by sketching the figure on a sheet of rectangularly ruled graph paper, mark some points on the figure, and then use these points as "end" control points for a piecewise continuous sequence of Bézier curves. Other control points needed between these "ends" will not lie on the final curve, but will "pull" the curve in certain directions.

EXAMPLE ■ 10 Use several Bézier curves to obtain a parametrization of a curve whose graph is the numeral "2."

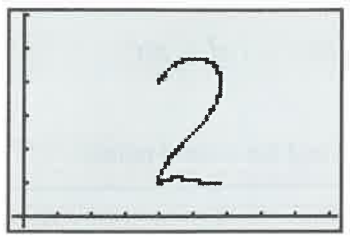
SOLUTION We begin with a crude drawing of the numeral "2" on a large sheet of graph paper. We draw it in a cursive style with no

Figure 9.12
$-3 \le x \le 92,\ -3 \le y \le 60$

(a)



(b)



line segments, and then subdivide the character into pieces that appear to be possible to obtain by a single Bézier curve. For the segments we have chosen, we try control points $P_0(40, 40)$, $P_1(48, 55)$, $P_2(70, 45)$, $P_3(50, 25)$ (repeated), $P_4(43, 18)$, $P_5(40, 13)$, $P_6(40, 10)$ (repeated), $P_7(46, 14)$, $P_8(53, 9)$, and $P_9(58, 10)$. Plotting the associated equations and the control points gives us the graph shown in Figure 9.12(a).

The use of a graphing utility makes it quite easy to experiment with modifications in the locations of the control points until we are satisfied that the Bézier curve produced represents the figure we have designed. If the result of the first attempt is not satisfactory, we move some control points and try again. Software for computer-aided geometric design allows the designer to use a mouse or trackball to drag a control point from one part of the computer screen to another. The software then recalculates the equations for the Bézier curves on the basis of the new coordinates of the control points and immediately plots the graph of the new curve.

EXERCISES 9.1

Exer. 1–24: (a) Find an equation in x and y whose graph contains the points on the curve C. (b) Sketch the graph of C and indicate the orientation.

1 $x = t - 2$, $y = 2t + 3$; $0 \le t \le 5$

2 $x = 1 - 2t$, $y = 1 + t$; $-1 \le t \le 4$

3 $x = t^2 + 1$, $y = t^2 - 1$; $-2 \le t \le 2$

4 $x = t^3 + 1$, $y = t^3 - 1$; $-2 \le t \le 2$

5 $x = 4t^2 - 5$, $y = 2t + 3$; t in $\mathbb{R}$

6 $x = t^3$, $y = t^2$; t in $\mathbb{R}$

7 $x = e^t$, $y = e^{-2t}$; t in $\mathbb{R}$

8 $x = \sqrt{t}$, $y = 3t + 4$; $t \ge 0$

9 $x = 2\sin t$, $y = 3\cos t$; $0 \le t \le 2\pi$

10 $x = \cos t - 2$, $y = \sin t + 3$; $0 \le t \le 2\pi$

11 $x = \sec t$, $y = \tan t$; $-\pi/2 < t < \pi/2$

12 $x = \cos 2t$, $y = \sin t$; $-\pi \le t \le \pi$

13 $x = t^2$, $y = 2\ln t$; $t > 0$

14 $x = \cos^3 t$, $y = \sin^3 t$; $0 \le t \le 2\pi$

15 $x = \sin t$, $y = \csc t$; $0 < t \le \pi/2$

16 $x = e^t$, $y = e^{-t}$; t in $\mathbb{R}$

17 $x = \cosh t$, $y = \sinh t$; t in $\mathbb{R}$

18 $x = 3\cosh t$, $y = 2\sinh t$; t in $\mathbb{R}$

19 $x = t$, $y = \sqrt{t^2 - 1}$; $|t| \ge 1$

20 $x = -2\sqrt{1 - t^2}$, $y = t$; $|t| \le 1$

21 $x = t$, $y = \sqrt{t^2 - 2t + 1}$; $0 \le t \le 4$

22 $x = 2t$, $y = 8t^3$; $-1 \le t \le 1$

23 $x = (t + 1)^3$, $y = (t + 2)^2$; $0 \le t \le 2$

24 $x = \tan t$, $y = 1$; $-\pi/2 < t < \pi/2$

Exer. 25–26: Curves C_1, C_2, C_3, and C_4 are given parametrically, for t in $\mathbb{R}$. Sketch their graphs and indicate orientations.

25 C_1: $x = t^2$, $y = t$
C_2: $x = t^4$, $y = t^2$
C_3: $x = \sin^2 t$, $y = \sin t$
C_4: $x = e^{2t}$, $y = -e^t$

26 C_1: $x = t$, $y = 1 - t$
C_2: $x = 1 - t^2$, $y = t^2$
C_3: $x = \cos^2 t$, $y = \sin^2 t$
C_4: $x = \ln t - t$, $y = 1 + t - \ln t$; $t > 0$

Exer. 27–28: The parametric equations specify the position of a moving point $P(x, y)$ at time t. Sketch the graph and indicate the motion of P as t increases.

27 (a) $x = \cos t$, $y = \sin t$; $0 \le t \le \pi$
(b) $x = \sin t$, $y = \cos t$; $0 \le t \le \pi$
(c) $x = t$, $y = \sqrt{1 - t^2}$; $-1 \le t \le 1$

28 (a) $x = t^2$, $y = 1 - t^2$; $0 \le t \le 1$
(b) $x = 1 - \ln t$, $y = \ln t$; $1 \le t \le e$
(c) $x = \cos^2 t$, $y = \sin^2 t$; $0 \le t \le 2\pi$

29 Show that

$$x = a\cos t + h, \quad y = b\sin t + k; \quad 0 \le t \le 2\pi$$

are parametric equations of an ellipse with center (h, k) and axes of lengths $2a$ and $2b$.

30 Show that

$$x = a\sec t + h, \quad y = b\tan t + k;$$
$$-\frac{\pi}{2} < t < \frac{3\pi}{2} \quad \text{and} \quad t \ne \frac{\pi}{2}$$

are parametric equations of a hyperbola with center (h, k), transverse axis of length $2a$, and conjugate axis of length $2b$. Determine the values of t for each branch.

31 If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are distinct points, show that

$$x = (x_2 - x_1)t + x_1, \quad y = (y_2 - y_1)t + y_1; \quad t \text{ in } \mathbb{R}$$

are parametric equations for the line l through P_1 and P_2.

32 Describe the difference between the graph of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ and the graph of

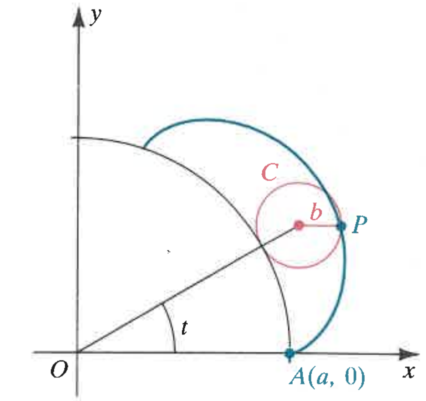
$$x = a\cosh t, \quad y = b\sinh t; \quad t \text{ in } \mathbb{R}.$$

(*Hint:* Use Theorem (6.42).)

33 A circle C of radius b rolls on the outside of the circle $x^2 + y^2 = a^2$, and $b < a$. Let P be a fixed point on C, and let the initial position of P be $A(a, 0)$, as shown in the figure. If the parameter t is the angle from the positive x-axis to the line segment from O to the center of C, show that parametric equations for the curve traced by P (an *epicycloid*) are

$$x = (a + b)\cos t - b\cos\left(\frac{a + b}{b}t\right),$$
$$y = (a + b)\sin t - b\sin\left(\frac{a + b}{b}t\right); \quad 0 \le t \le 2\pi.$$

Exercise 33



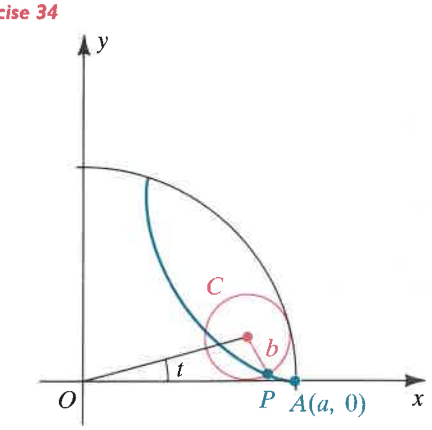
34 If the circle C of Exercise 33 rolls on the inside of the second circle (see figure), then the curve traced by P is a *hypocycloid*.

(a) Show that parametric equations for this curve are

$$x = (a - b)\cos t + b\cos\left(\frac{a - b}{b}t\right),$$
$$y = (a - b)\sin t - b\sin\left(\frac{a - b}{b}t\right); \quad 0 \le t \le 2\pi.$$

(b) If $b = \frac{1}{4}a$, show that $x = a\cos^3 t$, $y = a\sin^3 t$ and sketch the graph.

Exercise 34



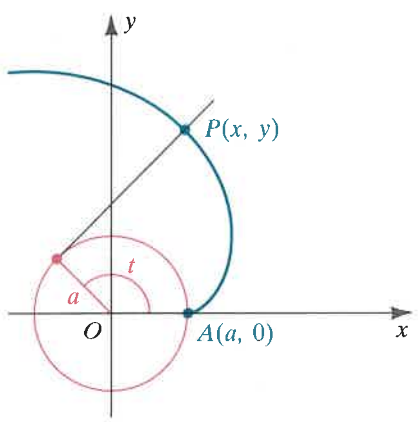
35 If $b = \frac{1}{3}a$ in Exercise 33, find the parametric equations for the epicycloid and sketch the graph.

36 The radius of circle B is one third that of circle A. How many revolutions will circle B make as it rolls around circle A until it reaches its starting point? (*Hint:* Use Exercise 35.)

37 If a string is unwound from around a circle of radius a and is kept tight in the plane of the circle, then a fixed point P on the string will trace a curve called the *involute of the circle.* Let the circle be chosen as in the figure. If the parameter t is the measure of the indicated angle and the initial position of P is $A(a, 0)$, show that parametric equations for the involute are

$$x = a(\cos t + t \sin t), \quad y = a(\sin t - t \cos t).$$

Exercise 37



38 Generalize the cycloid of Example 6 to the case where P is any point on a fixed line through the center C of the circle. If $b = d(C, P)$, show that

$$x = at - b\sin t, \quad y = a - b\cos t.$$

Sketch a typical graph if $b < a$ (a *curtate cycloid*) and if $b > a$ (a *prolate cycloid*). The term *trochoid* is sometimes used for either of these curves.

39 Refer to Example 5.

(a) Describe the Lissajous figure given by $f(t) = a\sin\omega t$ and $g(t) = b\cos\omega t$ for $t \geq 0$ and $a \neq b$.

(b) Suppose $f(t) = a\sin\omega_1 t$ and $g(t) = b\sin\omega_2 t$, where ω_1 and ω_2 are positive rational numbers, and write ω_2/ω_1 as m/n for positive integers m and n. Show that if $p = 2\pi n/\omega_1$, then $f(t+p) = f(t)$ and $g(t+p) = g(t)$. Conclude that the curve retraces itself every p units of time.

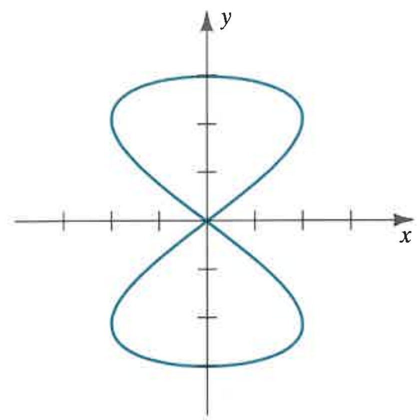
40 Shown in the figure is the Lissajous figure given by

$$x = 2\sin 3t, \quad y = 3\sin 1.5t; \quad t \geq 0.$$

(a) Find the period of the figure—that is, the length of the smallest t-interval that traces the curve.

(b) Find the maximum distance from the origin to a point on the graph.

Exercise 40



c **Exer. 41–44: Graph the curve.**

41 $x = 3\sin^5 t, \quad y = 3\cos^5 t; \quad 0 \leq t \leq 2\pi$

42 $x = 8\cos t - 2\cos 4t, \quad y = 8\sin t - 2\sin 4t; \quad 0 \leq t \leq 2\pi$

43 $x = 3t - 2\sin t, \quad y = 3 - 2\cos t; \quad -8 \leq t \leq 8$

44 $x = 2t - 3\sin t, \quad y = 2 - 3\cos t; \quad -8 \leq t \leq 8$

Exer. 45–48: Graph the given curves on the same coordinate axes and describe the shape of the resulting figure.

c **45** C_1: $x = 2\sin 3t, \ y = 3\cos 2t; \quad \frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$

C_2: $x = \frac{1}{4}\cos t + \frac{3}{4}, \ y = \frac{1}{4}\sin t + \frac{3}{2}; \quad 0 \leq t \leq 2\pi$

C_3: $x = \frac{1}{4}\cos t - \frac{3}{4}, \ y = \frac{1}{4}\sin t + \frac{3}{2}; \quad 0 \leq t \leq 2\pi$

C_4: $x = \frac{3}{4}\cos t, \ y = \frac{1}{4}\sin t; \quad 0 \leq t \leq 2\pi$

C_5: $x = \frac{1}{4}\cos t, \ y = \frac{1}{8}\sin t + \frac{3}{4}; \quad \pi \leq t \leq 2\pi$

46 C_1: $x = \frac{3}{2}\cos t + 1, \ y = \sin t - 1; \quad \frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$

C_2: $x = \frac{3}{2}\cos t + 1, \ y = \sin t + 1; \quad \frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$

C_3: $x = 1, \ y = 2\tan t; \quad \frac{-\pi}{4} \leq t \leq \frac{\pi}{4}$

47 C_1: $x = \tan t, \ y = 3\tan t; \quad 0 \leq t \leq \frac{\pi}{4}$

C_2: $x = 1 + \tan t, \ y = 3 - 3\tan t; \quad 0 \leq t \leq \frac{\pi}{4}$

C_3: $x = \frac{1}{2} + \tan t, \ y = \frac{3}{2}; \quad 0 \leq t \leq \frac{\pi}{4}$

48 C_1: $x = 1 + \cos t, \ y = 1 + \sin t; \quad \pi/3 \leq t \leq 2\pi$

C_2: $x = 1 + \tan t, \ y = 1; \quad 0 \leq t \leq \frac{\pi}{4}$

49 If a rock is thrown from a point 3 ft above the ground with a horizontal velocity of 90 ft/sec and a vertical velocity of 47 ft/sec, how far away will it land if nothing is obstructing its path? If there is a 7-ft high fence 9 ft in front of you, will the rock sail over the fence?

50 A basketball player shoots a ball with a speed of 25 ft/sec from a point 15 ft horizontally away from the center of the basket. The basket is 10 ft above the floor and the player releases the ball from a height of 8 ft. At what angle should the player shoot the ball?

51 If a projectile is launched at an angle θ to the horizontal, show that its horizontal range is $4M/(\tan\theta)$, where M is the maximum height reached by the projectile.

52 Anne kicks a soccer ball toward her brother Sasha with an initial velocity of 48 ft/sec at an angle of elevation of $\pi/6$. At the moment of the kick, he is 90 ft away and starts running to meet the ball. Sasha's top speed is 20 ft/sec.

(a) Can Sasha reach the ball before it hits the ground?

(b) If Sasha is 5 ft 6 in. tall, can he reach the ball in time to bounce it off the top of his head?

(c) For what range of angles can Anne kick the ball so that Sasha has time to hit the ball with his head?

c **Exer. 53–58: Plot the graph of the continuous piecewise Bézier curve for the given control points. Set the viewing window so that all control points appear within the window. If possible, use equally scaled axes and also plot the control points. For the first piece of each curve, use control points P_0, P_1, P_2, and P_3. If more points are given, use P_3, P_4, P_5, and P_6 for the second piece and P_6, P_7, P_8, and P_9 for the third piece.**

53 $P_0(10, 2)$, $P_1(2, 60)$, $P_2(100, 56)$, and $P_3(110, 10)$

54 $P_0(1, 32)$, $P_1(25, 85)$, $P_2(30, 1)$, and $P_3(3, 40)$

55 $P_0(5, 10)$, $P_1(4, 16)$, $P_2(28, 25)$, $P_3(30, 30)$, $P_4(1, 18)$, $P_5(18, 40)$, and $P_6(20, 16)$

56 $P_0(60, 40)$, $P_1(50, 30)$, $P_2(43, 10)$, $P_3(55, 10)$, $P_4(65, 10)$, $P_5(68, 25)$, and $P_6(50, 22)$

57 $P_0(30, 30)$, $P_1(58, 10)$, $P_2(12, 12)$, $P_3(45, 10)$, $P_4(40, 5)$, $P_5(66, 31)$, and $P_6(25, 30)$

58 $P_0(48, 20)$, $P_1(20, 15)$, $P_2(20, 50)$, $P_3(48, 45)$, $P_4(28, 47)$, $P_5(28, 18)$, $P_6(48, 20)$, $P_7(48, 36)$, $P_8(52, 32)$, and $P_9(40, 32)$

c **Exer. 59–60: Experiment with the locations for control points to obtain piecewise Bézier curves approximating the given shape or object. Give the final control points chosen, and sketch the resulting parametric curve.**

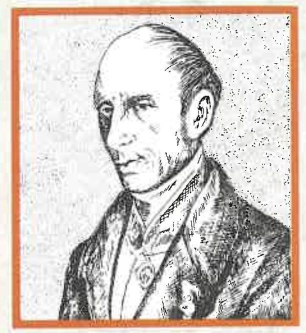
59 Find a piecewise Bézier curve with two components that approximates the letter "S" in a simple font. Improve the sketch by using a piecewise Bézier curve with three components.

60 Find a piecewise Bézier curve that approximates the Gateway Arch to the West. (See Exercise 35 of Section 6.8.)

Mathematicians and Their Times

AUGUSTIN-LOUIS CAUCHY

THE PEOPLE OF PARIS rose on July 14, 1789, to storm the Bastille and begin the struggle and violence of the French Revolution. The Revolution promised a new era of democracy, liberty, and equality to replace the supreme power of the king. It was tarnished by the Reign of Terror begun in 1793 by the Committee of Public Safety, which swept hundreds to the guillotine in its zeal to protect France's internal security.



