**Journal Title**: Calculus of a single variable /

**ISSN**:

**Volume**:
**Issue**:
**Month/Year**:
**Pages**: 82-141

**Article Author**:

**Article Title**: Chapter 1: Limits and Continuity

**Imprint**:

**Deliver to Middlebury College Patron:**

-Save to C:/Ariel Scan as a PDF
-Run Odyssey Helper
-Switch to Process Type: Document Delivery
-Process
-Switch back to Lending before closing.

**Call #:** **Laura's Desk**

**Location:**

**Item #:**

Michael Olinick (molinick)
Department of Mathematic s
Warner Hall
Middlebury, VT 05753

# INTRODUCTION

**IN THE GRACEFUL FLOW** of a river, the smooth passage of time, and the majestic ascent of a hot air balloon, we perceive *continuity* of motion. There are no abrupt changes, no jumping over intermediate points in the movement. Through the idea of limits, calculus provides the means to study continuity rigorously.

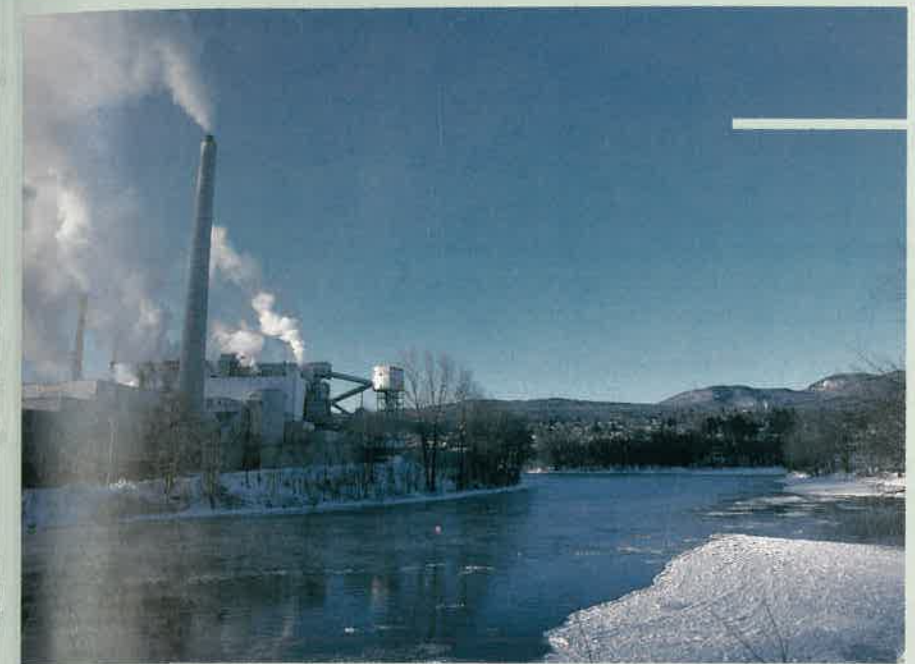
The concept of *limit* is the central idea of calculus. All the fundamental notions — continuous functions, derivatives, integrals, and convergent series — are limits in one sense or another. Limits are also essential for understanding the geometric properties of curves and surfaces, such as length, slope, area, and volume. The principal features of motion — velocity, speed, acceleration — are best comprehended as limits as well.

Paradoxes about limits play a vital role in the history of thought. Zeno (495 – 435 B.C.) posed many vexing paradoxes. In *Achilles and the Tortoise,* Zeno argues that a swift runner (Achilles) cannot overtake a slow opponent (the tortoise) if the latter has a head start. While Achilles runs to the tortoise's initial position, the tortoise moves forward to a new spot. While Achilles races to that spot, the tortoise moves to a further position. This process continues indefinitely, and Achilles always remains behind the tortoise! The paradox challenges our common sense that given enough time, a fast runner will pass a slower one. At the heart of the paradox lie sequences of numbers (the positions of the two racers) and the limit of these sequences.

As calculus developed in the eighteenth century, mathematicians treated the limit concept intuitively: Limits exist if a function's outputs get close to some value as its inputs get close to another value. In Section 1.1, we explore this intuitive definition, flawed by its use of the imprecise word *close.* A scientist may consider a measurement as being close to a value $L$ if it is within $10^{-6}$ cm of $L$. A marathon runner is close to the finish line when 100 yd are left in the race. An astronomer may measure closeness in terms of light-years