Among those in gravest danger were government officials suspected of loyalty to the king. Many fled to small villages to find safety. One such man was the father of Augustin-Louis Cauchy. A child of the Revolution, Cauchy was born on August 21, 1789. His earliest years were spent in fear and exile. Tumultuous political events continued to dominate France for most of his life: revolution in 1789, the rise and fall of Napoleon, the restoration of the Bourbon kings in 1814, more revolutions in 1830 and 1848, the overthrow of the second republic and establishment of the second empire by Napoleon III.

Cauchy's conservative political and religious attitudes put him at odds with many of his fellow scientists. His detractors called him self-righteous and arrogant, a narrow-minded bigot, and a smug hypocrite. His defenders regarded him as a pious believer in traditional religion, highly principled, and sincere but naive in his politics. All agree now, however, that Cauchy was one of the most influential mathematicians of the nineteenth century.

Cauchy's specific contributions to pure and applied mathematics were both deep and broad. He essentially created, for example, both the theory of functions of a complex variable and finite group theory. Even more important was his lead in raising the standards of rigor. Both Gauss and Cauchy were the "apostles of rigor" who transformed mathematics. "It is difficult to find an adequate simile for the magnitude of this advance," wrote E. T. Bell. "Suppose that for centuries an entire people had been worshipping false gods and that suddenly their error is revealed to them." Although Gauss preceded him, Cauchy had greater impact because of his gift for effective teaching, and his many textbooks and research papers that showed how calculus could be developed in a rigorous manner.

Cauchy died unexpectedly on May 23, 1857. Although estranged from many colleagues who charged him with judging scientists more on the basis of their political and religious views than on their scientific achievements, Cauchy was actively planning new charitable works. His last words were addressed to the Archbishop of Paris: "Men pass away, but their deeds abide."

9.2 ARC LENGTH AND SURFACE AREA

If a curve is described by an equation of the form $y = f(x)$, where f is a differentiable function, we know from earlier chapters how to find the slope of the tangent line at a point on the curve, the length of a segment of the curve, and the area of the surface of revolution obtained by revolving the curve about an axis. In this section, we discuss how to find these quantities when the curve is described by parametric equations.

The curve C given parametrically by

$$x = 2t, \quad y = t^2 - 1; \quad -1 \le t \le 2$$

can also be represented by an equation of the form $y = k(x)$, where k is a function defined on a suitable interval. In Example 1 of Section 9.1, we eliminated the parameter t, obtaining

$$y = k(x) = \tfrac{1}{4}x^2 - 1 \quad \text{for} \quad -2 \le x \le 4.$$

The slope of the tangent line at any point $P(x, y)$ on C is

$$k'(x) = \tfrac{1}{2}x, \quad \text{or} \quad k'(x) = \tfrac{1}{2}(2t) = t.$$

Since it is often difficult to eliminate a parameter, we next derive a formula that can be used to find the slope directly from the parametric equations.

Theorem 9.4

If a smooth curve C is given parametrically by $x = f(t)$, $y = g(t)$, then the slope dy/dx of the tangent line to C at $P(x, y)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \text{provided} \quad \frac{dx}{dt} \ne 0.$$

PROOF If $dx/dt \neq 0$ at $x = c$, then, since f is continuous at c, it follows from the intermediate value theorem (1.26) that $dx/dt > 0$ or $dx/dt < 0$ throughout an interval $[a, b]$, with $a < c < b$. Applying Theorem (6.6) or the analogous result for decreasing functions, we know that f has an inverse function f^{-1}, and we may consider $t = f^{-1}(x)$ for x in $[f(a), f(b)]$. Applying the chain rule to $y = g(t)$ and $t = f^{-1}(x)$, we obtain

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy/dt}{dx/dt},$$

where the last equality follows from Corollary (6.8). ■

EXAMPLE ■ 1 Let C be the curve with parametrization

$$x = 2t, \quad y = t^2 - 1; \quad -1 \le t \le 2.$$

Find the slopes of the tangent line and normal line to C at $P(x, y)$.

SOLUTION The curve C was considered in Example 1 of Section 9.1 (see Figure 9.2). Using Theorem (9.4) with $x = 2t$ and $y = t^2 - 1$, we find that the slope of the tangent line at $P(x, y)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t.$$

This result agrees with that of the discussion at the beginning of this section, where we used the form $y = k(x)$ to show that $m = \frac{1}{2}x = t$.

The slope of the normal line is the negative reciprocal $-1/t$, provided $t \neq 0$.

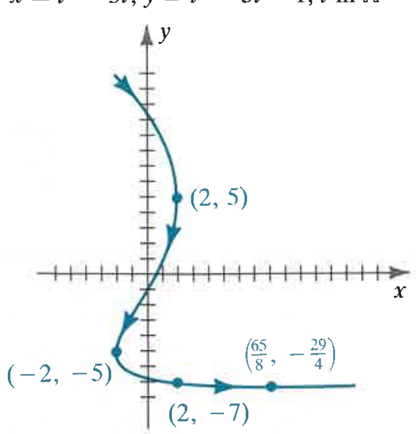
EXAMPLE ■ 2 Let C be the curve with parametrization

$$x = t^3 - 3t, \quad y = t^2 - 5t - 1; \quad t \text{ in } \mathbb{R}.$$

(a) Find an equation of the tangent line to C at the point corresponding to $t = 2$.

(b) For what values of t is the tangent line horizontal or vertical?

Figure 9.13
$x = t^3 - 3t$, $y = t^2 - 5t - 1$; t in $\mathbb{R}$



SOLUTION

(a) A portion of the graph of C is sketched in Figure 9.13, where we have also plotted several points and indicated the orientation. Using the parametric equations for C, we find that the point corresponding to $t = 2$ is $(2, -7)$. By Theorem (9.4),

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 5}{3t^2 - 3}.$$

The slope m of the tangent line at $(2, -7)$ is

$$m = \left.\frac{dy}{dx}\right]_{t=2} = \frac{2(2) - 5}{3(2^2) - 3} = -\frac{1}{9}.$$

Applying the point–slope form, we obtain an equation of the tangent line:

$$y + 7 = -\tfrac{1}{9}(x - 2), \quad \text{or} \quad x + 9y = -61$$

(b) The tangent line is horizontal if $dy/dx = 0$—that is, if $2t - 5 = 0$, or $t = \frac{5}{2}$. The corresponding point on C is $(\frac{65}{8}, -\frac{29}{4})$, as shown in Figure 9.13.

The tangent line is vertical if $3t^2 - 3 = 0$. Thus, there are vertical tangent lines at the points corresponding to $t = 1$ and $t = -1$—that is, at $(-2, 5)$ and $(2, 5)$.

If a curve C is parametrized by $x = f(t)$, $y = g(t)$ and if y' is a differentiable function of t, we can find d^2y/dx^2 by applying Theorem (9.4) to y' as follows.

***Second Derivative in Parametric Form* 9.5**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}$$

It is important to observe that

$$\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}.$$

EXAMPLE ■ 3 Let C be the curve with parametrization

$$x = e^{-t}, \quad y = e^{2t}; \quad t \text{ in } \mathbb{R}.$$

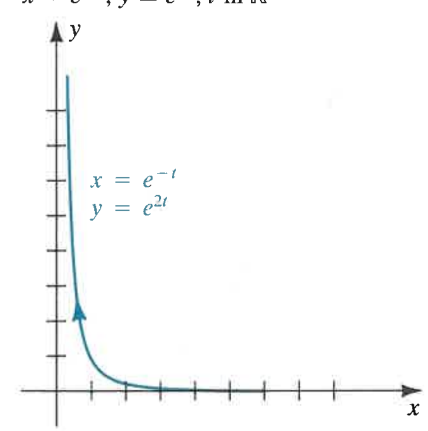
(a) Sketch the graph of C and indicate the orientation.

(b) Use (9.4) and (9.5) to find dy/dx and d^2y/dx^2.

(c) Find a function k that has the same graph as C, and use $k'(x)$ and $k''(x)$ to check the answers to part (b).

(d) Discuss the concavity of C.

Figure 9.14
$x = e^{-t}$, $y = e^{2t}$; t in $\mathbb{R}$



SOLUTION

(a) To help us sketch the graph, let us first eliminate the parameter. Using $x = e^{-t} = 1/e^t$, we see that $e^t = 1/x$. Substituting in $y = e^{2t} = (e^t)^2$ gives us

$$y = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2}.$$

Remembering that $x = e^{-t} > 0$ leads to the graph in Figure 9.14. Note that the point $(1, 1)$ corresponds to $t = 0$. If t increases in $(-\infty, 0]$, the point $P(x, y)$ approaches $(1, 1)$ from the right, as indicated by the arrowhead. If t increases in $[0, \infty)$, $P(x, y)$ moves up the curve, approaching the y-axis.

(b) By (9.4) and (9.5),

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t}}{-e^{-t}} = -2e^{3t}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{-6e^{3t}}{-e^{-t}} = 6e^{4t}.$$

(c) From part (a), a function k that has the same graph as C is given by

$$k(x) = \frac{1}{x^2} = x^{-2} \quad \text{for} \quad x > 0.$$

Differentiating twice yields

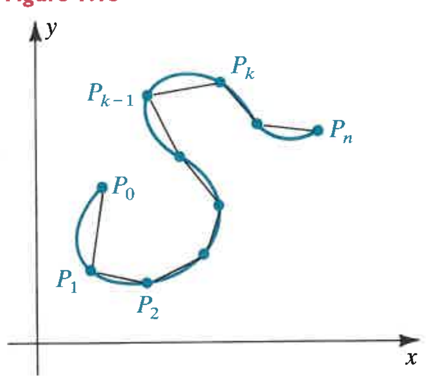
$$k'(x) = -2x^{-3} = -2(e^{-t})^{-3} = -2e^{3t}$$

$$k''(x) = 6x^{-4} = 6(e^{-t})^{-4} = 6e^{4t},$$

which is in agreement with part (b).

(d) Since $d^2y/dx^2 = 6e^{4t} > 0$ for every t, the curve C is concave upward at every point.

Figure 9.15



If a curve C is the graph of $y = f(x)$ and the function f is smooth on $[a, b]$, then the length of C is given by $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$ (see Definition (5.14)). We shall next obtain a formula for finding lengths of parametrized curves.

Suppose a smooth curve C is given parametrically by

$$x = f(t), \quad y = g(t); \quad a \le t \le b.$$

Furthermore, suppose C does not intersect itself—that is, different values of t between a and b determine different points on C. Consider a partition P of $[a, b]$ given by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. Let $\Delta t_k = t_k - t_{k-1}$ and let $P_k = (f(t_k), g(t_k))$ be the point on C that corresponds to t_k. If $d(P_{k-1}, P_k)$ is the length of the line segment $P_{k-1}P_k$, then the length L_P of the broken line in Figure 9.15 is

$$L_P = \sum_{k=1}^{n} d(P_{k-1}, P_k).$$

As in Section 5.5, we define

$$L = \lim_{\|P\| \to 0} L_P$$

and call L the **length of C** from P_0 to P_n if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|L_P - L| < \epsilon$ for every partition P with $\|P\| < \delta$.

By the distance formula,

$$d(P_{k-1}, P_k) = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}.$$

By the mean value theorem (3.12), there exist numbers w_k and z_k in the open interval (t_{k-1}, t_k) such that

$$f(t_k) - f(t_{k-1}) = f'(w_k)\Delta t_k$$

$$g(t_k) - g(t_{k-1}) = g'(z_k)\Delta t_k.$$

Substituting these in the formula for $d(P_{k-1}, P_k)$ and removing the common factor $(\Delta t_k)^2$ from the radicand gives us

$$d(P_{k-1}, P_k) = \sqrt{[f'(w_k)]^2 + [g'(z_k)]^2}\,\Delta t_k.$$

Consequently,

$$L = \lim_{\|P\| \to 0} L_P = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{[f'(w_k)]^2 + [g'(z_k)]^2}\,\Delta t_k,$$

provided the limit exists. If $w_k = z_k$ for every k, then the sums are Riemann sums for the function defined by $\sqrt{[f'(t)]^2 + [g'(t)]^2}$. The limit of these sums is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

The limit exists even if $w_k \ne z_k$; however, the proof requires advanced methods and is omitted. The next theorem summarizes this discussion.

Theorem 9.6

If a smooth curve C is given parametrically by $x = f(t)$, $y = g(t)$; $a \le t \le b$, and if C does not intersect itself, except possibly for $t = a$ and $t = b$, then the length L of C is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The integral formula in Theorem (9.6) is not necessarily true if C intersects itself. For example, if C has the parametrization $x = \cos t$, $y = \sin t$; $0 \le t \le 4\pi$, then the graph is a unit circle with center at the origin. If t varies from 0 to 4π, the circle is traced twice and hence intersects itself infinitely many times. If we use Theorem (9.6) with $a = 0$ and $b = 4\pi$, we obtain the incorrect value 4π for the length of C. The correct value 2π can be obtained by using the t-interval $[0, 2\pi]$. Note that in this case the curve intersects itself only at the points corresponding to $t = 0$ and $t = 2\pi$, which is allowable by the theorem.

If a curve C is given by $y = k(x)$, with k' continuous on $[a, b]$, then parametric equations for C are

$$x = t, \quad y = k(t); \quad a \le t \le b.$$

In this case,

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = k'(t) = k'(x), \qquad dt = dx,$$

and from Theorem (9.6),

$$L = \int_a^b \sqrt{1 + [k'(x)]^2}\, dx.$$

This result agrees with the arc length formula given in Definition (5.14).

EXAMPLE 4 Find the length of one arch of the cycloid that has the parametrization

$$x = t - \sin t, \quad y = 1 - \cos t; \quad t \text{ in } \mathbb{R}.$$

SOLUTION The graph has the shape illustrated in Figure 9.6. The radius a of the circle is 1. One arch is obtained if t varies from 0 to 2π. Applying Theorem (9.6) yields

$$L = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt.$$

Since $\cos^2 t + \sin^2 t = 1$, the integrand reduces to

$$\sqrt{2 - 2\cos t} = \sqrt{2}\sqrt{1-\cos t}.$$

Thus, $$L = \int_0^{2\pi} \sqrt{2}\sqrt{1-\cos t}\, dt.$$

By a half-angle formula, $\sin^2 \frac{1}{2}t = \frac{1}{2}(1-\cos t)$, or, equivalently,

$$1 - \cos t = 2\sin^2 \tfrac{1}{2}t.$$

Hence, $$\sqrt{1-\cos t} = \sqrt{2\sin^2 \tfrac{1}{2}t} = \sqrt{2}\left|\sin \tfrac{1}{2}t\right|.$$

The absolute value sign may be deleted, since if $0 \le t \le 2\pi$, then we have $0 \le \frac{1}{2}t \le \pi$ and hence $\sin \frac{1}{2}t \ge 0$. Consequently,

$$L = \int_0^{2\pi} \sqrt{2}\sqrt{2}\sin \tfrac{1}{2}t\, dt = 2\int_0^{2\pi} \sin \tfrac{1}{2}t\, dt$$
$$= -4\left[\cos \tfrac{1}{2}t\right]_0^{2\pi} = -4(-1-1) = 8.$$

To remember Theorem (9.6), recall that if ds is the differential of arc length, then, by Theorem (5.17),

$$(ds)^2 = (dx)^2 + (dy)^2.$$

Assuming that ds and dt are positive, we have the following.

Parametric Differential of Arc Length 9.7

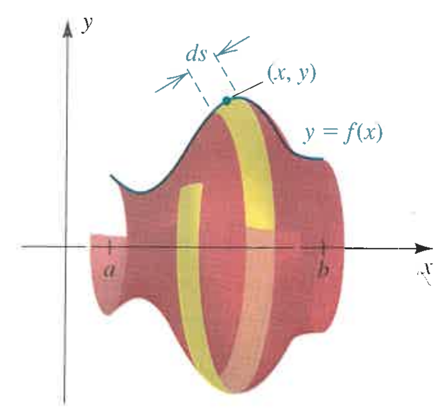
$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Using (9.7), we can rewrite the formula for arc length in Theorem (9.6) as

$$L = \int_{t=a}^{t=b} ds.$$

The limits of integration specify that the independent variable is t, not s

Figure 9.16



If a function f is smooth and nonnegative for $a \le x \le b$, then, by Definition (5.19), the area S of the surface that is generated by revolving the graph of $y = f(x)$ about the x-axis (see Figure 9.16) is given by

$$S = \int_{x=a}^{x=b} 2\pi y\, ds,$$

where $ds = \sqrt{1 + [f'(x)]^2}\, dx$. We can regard $2\pi y\, ds$ as the surface area of a frustum of a cone of slant height ds and average radius y (see (5.18)).

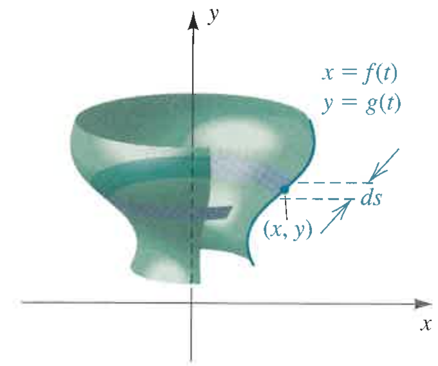
If a curve C is given parametrically by $x = f(t)$, $y = g(t)$; $a \le t \le b$ and if $g(t) \ge 0$ throughout $[a, b]$, we can use an argument similar to that given in Section 5.5 to show that the area of the surface generated by revolving C about the y-axis is $S = \int_{t=a}^{t=b} 2\pi y\, ds$, where ds is the parametric differential of arc length (9.7). Let us state this for reference as follows.

Theorem 9.8

Let a smooth curve C be given by $x = f(t)$, $y = g(t)$; $a \le t \le b$, and suppose C does not intersect itself, except possibly at the point corresponding to $t = a$ and $t = b$. If $g(t) \ge 0$ throughout $[a, b]$, then the area S of the surface of revolution obtained by revolving C about the x-axis is

$$S = \int_{t=a}^{t=b} 2\pi y\, ds = \int_a^b 2\pi g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Figure 9.17



The formula for S in Theorem (9.8) can be extended to the case in which $y = g(t)$ is negative for some t in $[a, b]$ by replacing the variable y that precedes ds by $|y|$.

If the curve C in Theorem (9.8) is revolved about the y-axis and if $x = f(t) \ge 0$ for $a \le t \le b$ (see Figure 9.17), then

$$S = \int_{t=a}^{t=b} 2\pi x\, ds = \int_a^b 2\pi f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In this case, we may regard $2\pi x\, ds$ as the surface area of a frustum of a cone of slant height ds and average radius x.

EXAMPLE 5 Verify that the surface area of a sphere of radius a is $4\pi a^2$.

SOLUTION If C is the upper half of the circle $x^2 + y^2 = a^2$, then the spherical surface may be obtained by revolving C about the x-axis. Parametric equations for C are

$$x = a\cos t, \quad y = a\sin t; \quad 0 \le t \le \pi.$$

Applying Theorem (9.8) and using the identity $\sin^2 t + \cos^2 t = 1$, we have

$$S = \int_0^{\pi} 2\pi a \sin t\sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = 2\pi a^2 \int_0^{\pi} \sin t\, dt$$
$$= -2\pi a^2 \left[\cos t\right]_0^{\pi} = -2\pi a^2[-1-1] = 4\pi a^2.$$

EXERCISES 9.2

Exer. 1–8: Find the slopes of the tangent line and the normal line at the point on the curve that corresponds to $t = 1$.

1 $x = t^2 + 1$, $y = t^2 - 1$; $-2 \le t \le 2$

2 $x = t^3 + 1$, $y = t^3 - 1$; $-2 \le t \le 2$

3 $x = 4t^2 - 5$, $y = 2t + 3$; t in $\mathbb{R}$

4 $x = t^3$, $y = t^2$; t in $\mathbb{R}$

5 $x = e^t$, $y = e^{-2t}$; t in $\mathbb{R}$

6 $x = \sqrt{t}$, $y = 3t + 4$; $t \ge 0$

7 $x = 2\sin t$, $y = 3\cos t$; $0 \le t \le 2\pi$

8 $x = \cos t - 2$, $y = \sin t + 3$; $0 \le t \le 2\pi$

Exer. 9–10: Let C be the curve with the given parametrization, for t in $\mathbb{R}$. Find the points on C at which the slope of the tangent line is m.

9 $x = -t^3$, $y = -6t^2 - 18t$; $m = 2$

10 $x = t^2 + t$, $y = 5t^2 - 3$; $m = 4$

Exer. 11–18: (a) Find the points on the curve C at which the tangent line is either horizontal or vertical. (b) Find d^2y/dx^2. (c) Sketch the graph of C.

11 $x = 4t^2$, $y = t^3 - 12t$; t in $\mathbb{R}$

12 $x = t^3 - 4t$, $y = t^2 - 4$; t in $\mathbb{R}$

13 $x = t^3 + 1$, $y = t^2 - 2t$; t in $\mathbb{R}$

14 $x = 12t - t^3$, $y = t^2 - 5t$; t in $\mathbb{R}$

15 $x = 3t^2 - 6t$, $y = \sqrt{t}$; $t \ge 0$

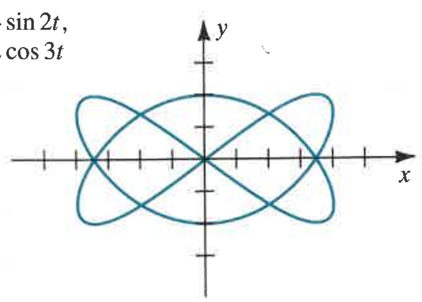
16 $x = \sqrt[3]{t}$, $y = \sqrt[3]{t} - t$; t in $\mathbb{R}$

17 $x = \cos^3 t$, $y = \sin^3 t$; $0 \le t \le 2\pi$

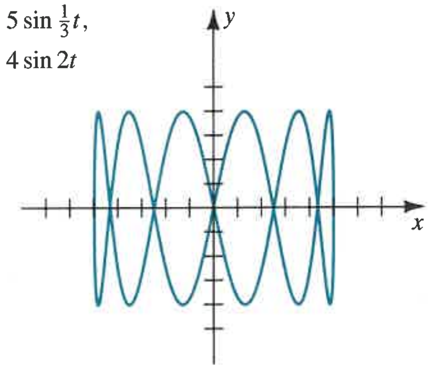
18 $x = \cosh t$, $y = \sinh t$; t in $\mathbb{R}$

Exer. 19–20: Shown is a Lissajous figure (see Example 5 of Section 9.1). Determine where the tangent line is horizontal or vertical.

19 $x = 4\sin 2t$, $y = 2\cos 3t$



20 $x = 5\sin \frac{1}{3}t$, $y = 4\sin 2t$



Exer. 21–26: Find the length of the curve.

21 $x = 5t^2$, $y = 2t^3$; $0 \le t \le 1$

22 $x = 3t$, $y = 2t^{3/2}$; $0 \le t \le 4$

23 $x = e^t \cos t$, $y = e^t \sin t$; $0 \le t \le \pi/2$

24 $x = \cos 2t$, $y = \sin^2 t$; $0 \le t \le \pi$

25 $x = t\cos t - \sin t$, $y = t\sin t + \cos t$; $0 \le t \le \pi/2$

26 $x = \cos^3 t$, $y = \sin^3 t$; $0 \le t \le \pi/2$

c **Exer. 27–28: Use Simpson's rule, with $n = 3$, to approximate the length of the curve.**

27 $x = 2\cos t$, $y = 3\sin t$; $0 \le t \le 2\pi$

28 $x = 4t^3 - t$, $y = 2t^2$; $0 \le t \le 1$

Exer. 29–34: Find the area of the surface generated by revolving the curve about the x-axis.

29 $x = t^2$, $y = 2t$; $0 \le t \le 4$

30 $x = 4t$, $y = t^3$; $1 \le t \le 2$

31 $x = t^2$, $y = t - \frac{1}{3}t^3$; $0 \le t \le 1$

32 $x = 4t^2 + 1$, $y = 3 - 2t$; $-2 \le t \le 0$

33 $x = t - \sin t$, $y = 1 - \cos t$; $0 \le t \le 2\pi$

34 $x = t$, $y = \frac{1}{3}t^3 + \frac{1}{4}t^{-1}$; $1 \le t \le 2$

Exer. 35–38: Find the area of the surface generated by revolving the curve about the y-axis.

35 $x = 4t^{1/2}$, $y = \frac{1}{2}t^2 + t^{-1}$; $1 \le t \le 4$

36 $x = 3t$, $y = t + 1$; $0 \le t \le 5$

37 $x = e^t \sin t$, $y = e^t \cos t$; $0 \le t \le \pi/2$

38 $x = 3t^2$, $y = 2t^3$; $0 \le t \le 1$

c **Exer. 39–40: Use Simpson's rule, with $n = 2$, to approximate the area of the surface generated by revolving the curve about the given axis.**

39 $x = \cos(t^2)$, $y = \sin^2 t$; $0 \le t \le 1$; the x-axis

40 $x = t^2 + 2t$, $y = t^4$; $0 \le t \le 1$; the y-axis

41 Prove that the tangent line at the initial control point P_0 on any Bézier curve will pass through the second control point P_1.

42 Prove that the tangent line at the final control point P_3 on any Bézier curve will pass through the third control point P_2.

c 43 Approximate the arc length for the Bézier curve in Exercise 53 of Section 9.1. Approximate the length of the piecewise *linear* curve made up of the line segments from P_0 to P_1, from P_1 to P_2, and from P_2 to P_3. Compare the two lengths.

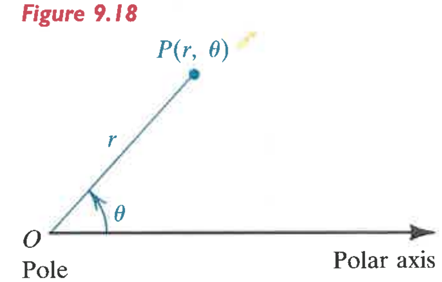
c 44 Work Exercise 43 for the Bézier curve in Exercise 54 of Section 9.1. Make a conjecture about a general result concerning these two lengths.

9.3 POLAR COORDINATES

The polar coordinate system, which we discuss in this section, provides a useful alternative to the rectangular coordinate system in investigating plane curves, particularly circles, ellipses, spirals, and other curves with similar symmetries.

In a rectangular coordinate system, the ordered pair (a, b) denotes the point whose directed distances from the x- and y-axes are b and a, respectively. Another method for representing points is to use *polar coordinates*. We begin with a fixed point O (the **origin**, or **pole**) and a directed half-line (the **polar axis**) with endpoint O. Next we consider any point P in the plane different from O. If, as illustrated in Figure 9.18, $r = d(O, P)$ and θ denotes the measure of any angle determined by the polar axis and OP, then r and θ are **polar coordinates** of P, and the symbols (r, θ) or $P(r, \theta)$ are used to denote P. As usual, θ is considered positive if the angle is generated by a counterclockwise rotation of the polar axis and negative if the rotation is clockwise. Either radian or degree measure may be used for θ.

Figure 9.18



The polar coordinates of a point are not unique. For example, $(3, \pi/4)$, $(3, 9\pi/4)$, and $(3, -7\pi/4)$ all represent the same point (see Figure 9.19).

Figure 9.19

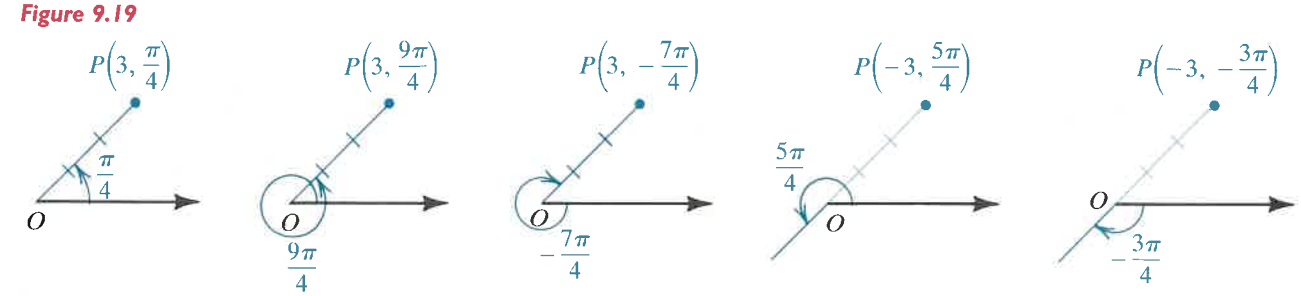
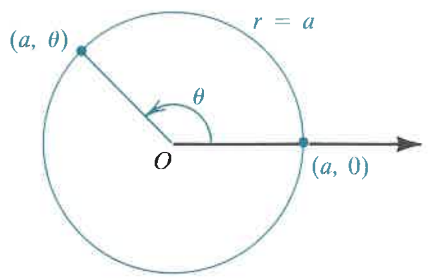


Figure 9.20

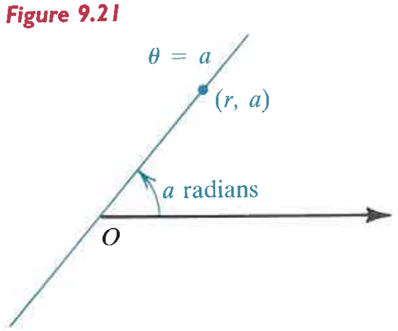


We shall also allow r to be negative. In this case, instead of measuring $|r|$ units along the terminal side of the angle θ, we measure along the half-line with endpoint O that has direction *opposite* that of the terminal side. The points corresponding to the pairs $(-3, 5\pi/4)$ and $(-3, -3\pi/4)$ are also plotted in Figure 9.19.

We agree that the pole O has polar coordinates $(0, \theta)$ for *any* θ. An assignment of ordered pairs of the form (r, θ) to points in a plane is a **polar coordinate system**, and the plane is an **$r\theta$-plane**.

A **polar equation** is an equation in r and θ. A **solution** of a polar equation is an ordered pair (a, b) that leads to equality if a is substituted for r and b for θ. The **graph** of a polar equation is the set of all points (in an $r\theta$-plane) that correspond to the solutions. The simplest polar equations are $r = a$ and $\theta = a$, where a is a nonzero real number. Since the solutions of the polar equation $r = a$ are of the form (a, θ) for *any* angle θ, it follows that the graph is a circle of radius $|a|$ with center at the pole. A graph for $a > 0$ is sketched in Figure 9.20. The same graph is obtained for $r = -a$.

Figure 9.21



The advantages of using polar coordinates to represent naturally occurring curves is already becoming apparent. In the xy-coordinate system, the equation for the circle of radius a with center at the origin is a quadratic expression in both variables, $x^2 + y^2 = a^2$. In polar coordinates, we have one of the simplest possible equations, a variable equals a constant, for the same circle.

The solutions of the polar equation $\theta = a$ are of the form (r, a) for *any* real number r. Since the (angle) coordinate a is constant, the graph is a line through the origin, as illustrated in Figure 9.21 for the case $0 < a < \pi/2$.

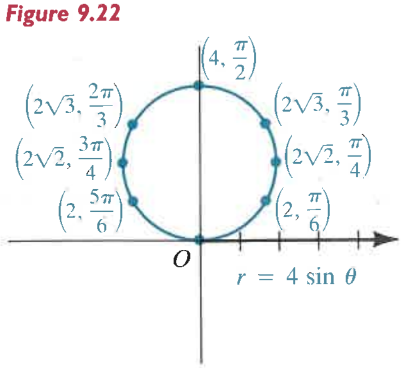
In the following examples, we obtain the graphs of polar equations by plotting points. As you proceed through this section, you should try to recognize forms of polar equations so that you will be able to sketch their graphs by plotting few, if any, points.

EXAMPLE ■ 1 Sketch the graph of the polar equation $r = 4\sin\theta$.

SOLUTION The following table displays some solutions of the equation. We have included a third row in the table that contains one-decimal-place approximations to r.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	2	$2\sqrt{2}$	$2\sqrt{3}$	4	$2\sqrt{3}$	$2\sqrt{2}$	2	0
r (approx.)	0	2	2.8	3.4	4	3.4	2.8	2	0

Figure 9.22



The points in an $r\theta$-plane that correspond to the pairs in the table appear to lie on a circle of radius 2, and we draw the graph accordingly (see Figure 9.22). As an aid to plotting points, we have extended the polar axis in the negative direction and introduced a vertical line through the pole.

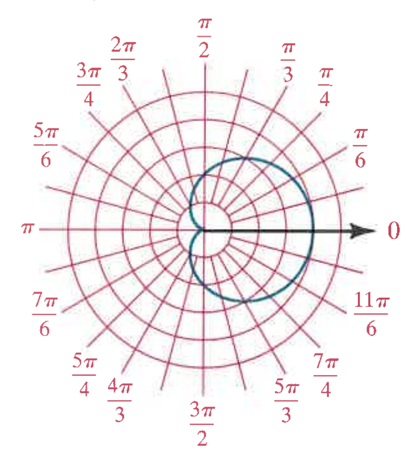
The proof that the graph of $r = 4\sin\theta$ is a circle is given in Example 6. Additional points obtained by letting θ vary from π to 2π lie on the same circle. For example, the solution $(-2, 7\pi/6)$ gives us the same point as $(2, \pi/6)$; the point corresponding to $(-2\sqrt{2}, 5\pi/4)$ is the same as that obtained from $(2\sqrt{2}, \pi/4)$; and so on. If we let θ increase through all real numbers, we obtain the same points again and again because of the periodicity of the sine function.

EXAMPLE ■ 2 Sketch the graph of the polar equation $r = 2 + 2\cos\theta$.

SOLUTION Since the cosine function decreases from 1 to -1 as θ varies from 0 to π, it follows that r decreases from 4 to 0 in this θ-interval. The following table exhibits some solutions of $r = 2 + 2\cos\theta$, together with one-decimal-place approximations to r.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0
r (approx.)	4	3.7	3.4	3	2	1	0.6	0.3	0

Figure 9.23
$r = 2 + 2\cos\theta$



Plotting points in an $r\theta$-plane leads to the upper half of the graph sketched in Figure 9.23. (We have used polar coordinate graph paper, which displays lines through O at various angles and concentric circles with centers at the pole.)

If θ increases from π to 2π, then $\cos\theta$ increases from -1 to 1 and, consequently, r increases from 0 to 4. Plotting points for $\pi \le \theta \le 2\pi$ gives us the lower half of the graph.

The same graph may be obtained by taking other intervals of length 2π for θ.

The heart-shaped graph in Example 2 is a **cardioid**. In general, the graph of any of the following polar equations, with $a \neq 0$, is a cardioid:

$$r = a(1 + \cos\theta), \qquad r = a(1 + \sin\theta),$$
$$r = a(1 - \cos\theta), \qquad r = a(1 - \sin\theta)$$

If a and b are not zero, then the graphs of the following polar equations are **limaçons**:

$$r = a + b\cos\theta, \qquad r = a + b\sin\theta$$

Note that the special limaçons in which $|a| = |b|$ are cardioids. Some limaçons contain a loop, as shown in the next example.

EXAMPLE ■ 3 Sketch the graph of the polar equation $r = 2 + 4\cos\theta$.

SOLUTION Coordinates of some points in an $r\theta$-plane that correspond to $0 \le \theta \le \pi$ are listed in the following table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	6	$2+2\sqrt{3}$	$2+2\sqrt{2}$	4	2	0	$2-2\sqrt{2}$	$2-2\sqrt{3}$	-2
r (approx.)	6	5.4	4.8	4	2	0	-0.8	-1.4	-2

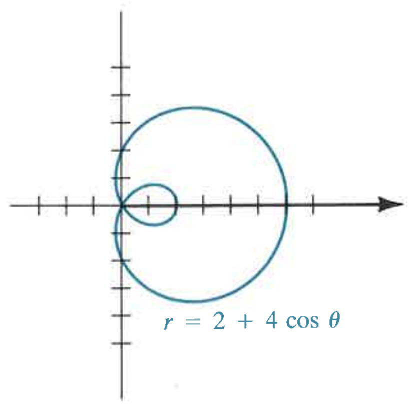


Figure 9.24

Note that $r = 0$ at $\theta = 2\pi/3$. The values of r are negative if $2\pi/3 < \theta \le \pi$, and this leads to the lower half of the small loop in Figure 9.24. Letting θ range from π to 2π gives us the upper half of the small loop and the lower half of the large loop.

EXAMPLE ■ 4 Sketch the graph of the polar equation $r = a\sin 2\theta$ for $a > 0$.

SOLUTION Instead of tabulating solutions, let us reason as follows. If θ increases from 0 to $\pi/4$, then 2θ varies from 0 to $\pi/2$ and hence $\sin 2\theta$ increases from 0 to 1. It follows that r increases from 0 to a in the θ-interval $[0, \pi/4]$. If we next let θ increase from $\pi/4$ to $\pi/2$, then 2θ changes from $\pi/2$ to π and hence $\sin 2\theta$ decreases from 1 to 0. Thus, r decreases from a to 0 in the θ-interval $[\pi/4, \pi/2]$. The corresponding points on the graph constitute the first-quadrant loop illustrated in Figure 9.25. Note that the point $P(r, \theta)$ traces the loop in a *counterclockwise* direction (indicated by the arrows) as θ increases from 0 to $\pi/2$.

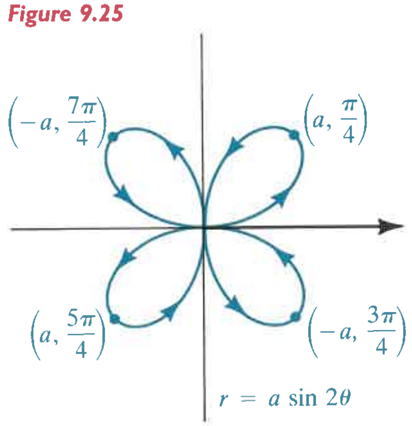


Figure 9.25

If $\pi/2 \le \theta \le \pi$, then $\pi \le 2\theta \le 2\pi$ and, therefore, $r = a\sin 2\theta \le 0$. Thus, *if* $\pi/2 < \theta < \pi$, *then* r *is negative and the points* $P(r, \theta)$ *are in the fourth quadrant*. If θ increases from $\pi/2$ to π, then we can show, by plotting points, that $P(r, \theta)$ traces (in a counterclockwise direction) the loop shown in the fourth quadrant.

Similarly, for $\pi \le \theta \le 3\pi/2$ we get the loop in the third quadrant, and for $3\pi/2 \le \theta \le 2\pi$ we get the loop in the second quadrant. Both loops are traced in a counterclockwise direction as θ increases. You should verify these facts by plotting some points with, say, $a = 1$. In Figure 9.25, we have plotted only those points on the graph that correspond to the largest numerical values of r.

The graph in Example 4 is a **four-leafed rose**. In general, a polar equation of the form

$$r = a\sin n\theta \quad \text{or} \quad r = a\cos n\theta$$

for any positive integer n greater than 1 and any nonzero real number a has a graph that consists of a number of loops through the origin. If n is even, there are $2n$ loops, and if n is odd, there are n loops (see Exercises 15–18).

The graph of the polar equation $r = a\theta$ for any nonzero real number a is a **spiral of Archimedes**. The case $a = 1$ is considered in the next example.

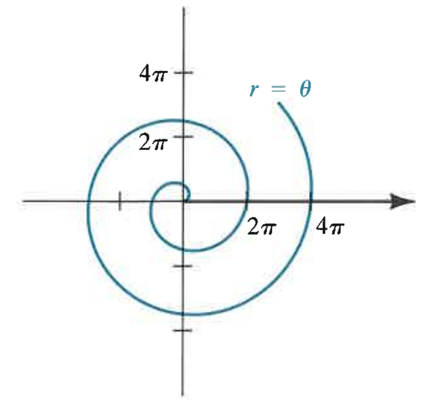


Figure 9.26

EXAMPLE ■ 5 Sketch the graph of the polar equation $r = \theta$ for $\theta \ge 0$.

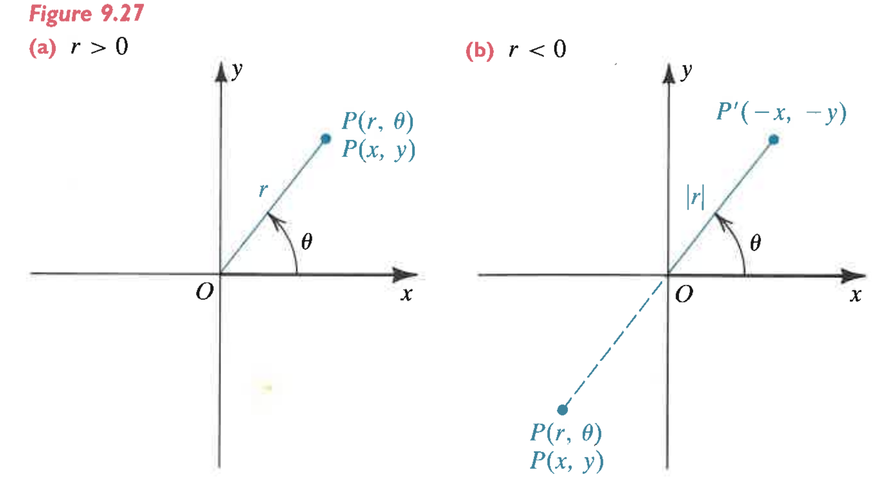
SOLUTION The graph consists of all points that have polar coordinates of the form (c, c) for every real number $c \ge 0$. Thus, the graph contains the points $(0, 0)$, $(\pi/2, \pi/2)$, (π, π), and so on. As θ increases, r increases at the same rate, and the spiral winds around the origin in a counterclockwise direction, intersecting the polar axis at $0, 2\pi, 4\pi, \ldots$, as illustrated in Figure 9.26.

If θ is allowed to be negative, then as θ decreases through negative values, the resulting spiral winds around the origin and is the symmetric image, with respect to the vertical axis, of the curve sketched in Figure 9.26.

Spirals seen in nature, such as those apparent in the curl of an ocean wave, may have quite complicated equations in the rectangular xy-coordinate system, but much simpler and more elegant representations in polar coordinates. For the spiral of Archimedes in Example 5, the corresponding equation in xy-coordinates is a quite complex expression, $\sqrt{x^2 + y^2} = \tan^{-1}(y/x)$. The graphs of polar coordinates illustrating other spirals are given in Exercises 21, 24, and 66.

Let us next superimpose an xy-plane on an $r\theta$-plane so that the positive x-axis coincides with the polar axis. Any point P in the plane may then be assigned rectangular coordinates (x, y) or polar coordinates (r, θ). If $r > 0$, we have a situation similar to that illustrated in Figure 9.27(a). If $r < 0$, we have that shown in Figure 9.27(b), where, for later purposes,

Figure 9.27



we have also plotted the point P' having polar coordinates $(|r|, \theta)$ and rectangular coordinates $(-x, -y)$.

The following result specifies relationships between (x, y) and (r, θ), where it is assumed that the positive x-axis coincides with the polar axis.

Relationships between Rectangular and Polar Coordinates 9.9

> The rectangular coordinates (x, y) and polar coordinates (r, θ) of a point P are related as follows:
>
> **(i)** $x = r\cos\theta, \quad y = r\sin\theta$
>
> **(ii)** $r^2 = x^2 + y^2, \quad \tan\theta = y/x \quad \text{if} \quad x \neq 0$

PROOF Although we have pictured θ as an acute angle in Figure 9.27, the discussion that follows is valid for all angles. If $r > 0$ as in Figure 9.27(a), then $\cos\theta = x/r$, $\sin\theta = y/r$, and hence

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

If $r < 0$, then $|r| = -r$, and from Figure 9.27(b) we see that

$$\cos\theta = \frac{-x}{|r|} = \frac{-x}{-r} = \frac{x}{r}, \qquad \sin\theta = \frac{-y}{|r|} = \frac{-y}{-r} = \frac{y}{r}.$$

Multiplication by r gives us relationship (i), and therefore these formulas hold if r is either positive or negative. If $r = 0$, then the point is the pole and we again see that the formulas in (i) are true.

The formulas in (ii) follow readily from Figure 9.27. ■

We may use this result to change from one system of coordinates to the other. A more important use is for transforming a polar equation to an equation in x and y, and vice versa, illustrated in Examples 6–8.

EXAMPLE ■ 6 Find an equation in x and y that has the same graph as the polar equation $r = a\sin\theta$, with $a \neq 0$. Sketch the graph.

SOLUTION From (9.9)(i), a relationship between $\sin\theta$ and y is given by $y = r\sin\theta$. To introduce this expression into the equation $r = a\sin\theta$, we multiply both sides by r, obtaining

$$r^2 = ar\sin\theta.$$

Next, using $r^2 = x^2 + y^2$ and $y = r\sin\theta$, we have

$$x^2 + y^2 = ay,$$

or

$$x^2 + y^2 - ay = 0.$$

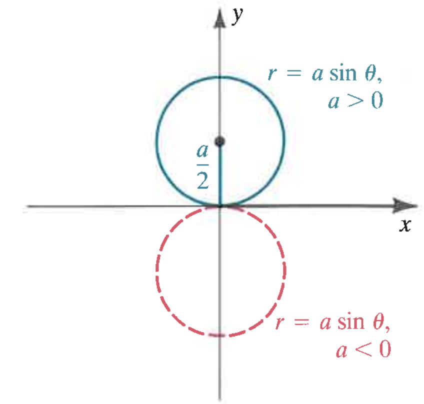
Completing the square in y gives us

$$x^2 + y^2 - ay + \left(\frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2,$$

or

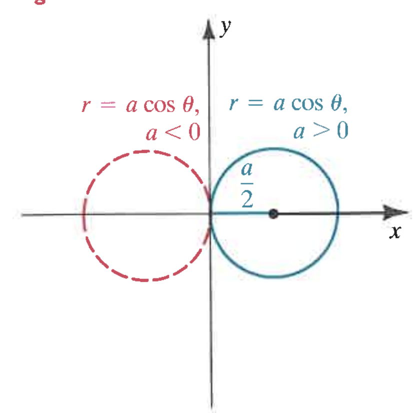
$$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2.$$

Figure 9.28



In the xy-plane, the graph of the last equation is a circle with center $(0, a/2)$ and radius $|a|/2$, as illustrated in Figure 9.28 for the case $a > 0$ (the solid circle) and $a < 0$ (the dashed circle).

Figure 9.29



Using the same method as in the preceding example, we can show that the graph of $r = a\cos\theta$, with $a \neq 0$, is a circle of radius $a/2$ of the type illustrated in Figure 9.29.

EXAMPLE ■ 7 Find a polar equation for the hyperbola given by $x^2 - y^2 = 16$.

SOLUTION Using the formulas $x = r\cos\theta$ and $y = r\sin\theta$, we obtain the following polar equations:

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 16$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 16$$
$$r^2(\cos^2\theta - \sin^2\theta) = 16$$
$$r^2\cos 2\theta = 16$$
$$r^2 = \frac{16}{\cos 2\theta} \quad \text{or} \quad r^2 = 16\sec 2\theta$$

The division by $\cos 2\theta$ is allowable because $\cos 2\theta \neq 0$. (Note that if $\cos 2\theta = 0$, then $r^2\cos 2\theta \neq 16$.)

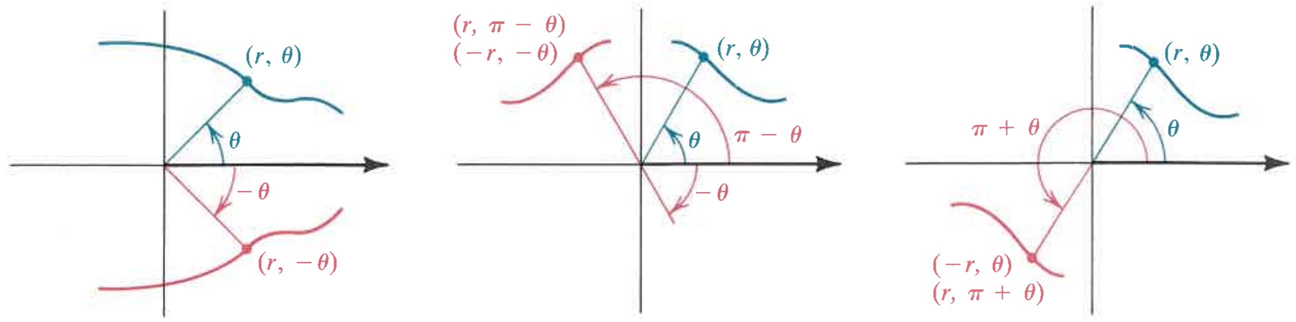
EXAMPLE ■ 8 Find a polar equation of an arbitrary line.

SOLUTION Every line in an xy-coordinate plane is the graph of a linear equation $ax + by = c$. Using the formulas $x = r\cos\theta$ and $y = r\sin\theta$ gives us the following equivalent polar equations:

$$ar\cos\theta + br\sin\theta = c$$
$$r(a\cos\theta + b\sin\theta) = c$$
$$r = \frac{c}{a\cos\theta + b\sin\theta}$$

Figure 9.30
Symmetries of graphs of polar equations

(a) Polar axis (b) Line $\theta = \pi/2$ (c) Pole



If we superimpose an xy-plane on an $r\theta$-plane, then the graph of a polar equation may be symmetric with respect to the x-axis (the polar axis), the y-axis (the line $\theta = \pi/2$), or the origin (the pole). Some typical symmetries are illustrated in Figure 9.30. The next result states tests for symmetry using polar coordinates.

Tests for Symmetry 9.10

(i) The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if substitution of $-\theta$ for θ leads to an equivalent equation.

(ii) The graph of $r = f(\theta)$ is symmetric with respect to the vertical line $\theta = \pi/2$ if substitution of either (a) $\pi - \theta$ for θ or (b) $-r$ for r and $-\theta$ for θ leads to an equivalent equation.

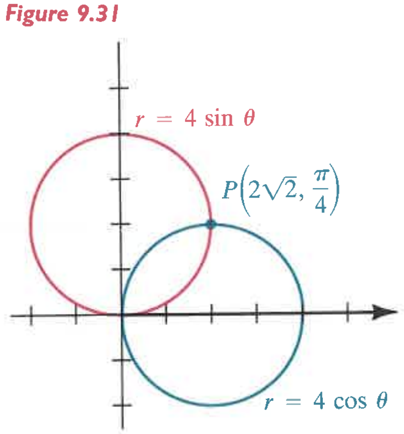
(iii) The graph of $r = f(\theta)$ is symmetric with respect to the pole if substitution of either (a) $-r$ for r or (b) $\pi + \theta$ for θ leads to an equivalent equation.

To illustrate, since $\cos(-\theta) = \cos\theta$, the graph of the polar equation $r = 2 + 4\cos\theta$ in Example 3 is symmetric with respect to the polar axis, by test (i). Since $\sin(\pi - \theta) = \sin\theta$, the graph in Example 1 is symmetric with respect to the line $\theta = \pi/2$, by test (ii). The graph in Example 4 is symmetric to the polar axis, the line $\theta = \pi/2$, and the pole. Other tests for symmetry may be stated; however, those we have listed are among the easiest to apply.

Unlike the graph of an equation in x and y, the graph of a polar equation $r = f(\theta)$ can be symmetric with respect to the polar axis, the line $\theta = \pi/2$, or the pole *without* satisfying one of the preceding tests for symmetry. This is true because of the many different ways of specifying a point in polar coordinates.

Another difference between rectangular and polar coordinate systems is that the points of intersection of two graphs cannot always be found by solving the polar equations simultaneously. To illustrate, from Example 1, the graph of $r = 4\sin\theta$ is a circle of diameter 4 with center at $(2, \pi/2)$ (see Figure 9.31). Similarly, the graph of $r = 4\cos\theta$ is a circle of diameter

Figure 9.31



4, with center at $(2, 0)$ on the polar axis. Referring to Figure 9.31, we see that the coordinates of the point of intersection $P(2\sqrt{2}, \pi/4)$ in quadrant I satisfy both equations; however, the origin O, which is on each circle, *cannot* be found by solving the equations simultaneously. Thus, in searching for points of intersection of polar graphs, it is sometimes necessary to refer to the graphs themselves, *in addition* to solving the two equations simultaneously. An alternative method is to use different (equivalent) equations for the graphs.

Tangent lines to graphs of polar equations may be found by means of the next theorem.

Theorem 9.11

The slope m of the tangent line to the graph of $r = f(\theta)$ at the point $P(r, \theta)$ is

$$m = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

PROOF If (x, y) are the rectangular coordinates of $P(r, \theta)$, then, by Theorem (9.9),

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta.$$

These may be considered as parametric equations for the graph with parameter θ. Applying Theorem (9.4), we find that the slope of the tangent line at (x, y) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f(\theta)(-\sin\theta) + f'(\theta)\cos\theta}$$
$$= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

which is equivalent to the formula for m in Theorem (9.11). ■

Horizontal tangent lines occur if the numerator in the formula for m is 0 and the denominator is not 0. Vertical tangent lines occur if the denominator is 0 and the numerator is not 0. The case $0/0$ requires further investigation.

To find the slopes of the tangent lines at the pole, we must determine the values of θ for which $r = f(\theta) = 0$. For such values (and with $r = 0$ and $dr/d\theta \neq 0$), the formula in Theorem (9.11) reduces to $m = \tan\theta$. These remarks are illustrated in the next example.

EXAMPLE 9 For the cardioid $r = 2 + 2\cos\theta$ with $0 \le \theta < 2\pi$, find

(a) the slope of the tangent line at $\theta = \pi/6$

(b) the points at which the tangent line is horizontal

(c) the points at which the tangent line is vertical

SOLUTION

(a) The graph of $r = 2 + 2\cos\theta$ was considered in Example 2 and is resketched in Figure 9.32. Applying Theorem (9.11), we find that the slope m of the tangent line is

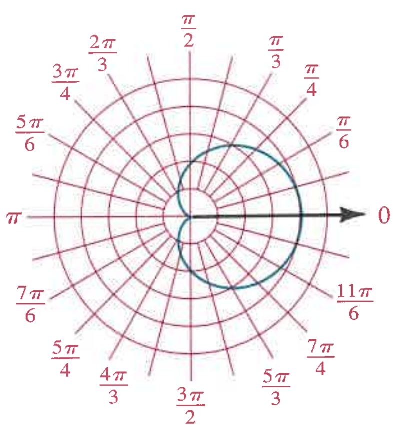


Figure 9.32
$r = 2 + 2\cos\theta$

$$m = \frac{(-2\sin\theta)\sin\theta + (2 + 2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (2 + 2\cos\theta)\sin\theta}$$

$$= \frac{2(\cos^2\theta - \sin^2\theta) + 2\cos\theta}{-2(2\sin\theta\cos\theta) - 2\sin\theta}$$

$$= -\frac{\cos 2\theta + \cos\theta}{\sin 2\theta + \sin\theta}.$$

At $\theta = \pi/6$ (that is, at the point $(2 + \sqrt{3}, \pi/6)$),

$$m = -\frac{\cos(\pi/3) + \cos(\pi/6)}{\sin(\pi/3) + \sin(\pi/6)} = -\frac{(1/2) + (\sqrt{3}/2)}{(\sqrt{3}/2) + (1/2)} = -1.$$

(b) To find horizontal tangents, we let

$$\cos 2\theta + \cos\theta = 0.$$

This equation may be written as

$$2\cos^2\theta - 1 + \cos\theta = 0,$$

or

$$(2\cos\theta - 1)(\cos\theta + 1) = 0.$$

From $\cos\theta = \frac{1}{2}$, we obtain $\theta = \pi/3$ and $\theta = 5\pi/3$. The corresponding points are $(3, \pi/3)$ and $(3, 5\pi/3)$.

Using $\cos\theta = -1$ gives us $\theta = \pi$. The denominator in the formula for m is 0 at $\theta = \pi$, and hence further investigation is required. If $\theta = \pi$, then $r = 0$ and the formula for m in Theorem (9.11) reduces to $m = \tan\theta$. Thus, the slope at $(0, \pi)$ is $m = \tan\pi = 0$, and therefore the tangent line is horizontal at the pole.

(c) To find vertical tangent lines, we let

$$\sin 2\theta + \sin\theta = 0.$$

Equivalent equations are

$$2\sin\theta\cos\theta + \sin\theta = 0$$

and

$$\sin\theta(2\cos\theta + 1) = 0.$$

Letting $\sin\theta = 0$ and $\cos\theta = -\frac{1}{2}$ leads to the following values of θ: $0, \pi, 2\pi/3$, and $4\pi/3$. We found, in part (b), that π gives us a horizontal tangent. The remaining values result in the points $(4, 0)$, $(1, 2\pi/3)$, and $(1, 4\pi/3)$, at which the graph has vertical tangent lines.

EXERCISES 9.3

Exer. 1–26: Sketch the graph of the polar equation.

1 $r = 5$
2 $r = -2$
3 $\theta = -\pi/6$
4 $\theta = \pi/4$
5 $r = 3\cos\theta$
6 $r = -2\sin\theta$
7 $r = 4 - 4\sin\theta$
8 $r = -6(1 + \cos\theta)$
9 $r = 2 + 4\sin\theta$
10 $r = 1 + 2\cos\theta$
11 $r = 2 - \cos\theta$
12 $r = 5 + 3\sin\theta$
13 $r = 4\csc\theta$
14 $r = -3\sec\theta$
15 $r = 8\cos 3\theta$
16 $r = 2\sin 4\theta$
17 $r = 3\sin 2\theta$
18 $r = 8\cos 5\theta$
19 $r^2 = 4\cos 2\theta$ (lemniscate)
20 $r^2 = -16\sin 2\theta$
21 $r = e^{\theta}$, $\theta \ge 0$ (logarithmic spiral)
22 $r = 6\sin^2(\theta/2)$
23 $r = 2\theta$, $\theta \ge 0$
24 $r\theta = 1$, $\theta > 0$ (spiral)
25 $r = 2 + 2\sec\theta$ (conchoid)
26 $r = 1 - \csc\theta$

Exer. 27–36: Find a polar equation that has the same graph as the equation in x and y.

27 $x = -3$
28 $y = 2$
29 $x^2 + y^2 = 16$
30 $x^2 = 8y$
31 $2y = -x$
32 $y = 6x$
33 $y^2 - x^2 = 4$
34 $xy = 8$
35 $(x^2 + y^2)\tan^{-1}(y/x) = ay$, $a > 0$ (cochleoid, or Ouija board curve)
36 $x^3 + y^3 - 3axy = 0$ (Folium of Descartes)

Exer. 37–50: Find an equation in x and y that has the same graph as the polar equation and use it to help sketch the graph in an $r\theta$-plane.

37 $r\cos\theta = 5$
38 $r\sin\theta = -2$
39 $r = -3\csc\theta$
40 $r = 4\sec\theta$
41 $r^2\cos 2\theta = 1$
42 $r^2\sin 2\theta = 4$
43 $r(\sin\theta - 2\cos\theta) = 6$
44 $r(3\cos\theta - 4\sin\theta) = 12$
45 $r(\sin\theta + r\cos^2\theta) = 1$
46 $r(r\sin^2\theta - \cos\theta) = 3$
47 $r = 8\sin\theta - 2\cos\theta$
48 $r = 2\cos\theta - 4\sin\theta$
49 $r = \tan\theta$
50 $r = 6\cot\theta$

Exer. 51–60: Find the slope of the tangent line to the graph of the polar equation at the point corresponding to the given value of θ.

51 $r = 2\cos\theta$; $\theta = \pi/3$
52 $r = -2\sin\theta$; $\theta = \pi/6$
53 $r = 4(1 - \sin\theta)$; $\theta = 0$
54 $r = 1 + 2\cos\theta$; $\theta = \pi/2$
55 $r = 8\cos 3\theta$; $\theta = \pi/4$
56 $r = 2\sin 4\theta$; $\theta = \pi/4$
57 $r^2 = 4\cos 2\theta$; $\theta = \pi/6$
58 $r^2 = -2\sin 2\theta$; $\theta = 3\pi/4$
59 $r = 2^{\theta}$; $\theta = \pi$
60 $r\theta = 1$; $\theta = 2\pi$

61 If $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ are points in an $r\theta$-plane, use the law of cosines to prove that

$$[d(P_1, P_2)]^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1).$$

62 If a and b are nonzero real numbers, prove that the graph of $r = a\sin\theta + b\cos\theta$ is a circle, and find its center and radius.

63 If the graphs of the polar equations $r = f(\theta)$ and $r = g(\theta)$ intersect at $P(r, \theta)$, prove that the tangent lines at P are perpendicular if and only if

$$f'(\theta)g'(\theta) + f(\theta)g(\theta) = 0.$$

(The graphs are said to be *orthogonal* at P.)

64 Use Exercise 63 to prove that the graphs of each pair of equations are orthogonal at their point of intersection:

(a) $r = a\sin\theta$, $r = a\cos\theta$

(b) $r = a\theta$, $r\theta = a$

65 If $\cos\theta \ne 0$, show that the slope of the tangent line to the graph of $r = f(\theta)$ is

$$m = \frac{(dr/d\theta)\tan\theta + r}{(dr/d\theta) - r\tan\theta}.$$

66 A logarithmic spiral has a polar equation of the form $r = ae^{b\theta}$ for nonzero constants a and b (see Exercise 21). A famous *four bugs problem* illustrates such a curve. Four bugs A, B, C, and D are placed at the four corners of a square. The center of the square corresponds to the pole. The bugs begin to crawl simultaneously—bug A crawls toward B, B toward C, C toward D, and D toward A, as shown in the figure. Assume that all bugs crawl at the same rate, that they move directly toward the next bug at all times, and that they approach one another but never meet. (The bugs are infinitely small!) At any instant, the positions of the bugs are the vertices of a square, which shrinks and rotates toward the center of the original square as the bugs continue to crawl. If the position of bug A has polar coordinates (r, θ), then the position of bug B has coordinates $(r, \theta + \pi/2)$.

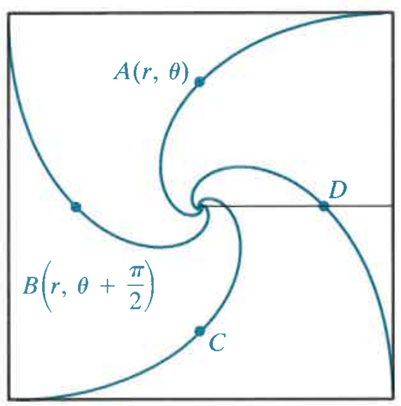
(a) Show that the line through A and B has slope

$$\frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}.$$

(b) The line through A and B is tangent to the path of bug A. Use the formula in Exercise 65 to conclude that $dr/d\theta = -r$.

(c) Prove that the path of bug A is a logarithmic spiral. (*Hint:* Solve the differential equation in part (b) by separating variables.)

Exercise 66



c **Exer. 67–68: Graph the polar equation for the given values of θ, and use the graph to determine symmetries.**

67 $r = 2\sin^2\theta\tan^2\theta; \quad -\pi/3 \le \theta \le \pi/3$

68 $r = \dfrac{4}{1 + \sin^2\theta}; \quad 0 \le \theta \le 2\pi$

c **Exer. 69–70: Graph the polar equations on the same coordinate plane, and estimate the points of intersection of the graphs.**

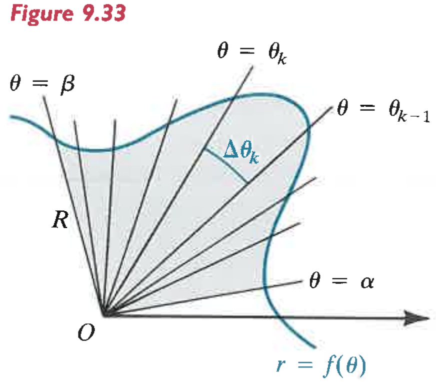
69 $r = 8\cos 3\theta, \quad r = 4 - 2.5\cos\theta$

70 $r = 2\sin^2\theta, \quad r = \frac{3}{4}(\theta + \cos^2\theta)$

9.4 INTEGRALS IN POLAR COORDINATES

The areas of certain regions bounded by graphs of polar equations can be found by using limits of sums of areas of circular sectors. We shall call a region R in the $r\theta$-plane an R_θ **region** (for integration with respect to θ) if R is bounded by lines $\theta = \alpha$ and $\theta = \beta$ for $0 \le \alpha < \beta \le 2\pi$ and by the graph of a polar equation $r = f(\theta)$, where f is continuous and $f(\theta) \ge 0$ on $[\alpha, \beta]$. An R_θ region is illustrated in Figure 9.33.

Figure 9.33

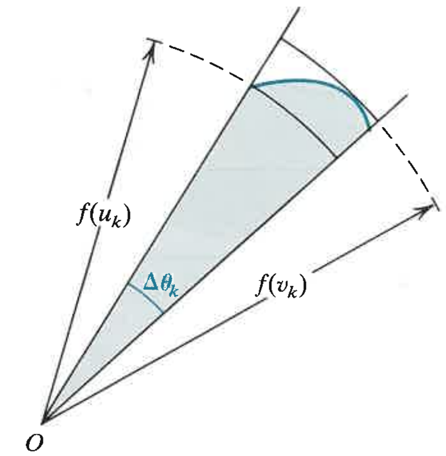


Let P denote a partition of $[\alpha, \beta]$ determined by

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = \beta$$

and let $\Delta\theta_k = \theta_k - \theta_{k-1}$ for $k = 1, 2, \ldots, n$. The lines $\theta = \theta_k$ divide R into wedge-shaped subregions. If $f(u_k)$ is the minimum value and $f(v_k)$ is the maximum value of f on $[\theta_{k-1}, \theta_k]$, then, as illustrated in Figure 9.34, the area ΔA_k of the kth subregion is between the areas of the inscribed and circumscribed circular sectors having central angle $\Delta\theta_k$ and radii $f(u_k)$ and $f(v_k)$, respectively. Hence, by the formula for finding the area of a circular sector (page 38),

Figure 9.34



$$\tfrac{1}{2}[f(u_k)]^2\Delta\theta_k \le \Delta A_k \le \tfrac{1}{2}[f(v_k)]^2\Delta\theta_k.$$

Summing from $k = 1$ to $k = n$ and using the fact that the sum of the ΔA_k is the area A of R, we obtain

$$\sum_{k=1}^{n} \tfrac{1}{2}[f(u_k)]^2\Delta\theta_k \le A \le \sum_{k=1}^{n} \tfrac{1}{2}[f(v_k)]^2\Delta\theta_k.$$

The limits of the sums, as the norm $\|P\|$ of the subdivision approaches zero, both equal the integral $\int_\alpha^\beta \frac{1}{2}[f(\theta)]^2\,d\theta$. This gives us the following result.

Theorem 9.12

If f is continuous and $f(\theta) \ge 0$ on $[\alpha, \beta]$, where $0 \le \alpha < \beta \le 2\pi$, then the area A of the region bounded by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is

$$A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta = \int_\alpha^\beta \tfrac{1}{2}r^2\,d\theta.$$

The integral in Theorem (9.12) may be interpreted as a limit of sums by writing

$$A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \tfrac{1}{2}[f(w_k)]^2\Delta\theta_k$$

for *any* number w_k in the subinterval $[\theta_{k-1}, \theta_k]$ of $[\alpha, \beta]$. Figure 9.35 is a geometric illustration of a typical Riemann sum.

Figure 9.35

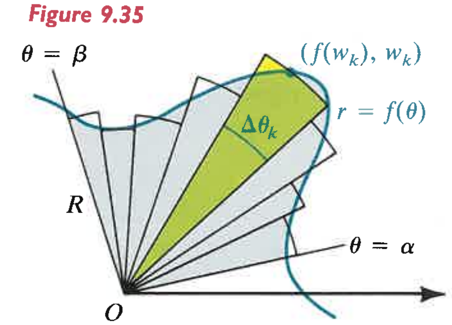
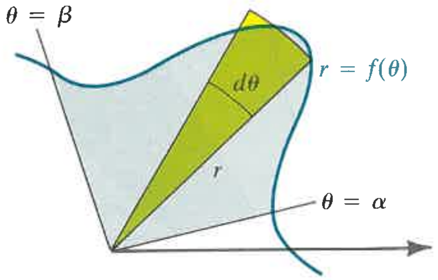


Figure 9.36



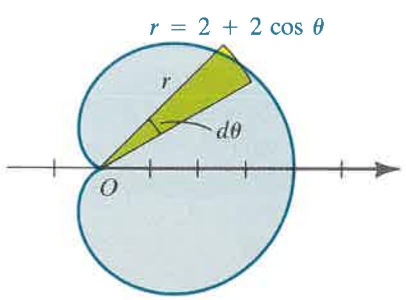
The following guidelines may be useful for remembering this limit of sums formula (see Figure 9.36).

Guidelines for Finding the Area of an R_θ Region 9.13

1. Sketch the region, labeling the graph of $r = f(\theta)$. Find the smallest value $\theta = \alpha$ and the largest value $\theta = \beta$ for points (r, θ) in the region.
2. Sketch a typical circular sector and label its central angle $d\theta$.
3. Express the area of the sector in guideline (2) as $\frac{1}{2}r^2\,d\theta$.
4. Apply the limit of sums operator $\int_\alpha^\beta$ to the expression in guideline (3) and evaluate the integral.

EXAMPLE 1 Find the area of the region bounded by the cardioid $r = 2 + 2\cos\theta$.

Figure 9.37



SOLUTION Following guideline (1), we first sketch the region as in Figure 9.37. The cardioid is obtained by letting θ vary from 0 to 2π; however, using symmetry we may find the area of the top half and multiply by 2. Thus, we use $\alpha = 0$ and $\beta = \pi$ for the smallest and largest values of θ. As in guideline (2), we sketch a typical circular sector and label its central angle $d\theta$. To apply guideline (3), we refer to the figure, obtaining the following:

radius of circular sector: $r = 2 + 2\cos\theta$

area of sector: $\frac{1}{2}r^2\,d\theta = \frac{1}{2}(2 + 2\cos\theta)^2\,d\theta$

We next use guideline (4), with $\alpha = 0$ and $\beta = \pi$, remembering that applying $\int_0^\pi$ to the expression $\frac{1}{2}(2 + 2\cos\theta)^2\,d\theta$ represents taking a limit of sums of areas of circular sectors, *sweeping out* the region by letting θ vary from 0 to π. Thus,

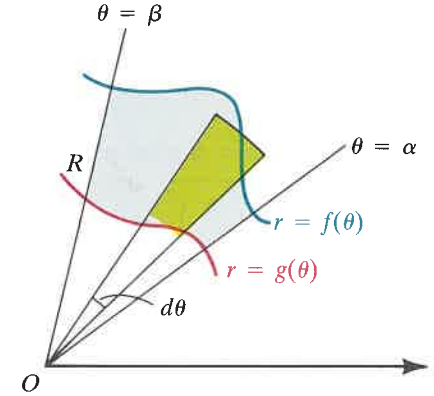
$$A = 2\int_0^\pi \tfrac{1}{2}(2 + 2\cos\theta)^2\,d\theta$$
$$= \int_0^\pi (4 + 8\cos\theta + 4\cos^2\theta)\,d\theta.$$

Using the fact that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ yields

$$A = \int_0^\pi (6 + 8\cos\theta + 2\cos 2\theta)\,d\theta$$
$$= \Big[6\theta + 8\sin\theta + \sin 2\theta\Big]_0^\pi = 6\pi.$$

We could also have found the area by using $\alpha = 0$ and $\beta = 2\pi$.

Figure 9.38



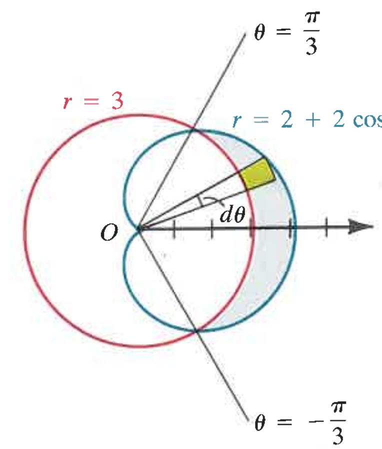
A region R between the graphs of two polar equations $r = f(\theta)$ and $r = g(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is sketched in Figure 9.38. We may find the area A of R by subtracting the area of the *inner* region bounded by $r = g(\theta)$ from the area of the *outer* region bounded by $r = f(\theta)$ as follows:

$$A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]^2\,d\theta - \int_\alpha^\beta \tfrac{1}{2}[g(\theta)]^2\,d\theta$$

We use this technique in the next example.

EXAMPLE 2 Find the area A of the region R that is inside the cardioid $r = 2 + 2\cos\theta$ and outside the circle $r = 3$.

Figure 9.39



SOLUTION Figure 9.39 shows the region R and circular sectors that extend from the pole to the graphs of the two polar equations. The points of intersection $(3, -\pi/3)$ and $(3, \pi/3)$ can be found by solving the equations simultaneously. Since the angles α and β in Guidelines (9.13) are nonnegative, we shall find the area of the top half of R (using $\alpha = 0$ and $\beta = \pi/3$) and then double the result. Subtracting the area of the inner region (bounded by $r = 3$) from the area of the outer region (bounded by $r = 2 + 2\cos\theta$), we obtain

$$A = 2\left[\int_0^{\pi/3} \tfrac{1}{2}(2 + 2\cos\theta)^2\,d\theta - \int_0^{\pi/3} \tfrac{1}{2}(3)^2\,d\theta\right]$$
$$= \int_0^{\pi/3} (4\cos^2\theta + 8\cos\theta - 5)\,d\theta.$$

As in Example 1, the integral may be evaluated by using the substitution $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$. It can be shown that

$$A = \tfrac{9}{2}\sqrt{3} - \pi \approx 4.65.$$

If a curve C is the graph of a polar equation $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we can find its length L by using parametric equations. Thus, as in the proof of Theorem (9.11), a parametrization for C is

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta; \quad \alpha \le \theta \le \beta.$$

Differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta$$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta.$$

Using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, we can show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

Substitution in Theorem (9.6) with $t = \theta$, $a = \alpha$, and $b = \beta$ gives us

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

As an aid to remembering this formula, we may use the differential of arc length $ds = \sqrt{(dx)^2 + (dy)^2}$ in (9.7). The preceding manipulations give us the following.

***Differential of Arc Length in Polar Coordinates* 9.14**

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

We may now write the formula for L as

$$L = \int_{\theta=\alpha}^{\theta=\beta} ds.$$

The limits of integration specify that the independent variable is θ, not s.

EXAMPLE 3 Find the length of the cardioid $r = 1 + \cos\theta$.

SOLUTION The cardioid is sketched in Figure 9.40. Making use of symmetry, we shall find the length of the upper half and double the result. Applying (9.14), we have

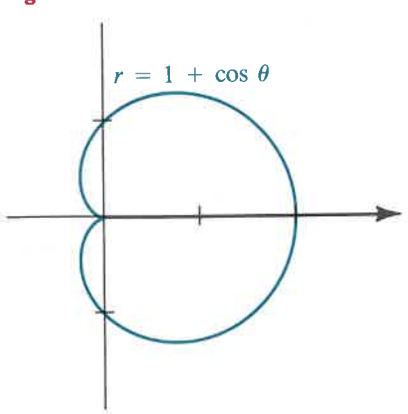


Figure 9.40

$$ds = \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= \sqrt{2}\sqrt{1+\cos\theta}\, d\theta.$$

Hence,

$$L = 2\int_{\theta=0}^{\theta=\pi} ds = 2\int_0^\pi \sqrt{2}\sqrt{1+\cos\theta}\, d\theta.$$

The last integral may be evaluated by employing the trigonometric identity $\cos^2\frac{1}{2}\theta = \frac{1}{2}(1+\cos\theta)$, or, equivalently, $1 + \cos\theta = 2\cos^2\frac{1}{2}\theta$. Thus,

$$L = 2\sqrt{2}\int_0^\pi \sqrt{2\cos^2\tfrac{1}{2}\theta}\, d\theta$$

$$= 4\int_0^\pi \cos\tfrac{1}{2}\theta\, d\theta$$

$$= 8\left[\sin\tfrac{1}{2}\theta\right]_0^\pi = 8.$$

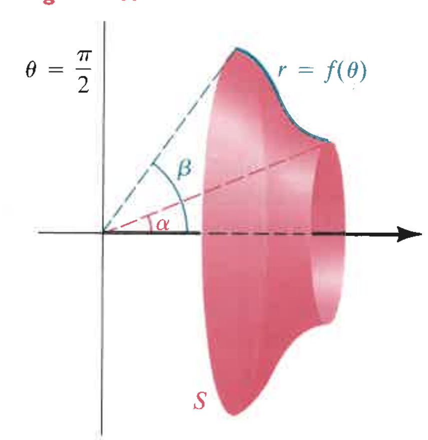


Figure 9.41

In the solution to Example 3, it was legitimate to replace $\sqrt{\cos^2\frac{1}{2}\theta}$ by $\cos\frac{1}{2}\theta$, because if $0 \le \theta \le \pi$, then $0 \le \frac{1}{2}\theta \le \pi/2$, and hence $\cos\frac{1}{2}\theta$ is *positive* on $[0, \pi]$. If we had *not* used symmetry, but had written L as $\int_0^{2\pi}\sqrt{r^2 + (dr/d\theta)^2}\, d\theta$, this simplification would not have been valid. Generally, in determining areas or arc lengths that involve polar coordinates, it is a good idea to use any symmetries that exist.

Let C be the graph of a polar equation $r = f(\theta)$ for $\alpha \le \theta \le \beta$. Let us obtain a formula for the area S of the surface generated by revolving C about the polar axis, as illustrated in Figure 9.41. Since parametric equations for C are

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta; \quad \alpha \le \theta \le \beta,$$

we may find S by using Theorem (9.8) with $\theta = t$. This gives us the following result, where the arc length differential ds is given by (9.14).

***Surfaces of Revolution in Polar Coordinates* 9.15**

About the polar axis: $S = \displaystyle\int_{\theta=\alpha}^{\theta=\beta} 2\pi y\, ds = \int_{\theta=\alpha}^{\theta=\beta} 2\pi r\sin\theta\, ds$

About the line $\theta = \pi/2$: $S = \displaystyle\int_{\theta=\alpha}^{\theta=\beta} 2\pi x\, ds = \int_{\theta=\alpha}^{\theta=\beta} 2\pi r\cos\theta\, ds$

When applying (9.15), *we must choose α and β so that the surface does not retrace itself when C is revolved*, as would be the case if the circle $r = \cos\theta$, with $0 \le \theta \le \pi$, were revolved about the polar axis.

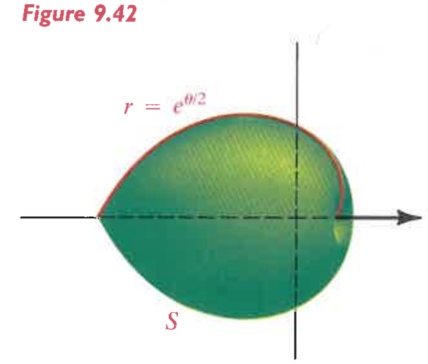


Figure 9.42

EXAMPLE 4 The part of the spiral $r = e^{\theta/2}$ from $\theta = 0$ to $\theta = \pi$ is revolved about the polar axis. Find the area of the resulting surface.

SOLUTION The surface is illustrated in Figure 9.42. By (9.14), the polar differential of arc length in polar coordinates is

$$ds = \sqrt{(e^{\theta/2})^2 + (\tfrac{1}{2}e^{\theta/2})^2}\, d\theta = \sqrt{\tfrac{5}{4}e^\theta}\, d\theta = \frac{\sqrt{5}}{2}e^{\theta/2}\, d\theta.$$

Hence, by (9.15),

$$S = \int_{\theta=0}^{\theta=\pi} 2\pi y\, ds = \int_{\theta=0}^{\theta=\pi} 2\pi r \sin\theta\, ds$$
$$= \int_0^{\pi} 2\pi e^{\theta/2} \sin\theta \left(\frac{\sqrt{5}}{2} e^{\theta/2}\right) d\theta$$
$$= \sqrt{5}\pi \int_0^{\pi} e^{\theta} \sin\theta\, d\theta.$$

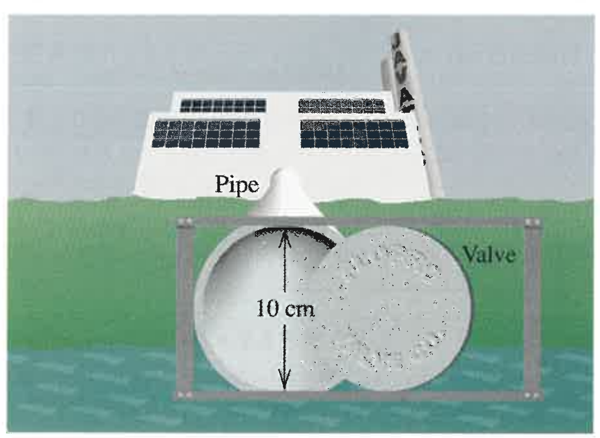
Using integration by parts or Formula 98 in the table of integrals (see Appendix II), we have

$$S = \frac{\sqrt{5}\pi}{2}\left[e^{\theta}(\sin\theta - \cos\theta)\right]_0^{\pi} = \frac{\sqrt{5}\pi}{2}(e^{\pi} + 1) \approx 84.8.$$

We have already seen many applications of calculus that involve the calculation of area. The next example illustrates an application in which the area is most naturally represented as an integral using polar coordinates.

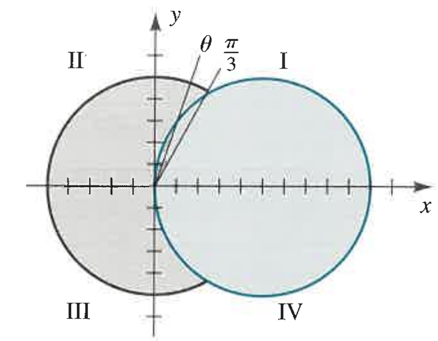
EXAMPLE ■ 5 An industrial plant releases its discharge through a circular pipe of diameter 10 cm. The flow is controlled by a valve consisting of a circular disk of the same diameter. Moving the valve back and forth across the pipe increases or decreases the rate of discharge. (See Figure 9.43.) If the flow of discharge is proportional to the area of the opening, what percentage of the maximum flow occurs when the center of the valve disk is 5 cm from the center of the pipe?

Figure 9.43



SOLUTION We set up a coordinate system with the pole at the center of the pipe. The polar coordinate equation of the pipe will then be $r = 5$. Since the radius of the valve disk is 5 cm, when its center is 5 cm from the center of the pipe, the center is at the point with rectangular coordinates $(5, 0)$. The polar coordinate equation for the valve disk is then

Figure 9.44



$r = 10\cos\theta$. The maximum flow is proportional to the area of the circular opening of the pipe, which is $\pi(5)^2$. With the valve blocking the flow, the actual flow is proportional to the shaded area shown in Figure 9.44.

To find this area A, we begin by noting that it is made up of the left semicircle of the pipe plus additional regions in quadrants I and IV. By symmetry, the regions in quadrants I and quadrant IV have the same area. Thus,

$$A = (\text{area of semicircle of radius } 5) + 2(\text{area of region in quadrant I}).$$

To find the area in quadrant I, we first determine the intersection point of the two circles. They intersect when $5 = 10\cos\theta$, or $\cos\theta = 1/2$. Thus, $\theta = \pi/3$ or $5\pi/3$. The first value, $\theta = \pi/3$, is the one in quadrant I.

The area of the region in quadrant I is the shaded area between the pipe and the valve as θ varies from $\pi/3$ to $\pi/2$. This area is

$$\int_{\pi/3}^{\pi/2} \tfrac{1}{2}[5^2 - (10\cos\theta)^2]\, d\theta$$
$$= \tfrac{1}{2}\int_{\pi/3}^{\pi/2} [25 - 100\cos^2\theta]\, d\theta$$
$$= \tfrac{1}{2}\int_{\pi/3}^{\pi/2} [25 - 50(1 + \cos 2\theta)]\, d\theta$$
$$= \tfrac{1}{2}\int_{\pi/3}^{\pi/2} [-25 - 50\cos 2\theta]\, d\theta = \tfrac{1}{2}\left[-25\theta - 25\sin 2\theta\right]_{\pi/3}^{\pi/2}$$
$$= \frac{1}{2}\left[\left(-\frac{25\pi}{2} - 0\right) - \left(-\frac{25\pi}{3} - \frac{25\sqrt{3}}{2}\right)\right]$$
$$= \frac{1}{2}\left(\frac{25\sqrt{3}}{2} - \frac{25\pi}{6}\right).$$

Thus, the total shaded area is

$$\frac{25\pi}{2} + 2\left[\frac{1}{2}\left(\frac{25\sqrt{3}}{2} - \frac{25\pi}{6}\right)\right] = 25\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right).$$

Since the entire area of the pipe is 25π, the fraction that is left uncovered is

$$\frac{25\left(\pi/3+\sqrt{3}/2\right)}{25\pi}=\frac{1}{3}+\frac{\sqrt{3}}{2\pi}\approx 0.61.$$

Hence, when the center of the valve is 5 cm from the center of the pipe, it restricts the flow to about 61% of the maximum possible flow.

EXERCISES 9.4

Exer. 1–6: Find the area of the region bounded by the graph of the polar equation.

1 $r=2\cos\theta$ **2** $r=5\sin\theta$

3 $r=1-\cos\theta$ **4** $r=6-6\sin\theta$

5 $r=\sin 2\theta$ **6** $r^2=9\cos 2\theta$

Exer. 7–8: Find the area of region *R*.

7 $R=\{(r,\theta): 0\le\theta\le\pi/2, 0\le r\le e^{\theta}\}$

8 $R=\{(r,\theta): 0\le\theta\le\pi, 0\le r\le 2\theta\}$

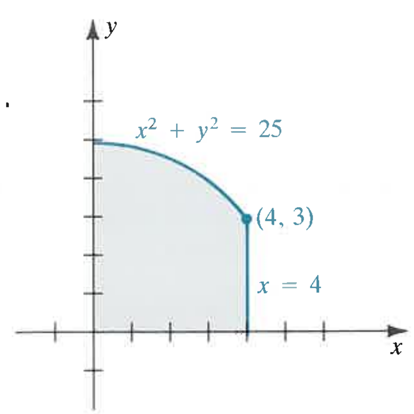
Exer. 9–12: Find the area of the region bounded by one loop of the graph of the polar equation.

9 $r^2=4\cos 2\theta$ **10** $r=2\cos 3\theta$

11 $r=3\cos 5\theta$ **12** $r=\sin 6\theta$

Exer. 13–16: Set up integrals in polar coordinates that can be used to find the area of the region shown in the figure.

13



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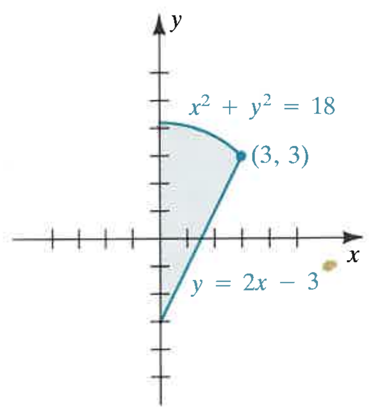

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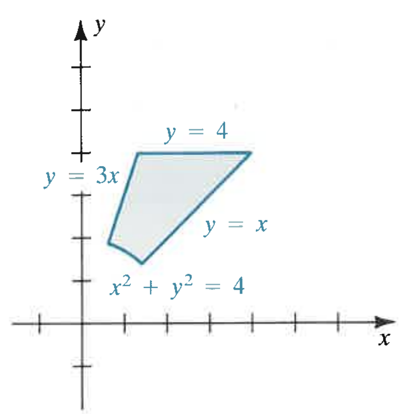

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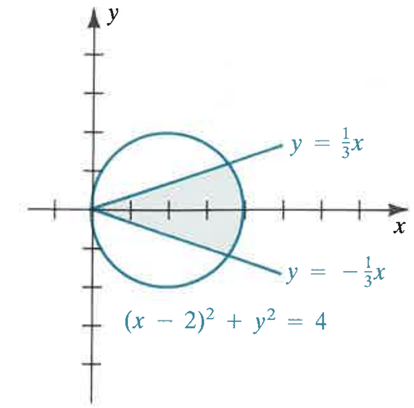
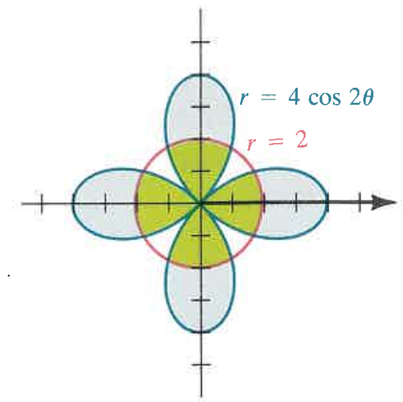


Exer. 17–18: Set up integrals in polar coordinates that can be used to find the area of (a) the blue region and (b) the green region.

17



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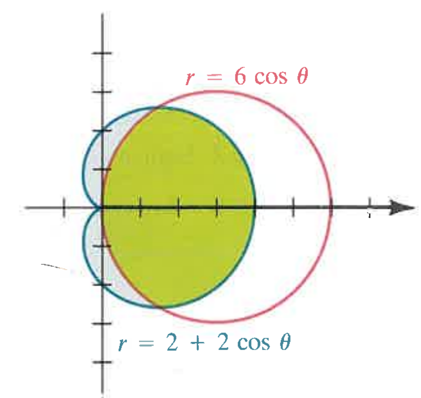


Exer. 19–22: Find the area of the region that is outside the graph of the first equation and inside the graph of the second equation.

19 $r=2+2\cos\theta,\quad r=3$

20 $r=2,\quad r=4\cos\theta$

21 $r=2,\quad r^2=8\sin 2\theta$

22 $r=1-\sin\theta,\quad r=3\sin\theta$

Exer. 23–26: Find the area of the region that is inside the graphs of *both* equations.

23 $r=\sin\theta,\quad r=\sqrt{3}\cos\theta$

24 $r=2(1+\sin\theta),\quad r=1$

25 $r=1+\sin\theta,\quad r=5\sin\theta$

26 $r^2=4\cos 2\theta,\quad r=1$

Exer. 27–32: Find the length of the curve.

27 $r=e^{-\theta}$ from $\theta=0$ to $\theta=2\pi$

28 $r=\theta$ from $\theta=0$ to $\theta=4\pi$

29 $r=\cos^2\frac{1}{2}\theta$ from $\theta=0$ to $\theta=\pi$

30 $r=2^{\theta}$ from $\theta=0$ to $\theta=\pi$

31 $r=\sin^3\frac{1}{3}\theta$

32 $r=2-2\cos\theta$

c **Exer. 33–34: Use Simpson's rule, with $n=2$, to approximate the length of the curve.**

33 $r=\theta+\cos\theta$ from $\theta=0$ to $\theta=\pi/2$

34 $r=\sin\theta+\cos^2\theta$ from $\theta=0$ to $\theta=\pi$

Exer. 35–38: Find the area of the surface generated by revolving the graph of the equation about the polar axis.

35 $r=2+2\cos\theta$ **36** $r^2=4\cos 2\theta$

37 $r=2a\sin\theta$ **38** $r=2a\cos\theta$

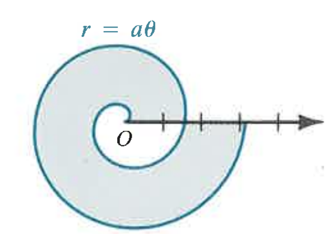
c **Exer. 39–40: Use the trapezoidal rule, with $n=4$, to approximate the area of the surface generated by revolving the graph of the polar equation about the line $\theta=\pi/2$. (Use symmetry when setting up the integral.)**

39 $r=\sin^2\theta$ **40** $r=\cos^2\theta$

41 A *torus* is the surface generated by revolving a circle about a nonintersecting line in its plane. Use polar coordinates to find the surface area of the torus generated by revolving the circle $x^2+y^2=a^2$ about the line $x=b$, where $0<a<b$.

42 Let OP be the ray from the pole to the point $P(r,\theta)$ on the spiral $r=a\theta$, where $a>0$. If the ray makes two revolutions (starting from $\theta=0$), find the area of the region swept out in the second revolution that was not swept out in the first revolution (see figure).

Exercise 42



43 The part of the spiral $r = e^{-\theta}$ from $\theta = 0$ to $\theta = \pi/2$ is revolved about the line $\theta = \pi/2$. Find the area of the resulting surface.

c **Exer. 44–45: Refer to Example 5.**

44 Determine how far from the center of the pipe the center of the valve disk should be in order to limit the flow to half the maximum possible flow.

45 Obtain a graph of the percentage of flow blocked as a function of the distance from the center of the valve to the center of the pipe.

9.5 TRANSLATION AND ROTATION OF AXES

In the study of plane curves, it is often helpful to consider new coordinate systems obtained by translating or rotating the original coordinate axes. In these new systems, the equation of a curve may be much simpler than it was in the original system. We define and illustrate the use of these translations and rotations in this section. One of our main goals is to show that the graph of the general second-degree equation in x and y

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is either a conic or a degenerate conic. You may wish to review the material on conic sections in the Precalculus Review.

TRANSLATION OF AXES

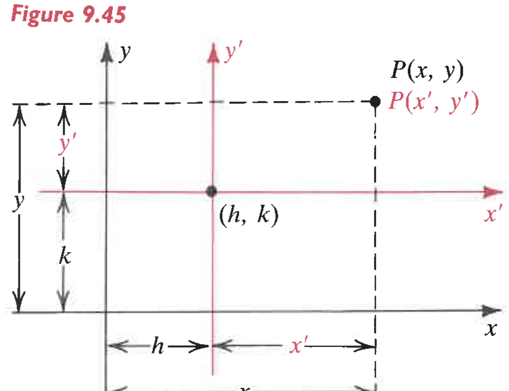
Figure 9.45 illustrates a **translation of axes**, where the x- and y-axes are shifted to positions—denoted by x' and y'—that are parallel to their original positions. Every point P in the plane then has two different ordered-pair representations: $P(x, y)$ in the xy-system and $P(x', y')$ in the $x'y'$-system. If the origin of the new $x'y'$-system has coordinates (h, k) in the xy-plane, as shown in Figure 9.45, we see that

$$x = x' + h \quad \text{and} \quad y = y' + k.$$

These formulas are true for all values of h and k. Equivalent formulas are

$$x' = x - h \quad \text{and} \quad y' = y - k.$$

Figure 9.45



We may summarize this discussion by the following.

Translation of Axes Formulas 9.16

If (x, y) are the coordinates of a point P in an xy-plane and if (x', y') are the coordinates of P in an $x'y'$-plane with origin at the point (h, k) of the xy-plane, then

(i) $x = x' + h, \quad y = y' + k$

(ii) $x' = x - h, \quad y' = y - k$

If a certain collection of points in the xy-plane is the graph of an equation in x and y, then to find an equation in x' and y' that has the same graph in the $x'y'$-plane, we substitute $x' + h$ for x and $y' + k$ for y. Conversely, if a set of points in the $x'y'$-plane is the graph of an equation in x' and y', then to find the corresponding equation in x and y, we substitute $x - h$ for x' and $y - k$ for y'.

To illustrate the use of (9.16), we begin by noting that

$$(x')^2 + (y')^2 = 1$$

is an equation of a circle of radius 1 with center at the origin O' of the $x'y'$-plane. Using translation of axes formulas (9.16)(ii), we see that

$$(x - h)^2 + (y - k)^2 = 1$$

is an equation of the same circle in the xy-plane with center (h, k).

The next example shows another application of translation of axes to simplify a second-degree equation.

EXAMPLE 1 Discuss and sketch the graph of

$$25x^2 + 250x - 16y^2 + 32y + 109 = 0.$$

SOLUTION We first complete the square in x and y by rewriting the equation as

$$25(x^2 + 10x) - 16(y^2 - 2y) = -109.$$

Then we add constant terms to obtain squares of binomials in x and y:

$$25(x^2 + 10x + 25) - 16(y^2 - 2y + 1) = -109 + 25(25) - 16(1),$$

which we can write as

$$25(x + 5)^2 - 16(y - 1)^2 = 500.$$

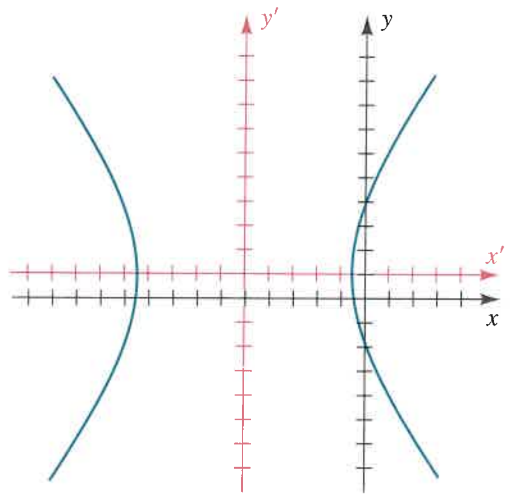
We now use the translation of axes formulas (9.16)(ii), with $x' = x + 5$ and $y' = y - 1$, in order to write the equation as

$$25(x')^2 - 16(y')^2 = 500$$

or, equivalently,

$$\frac{(x')^2}{20} - \frac{(y')^2}{\frac{125}{4}} = 1,$$

Figure 9.46
$25x^2 + 250x - 16y^2 + 32y + 109 = 0$,
$[(x')^2/20] - [4(y')^2/125] = 1$



which has the form

$$\frac{(x')^2}{a^2} - \frac{(y')^2}{b^2} = 1$$

with $a^2 = 20$ and $b^2 = \frac{125}{4}$. In this form, we recognize that the curve represented by the equation is a hyperbola with vertices on the x'-axis and center at the origin of the $x'y'$-coordinate system. From Theorem 38 in the Precalculus Review (page 75), the hyperbola has vertices with $x'y'$-coordinates $(\pm a, 0) = (\pm 2\sqrt{5}, 0)$. The foci have $x'y'$-coordinates $(\pm c, 0)$, where $c = \sqrt{a^2 + b^2} = \sqrt{20 + (125/4)} = \sqrt{205}/2$, and the asymptotes have equations $y' = \pm(b/a)x' = \pm(5/4)x'$. Figure 9.46 shows a sketch of the curve.

Since $x = x' - 5$ and $y = y' + 1$, we can translate the information about the center, the vertices, and the foci to xy-coordinates. The vertices, for example, have xy-coordinates $(-5 \pm 2\sqrt{5}, 1)$, and the asymptotes are the lines $(y - 1) = \pm\frac{5}{4}(x + 5)$.

In a similar manner, by completing the square and translating axes, we can replace

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

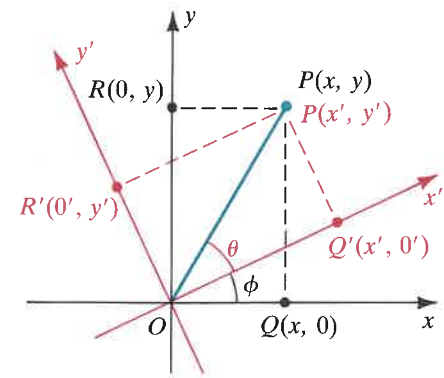
by an equivalent equation of the form

$$A'(x')^2 + B'x'y' + C'(y')^2 + F' = 0.$$

That is, we can eliminate the linear terms. Translation of axes, however, will still retain the second-degree term $x'y'$. To remove such a term, we must introduce rotation of axes.

ROTATION OF AXES

Figure 9.47



As we have seen, translation of axes gives a new coordinate system that may result in simpler representations for the equations of curves. Rotation of axes, which we now investigate, provides an additional way to build new coordinate systems in which we may obtain even further simplifications of such equations. We obtained the $x'y'$-plane used in a translation of axes by moving the origin O of the xy-plane to a new position $C(h, k)$ without changing the positive directions of the axes or the units of length. We now consider a new coordinate plane obtained by keeping the origin O fixed and rotating the x- and y-axes about O to another position, denoted by x' and y'. A transformation of this type is a **rotation of axes**.

Consider the rotation of axes in Figure 9.47, and let ϕ denote the acute angle through which the positive x-axis must be rotated in order to coincide with the positive x'-axis. If (x, y) are the coordinates of a point P relative to the xy-plane, then (x', y') will denote its coordinates relative to the new $x'y'$-plane.

Let the projection of P on the various axes be denoted as in Figure 9.47, and let θ denote angle POQ'. If $p = d(O, P)$, then

$$x' = p\cos\theta, \qquad y' = p\sin\theta$$
$$x = p\cos(\theta + \phi), \qquad y = p\sin(\theta + \phi).$$

Applying the addition formulas for the sine and cosine, we see that

$$x = p\cos\theta\cos\phi - p\sin\theta\sin\phi$$
$$y = p\sin\theta\cos\phi + p\cos\theta\sin\phi.$$

Using the fact that $x' = p\cos\theta$ and $y' = p\sin\theta$ gives us (i) of the next theorem. The formulas in (ii) may be obtained from (i) by solving for x' and y'.

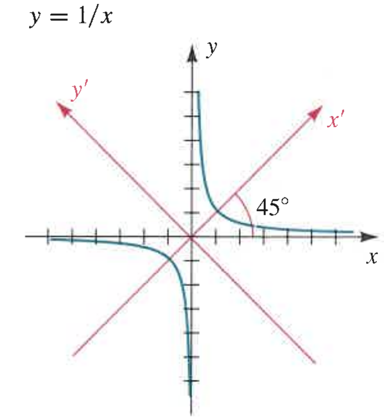
***Rotation of Axes Formulas* 9.17**

If the x- and y-axes are rotated about the origin O, through an acute angle ϕ, then the coordinates (x, y) and (x', y') of a point P in the xy- and $x'y'$-planes are related as follows:

(i) $x = x'\cos\phi - y'\sin\phi, \quad y = x'\sin\phi + y'\cos\phi$

(ii) $x' = x\cos\phi + y\sin\phi, \quad y' = -x\sin\phi + y\cos\phi$

Figure 9.48
$y = 1/x$



EXAMPLE 2 The graph of $xy = 1$, or, equivalently, $y = 1/x$, is sketched in Figure 9.48. If the coordinate axes are rotated through an angle of 45°, find an equation of the graph relative to the new $x'y'$-plane.

SOLUTION We let $\phi = 45°$ in rotation of axes formulas (9.17)(i):

$$x = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$

$$y = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y')$$

Substituting for x and y in the equation $xy = 1$ gives us

$$\frac{\sqrt{2}}{2}(x' - y') \cdot \frac{\sqrt{2}}{2}(x' + y') = 1.$$

This equation reduces to

$$\frac{(x')^2}{2} - \frac{(y')^2}{2} = 1,$$

which is an equation of a hyperbola with vertices $(\pm\sqrt{2}, 0)$ on the x'-axis. Note that the asymptotes for the hyperbola have equations $y' = \pm x'$ in the new system. These correspond to the original x- and y-axes.

Example 2 illustrates a method for eliminating a term of an equation that contains the product xy. This method can be used to transform any equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where $B \neq 0$, into an equation in x' and y' that contains no $x'y'$-term. Let us prove that this may always be done. If we rotate the axes through an angle ϕ, then using rotation of axes formulas (9.17)(i) to substitute for x and y gives us

$$\begin{aligned} &A\,(x'\cos\phi - y'\sin\phi)^2 + B(x'\cos\phi - y'\sin\phi)(x'\sin\phi + y'\cos\phi) \\ &+ C(x'\sin\phi + y'\cos\phi)^2 + D(x'\cos\phi - y'\sin\phi) \\ &+ E(x'\sin\phi + y'\cos\phi) + F \\ &= 0. \end{aligned}$$

By performing the multiplications and rearranging terms, we may write this equation in the form

$$A'(x')^2 + B'x'y' + C'(y')^2 + D'x' + E'y' + F' = 0$$

with

$$\begin{aligned} A' &= A\cos^2\phi + B\cos\phi\sin\phi + C\sin^2\phi \\ B' &= 2(C - A)\sin\phi\cos\phi + B(\cos^2\phi - \sin^2\phi) \\ C' &= A\sin^2\phi - B\sin\phi\cos\phi + C\cos^2\phi \\ D' &= D\cos\phi + E\sin\phi \\ E' &= -D\sin\phi + E\cos\phi \\ F' &= F. \end{aligned}$$

To eliminate the $x'y'$-term, we must select ϕ such that $B' = 0$—that is,

$$2(C - A)\sin\phi\cos\phi + B(\cos^2\phi - \sin^2\phi) = 0.$$

Using double-angle formulas, we may write this equation as

$$(C - A)\sin 2\phi + B\cos 2\phi = 0,$$

which is equivalent to

$$\cot 2\phi = \frac{A - C}{B}.$$

This formulation proves the next result.

Theorem 9.18

To eliminate the xy-term from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where $B \neq 0$, choose an angle ϕ such that

$$\cot 2\phi = \frac{A - C}{B} \quad \text{with} \quad 0 < 2\phi < \pi$$

and use the rotation of axes formulas.

The graph of any equation in x and y of the type displayed in the preceding theorem is a conic, except for certain degenerate cases.

In using Theorem (9.18), note that $\sin 2\phi > 0$, since $0 < 2\phi < \pi$. Moreover, because $\cot 2\phi = \cos 2\phi / \sin 2\phi$, the signs of $\cot 2\phi$ and $\cos 2\phi$ are always the same.

EXAMPLE 3 Discuss and sketch the graph of the equation

$$41x^2 - 24xy + 34y^2 - 25 = 0.$$

SOLUTION Use the notation of Theorem (9.18):

$$A = 41, \qquad B = -24, \qquad C = 34$$

$$\cot 2\phi = \frac{41 - 34}{-24} = -\frac{7}{24}$$

Since $\cot 2\phi$ is negative, we choose 2ϕ such that $\pi/2 < 2\phi < \pi$, and consequently, $\cos 2\phi = -\frac{7}{25}$. We now use the half-angle formulas to obtain

$$\sin\phi = \sqrt{\frac{1 - \cos 2\phi}{2}} = \sqrt{\frac{1 - (-\frac{7}{25})}{2}} = \frac{4}{5}$$

$$\cos\phi = \sqrt{\frac{1 + \cos 2\phi}{2}} = \sqrt{\frac{1 + (-\frac{7}{25})}{2}} = \frac{3}{5}.$$

Thus, the desired rotation of axes formulas are

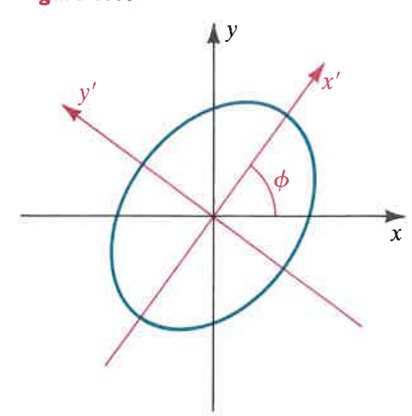
$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y' \quad \text{and} \quad y = \tfrac{4}{5}x' + \tfrac{3}{5}y'.$$

After substituting for x and y in the given equation and simplifying, we obtain the equation

$$(x')^2 + 2(y')^2 = 1.$$

The graph therefore is an ellipse with vertices at $(\pm 1, 0)$ on the x'-axis. Since $\tan\phi = \sin\phi/\cos\phi = (\frac{4}{5})/(\frac{3}{5}) = \frac{4}{3}$, we obtain $\phi = \tan^{-1}(\frac{4}{3})$. The angle ϕ is approximately 0.927 radian; to the nearest minute, $\phi \approx 53°8'$. The graph is sketched in Figure 9.49.

Figure 9.49



The next theorem states rules that we can apply to identify the type of conic *before* rotating the axes.

Identification Theorem 9.19

> The graph of the equation
> $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
> is either a conic or a degenerate conic. If the graph is a conic, then it is
> **(i)** a parabola if $B^2 - 4AC = 0$
> **(ii)** an ellipse if $B^2 - 4AC < 0$
> **(iii)** a hyperbola if $B^2 - 4AC > 0$

PROOF If the x- and y-axes are rotated through an angle ϕ, using the rotation of axes formulas gives us

$$A'(x')^2 + B'x'y' + C'(y')^2 + D'x' + E'y' + F' = 0.$$

Using the formulas for A', B', and C' on page 836, we can show that

$$(B')^2 - 4A'C' = B^2 - 4AC.$$

For a suitable rotation of axes, we obtain $B' = 0$ and

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0.$$

Except for degenerate cases, the graph of this equation is an ellipse if $A'C' > 0$ (A' and C' have the same sign), a hyperbola if $A'C' < 0$ (A' and C' have opposite signs), or a parabola if $A'C' = 0$ (either $A' = 0$ or $C' = 0$). However, if $B' = 0$, then $B^2 - 4AC = -4A'C'$, and hence the graph is an ellipse if $B^2 - 4AC < 0$, a hyperbola if $B^2 - 4AC > 0$, or a parabola if $B^2 - 4AC = 0$. ■

The expression $B^2 - 4AC$ is called the **discriminant** of the equation in the identification theorem (9.19). We say that this discriminant is **invariant** under a rotation of axes, because it is unchanged by any such rotation.

EXAMPLE ■ 4 Use the identification theorem (9.19) to determine if the graph of the equation

$$41x^2 - 24xy + 34y^2 - 25 = 0$$

is a parabola, an ellipse, or a hyperbola.

SOLUTION We considered this equation in Example 3, where we performed a rotation of axes. Since $A = 41$, $B = -24$, and $C = 34$, the discriminant is

$$B^2 - 4AC = 576 - 4(41)(34) = -5000 < 0.$$

Hence, by the identification theorem, the graph is an ellipse.

In some cases, after eliminating the xy-term, it may be necessary to translate the axes of the $x'y'$-coordinate system to obtain the graph, as illustrated in the next example.

EXAMPLE ■ 5 Discuss and sketch the graph of the equation

$$x^2 + 2\sqrt{3}xy + 3y^2 + 8\sqrt{3}x - 8y + 32 = 0.$$

SOLUTION Using $A = 1$, $B = 2\sqrt{3}$, and $C = 3$, we see that

$$B^2 - 4AC = 12 - 12 = 0.$$

By the identification theorem (9.19), the graph is a parabola.

To apply a rotation of axes, we calculate

$$\cot 2\phi = \frac{A - C}{B} = \frac{1 - 3}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

Hence $2\phi = 2\pi/3$, $\phi = \pi/3$, and

$$\sin\phi = \frac{\sqrt{3}}{2}, \qquad \cos\phi = \frac{1}{2}.$$

The rotation of axes formulas (9.17)(i) are as follows:

$$x = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' = \frac{1}{2}(x' - \sqrt{3}y')$$

$$y = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \frac{1}{2}(\sqrt{3}x' + y')$$

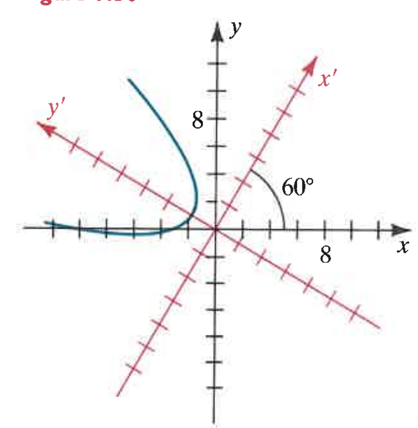
Substituting for x and y in the given equation and simplifying leads to

$$4(x')^2 - 16y' + 32 = 0,$$

or, equivalently, $$(x')^2 = 4(y' - 2).$$

The parabola is sketched in Figure 9.50, where each tic represents two units. Note that the vertex is at the point $(0, 2)$ in the $x'y'$-plane, and the graph is symmetric with respect to the y'-axis.

Figure 9.50



EXERCISES 9.5

Exer. 1–6: Find the vertices and the foci of the conic, and use the translation of axes formulas to sketch its graph.

1. $y^2 - 8x + 8y + 32 = 0$
2. $x = 2y^2 + 8y + 3$
3. $4x^2 + 9y^2 + 24x - 36y + 36 = 0$
4. $3x^2 + 4y^2 - 18x + 8y + 19 = 0$
5. $x^2 - 9y^2 + 8x + 90y - 210 = 0$
6. $4x^2 - y^2 - 40x - 8y + 88 = 0$

Exer. 7–19: (a) Use the identification theorem (9.19) to determine whether the graph of the equation is a parabola, an ellipse, or a hyperbola. (b) Use a suitable rotation of axes to find an equation for the graph in an $x'y'$-plane, and sketch the graph, labeling vertices.

7. $x^2 - 2xy + y^2 - 2\sqrt{2}x - 2\sqrt{2}y = 0$
8. $x^2 - 2xy + y^2 + 4x + 4y = 0$
9. $5x^2 - 8xy + 5y^2 = 9$
10. $x^2 - xy + y^2 = 3$

11 $11x^2 + 10\sqrt{3}xy + y^2 = 4$

12 $7x^2 - 48xy - 7y^2 = 225$

13 $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

14 $x^2 + 4xy + 4y^2 + 6\sqrt{5}x - 18\sqrt{5}y + 45 = 0$

15 $40x^2 - 36xy + 25y^2 - 8\sqrt{13}x - 12\sqrt{13}y = 0$

16 $18x^2 - 48xy + 82y^2 + 6\sqrt{10}x + 2\sqrt{10}y - 80 = 0$

17 $5x^2 + 6\sqrt{3}xy - y^2 + 8x - 8\sqrt{3}y - 12 = 0$

18 $15x^2 + 20xy - 4\sqrt{5}x + 8\sqrt{5}y - 100 = 0$

19 $32x^2 - 72xy + 53y^2 = 80$

c **Exer. 20–22: Graph the equation.**

20 $1.1x^2 - 1.3xy + y^2 - 2.9x - 1.9y = 0$

21 $2.1x^2 - 4xy + 1.5y^2 - 4x + y - 1 = 0$

22 $3.2x^2 - 4\sqrt{2}xy + 2.5y^2 + 2.1y + 3x - 2.1 = 0$

CHAPTER 9 REVIEW EXERCISES

Exer. 1–4: (a) Find an equation in x and y whose graph contains the points on the curve C. (b) Sketch the graph of C and indicate the orientation.

1 $x = \dfrac{1}{t} + 1, \quad y = \dfrac{2}{t} - t; \quad 0 < t \le 4$

2 $x = \cos^2 t - 2, \quad y = \sin t + 1; \quad 0 \le t \le 2\pi$

3 $x = \sqrt{t}, \quad y = 2^{-t}; \quad t \ge 0$

4 $x = 3\cos t + 2, \quad y = -3\sin t - 1; \quad 0 \le t \le 2\pi$

Exer. 5–6: Sketch the graphs of C_1, C_2, C_3, and C_4, and indicate their orientations.

5 $C_1: x = t, \quad y = \sqrt{16 - t^2}; \quad -4 \le t \le 4$
$C_2: x = -\sqrt{16 - t}, \quad y = -\sqrt{t}; \quad 0 \le t \le 16$
$C_3: x = 4\cos t, \quad y = 4\sin t; \quad 0 \le t \le 2\pi$
$C_4: x = e^t, \quad y = -\sqrt{16 - e^{2t}}; \quad t \le \ln 4$

6 $C_1: x = t^2, \quad y = t^3; \quad t \text{ in } \mathbb{R}$
$C_2: x = t^4, \quad y = t^6; \quad t \text{ in } \mathbb{R}$
$C_3: x = e^{2t}, \quad y = e^{3t}; \quad t \text{ in } \mathbb{R}$
$C_4: x = 1 - \sin^2 t, \quad y = \cos^3 t; \quad t \text{ in } \mathbb{R}$

Exer. 7–8: Let C be the given parametrized curve. (a) Express dy/dx in terms of t. (b) Find the values of t that correspond to horizontal or vertical tangent lines to the graph of C. (c) Express d^2y/dx^2 in terms of t.

7 $x = t^2, \quad y = 2t^3 + 4t - 1; \quad t \text{ in } \mathbb{R}$

8 $x = t - 2\sin t, \quad y = 1 - 2\cos t; \quad t \text{ in } \mathbb{R}$

Exer. 9–26: Sketch the graph of the polar equation.

9 $r = -4\sin\theta$

10 $r = 10\cos\theta$

11 $r = 6 - 3\cos\theta$

12 $r = 3 + 2\cos\theta$

13 $r^2 = 9\sin 2\theta$

14 $r^2 = -4\sin 2\theta$

15 $r = 3\sin 5\theta$

16 $r = 2\sin 3\theta$

17 $2r = \theta$

18 $r = e^{-\theta}, \quad \theta \ge 0$

19 $r = 8\sec\theta$

20 $r(3\cos\theta - 2\sin\theta) = 6$

21 $r = 4 - 4\cos\theta$

22 $r = 4\cos^2 \frac{1}{2}\theta$

23 $r = 6 - r\cos\theta$

24 $r = 6\cos 2\theta$

25 $r = \dfrac{8}{3 + \cos\theta}$

26 $r = \dfrac{8}{1 - 3\sin\theta}$

Exer. 27–32: Find a polar equation that has the same graph as the given equation.

27 $y^2 = 4x$

28 $x^2 + y^2 - 3x + 4y = 0$

29 $2x - 3y = 8$

30 $x^2 + y^2 = 2xy$

31 $y^2 = x^2 - 2x$

32 $x^2 = y^2 + 3y$

Exer. 33–38: Find an equation in x and y that has the same graph as the polar equation.

33 $r^2 = \tan\theta$

34 $r = 2\cos\theta + 3\sin\theta$

35 $r^2 = 4\sin 2\theta$

36 $r^2 = \sec 2\theta$

37 $\theta = \sqrt{3}$

38 $r = -6$

Exer. 39–40: Find the slope of the tangent line to the graph of the polar equation at the point corresponding to the given value of θ.

39 $r = \dfrac{3}{2 + 2\cos\theta}; \quad \theta = \pi/2$

40 $r = e^{3\theta}; \quad \theta = \pi/4$

41 Find the area of the region bounded by one loop of $r^2 = 4\sin 2\theta$.

42 Find the area of the region that is inside the graph of $r = 3 + 2\sin\theta$ and outside the graph of $r = 4$.

43 The position (x, y) of a moving point at time t is given by $x = 2\sin t$, $y = \sin^2 t$. Find the distance that the point travels from $t = 0$ to $t = \pi/2$.

44 Find the length of the spiral $r = 1/\theta$ from $\theta = 1$ to $\theta = 2$.

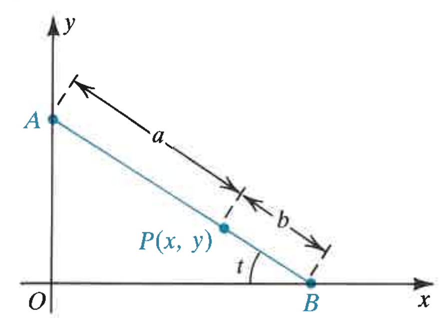
45 The curve with parametrization $x = 2t^2 + 1$, $y = 4t - 3$; $0 \le t \le 1$ is revolved about the y-axis. Find the area of the resulting surface.

46 The arc of the spiral $r = e^\theta$ from $\theta = 0$ to $\theta = 1$ is revolved about the line $\theta = \pi/2$. Find the area of the resulting surface.

47 Find the area of the surface generated by revolving the lemniscate $r^2 = a^2\cos 2\theta$ about the polar axis.

48 A line segment of fixed length has endpoints A and B on the y-axis and x-axis, respectively. A fixed point P on AB is selected with $d(A, P) = a$ and $d(B, P) = b$ (see figure). If A and B may slide freely along their respective axes, a curve C is traced by P. If t is the radian measure of angle ABO, find parametric equations for C with parameter t and describe C.

Exercise 48



Exer. 49–51: Find the vertices and the foci of the conic, and use the translation of axes formulas to sketch its graph.

49 $y^2 - 2y + 4x - 7 = 0$

50 $4x^2 + y^2 - 24x + 4y + 36 = 0$

51 $x^2 - 9y^2 + 8x + 7 = 0$

52 Use the discriminant to identify the graph of each equation. (Do not sketch the graph.)

(a) $2x^2 - 3xy + 4y^2 + 6x - 2y - 6 = 0$

(b) $3x^2 + 2xy - y^2 - 2x + y + 4 = 0$

(c) $x^2 - 6xy + 9y^2 + x - 3y + 5 = 0$

Exer. 53–54: Use a suitable rotation of axes to find an equation for the graph in an $x'y'$-plane, and sketch the graph, labeling vertices.

53 $x^2 - 8xy + 16y^2 - 12\sqrt{17}x - 3\sqrt{17}y = 0$

54 $8x^2 + 12xy + 17y^2 - 16\sqrt{5}x - 12\sqrt{5}y = 0$

EXTENDED PROBLEMS AND GROUP PROJECTS

1 Investigate the representation of conic sections in polar coordinates.

(a) Let F be a fixed point and l a fixed line in a plane. Prove that the set of all points P in the plane, such that the ratio $d(P, F)/d(P, Q)$ is a positive constant e with $d(P, Q)$ the distance from P to l, is a conic section. Show that the conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$.

(b) Prove the following theorem: A polar equation that has one of the four forms

$$r = \frac{de}{1 \pm e\cos\theta}, \qquad r = \frac{de}{1 \pm e\sin\theta}$$

is a conic section. Show that the conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, or a hyperbola if $e > 1$.

(c) Describe and sketch the graph of each of the following polar equations:

(i) $r = 10/(3 + 2\cos\theta)$

(ii) $r = 10/(2 + 3\sin\theta)$

(iii) $r = 15/(4 - 4\cos\theta)$

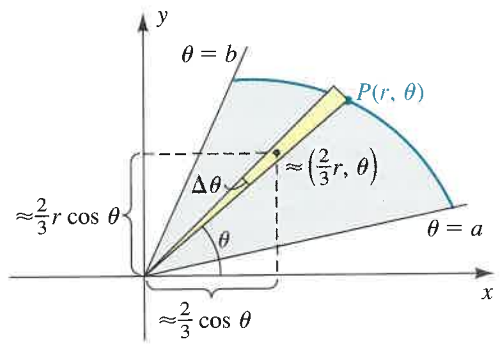
(d) Find a polar equation of the conic with a focus at the pole, eccentricity $e = 1/2$, and directrix $r = -3\sec\theta$.

2 Develop formulas in polar coordinates for the center of mass using the following as an outline for a possible approach.

(a) Show that the center of mass of a triangle is located on each median, two thirds of the way from a vertex to the opposite side.

(b) Consider a thin triangle as shown in the figure. Discuss why it is reasonable to assume that its center of mass has polar coordinates that are approximately $(\frac{2}{3}r, \theta)$.

Problem 2



(c) Consider a region in the plane bounded by the lines $\theta = a$, $\theta = b$, and the curve $r = f(\theta)$. Slice the region into triangular wedges. Show that the moment of the region about the x-axis is approximately $\sum \frac{1}{3}r^3 \sin\theta\, \Delta\theta$.

(d) Show that the sum in part (c) is a Riemann sum and find the form for the definite integral that is the limit as $\Delta\theta \to 0$.

(e) Show that the coordinates of the center of mass are given by

$$\bar{x} = \frac{\int_a^b \frac{2}{3}r^3 \cos\theta\, d\theta}{\int_a^b r^2\, d\theta}, \qquad \bar{y} = \frac{\int_a^b \frac{2}{3}r^3 \sin\theta\, d\theta}{\int_a^b r^2\, d\theta}.$$

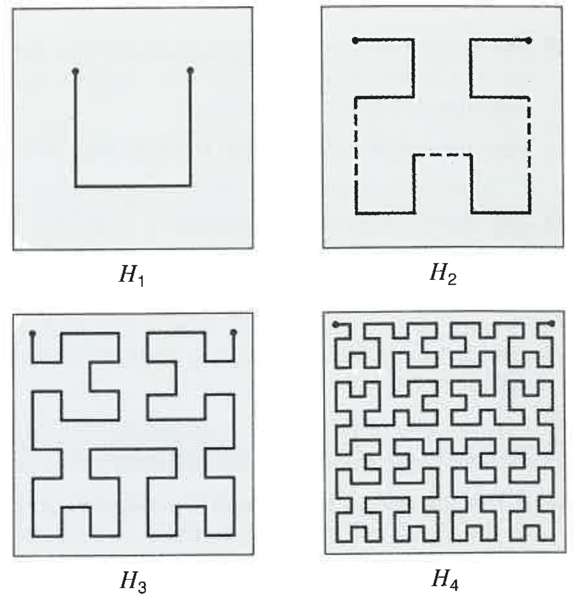
(f) Show that the formulas in part (e) give the correct answer if the region is a circle centered at the origin.

(g) Find the center of mass of the region enclosed by a semicircle of radius a.

(h) Find the center of mass of the region enclosed by the cardioid $r = a(1 + \sin\theta)$.

3 Investigate *space-filling curves*, which are curves whose graphs fill a solid region of the plane. The discovery of such curves in the late nineteenth century dramatically revealed that our intuitive understanding of the "one dimensionality" of curves was deeply flawed. This revelation led to an intensive research into a precise definition of *dimension* and the properties of curves that has actively continued to the present.

Problem 3



The first such curve was discovered in 1890 by Giuseppe Peano (1858–1932).* In this problem, we consider an example published in 1891 by David Hilbert (1862–1943). The Hilbert curve is the limit of a sequence of curves $H_1, H_2, H_3, \ldots$, each of which is contained in the unit square S of side 1. The first four curves in this sequence are shown in the figure.

(a) Examine the first stage, H_1, which is made up of three line segments. Show that the curve H_1 can be given a simple parametric representation, $(f_1(t), g_1(t))$, for $0 \le t \le 1$.

(b) Show that if the unit square is divided into four congruent squares, then H_1 passes through the center of each of these squares.

(c) Show that every point in the unit square is within a distance of $\sqrt{2}/4$ units of a point on the curve H_1.

To obtain H_2, a smaller copy of H_1 is placed in each of the four small squares together with extending segments, connecting the copies of H_1 to form a curve made up of line segments.

*For a description of Peano's curve and other attempts to create space-filling curves, see Heinz-Otto Peitgen, Hartmut Jurgens, and Dietmar Saupe, *Fractals for the Classroom, Part One: Introduction to Fractals and Chaos*, New York: Springer-Verlag, 1991.

(d) Determine the number of line segments in H_2. Show that it is possible to give a parametric representation of H_2. Show that if the unit square is divided into 16 congruent smaller squares, then H_2 passes through the center of each of these squares, and that every point in the unit square is within $\sqrt{2}/8$ units of a point on H_2.

At each stage in the construction, the unit square is divided into four congruent subsquares and a shrunken copy of H_n is placed inside each of the four subsquares; these are joined to form H_{n+1}.

(e) Show that if the unit square is divided into 4^n congruent subsquares, then the curve H_n passes through the center of each of these subsquares. Show that each point of the unit square is within $\sqrt{2}/2^{n+1}$ units of a point on H_n. Show that each H_n has a parametric representation $(f_n(t), g_n(t))$ for $0 \le t \le 1$.

(f) Let $H = \lim_{n\to\infty} H_n$. Show that H has a parametric representation $(f(t), g(t))$, where $f(t) = \lim_{n\to\infty} f_n(t)$ and $g(t) = \lim_{n\to\infty} g_n(t)$. Prove that $f(t)$ and $g(t)$ are continuous functions and that the graph of H passes through every point in the unit square.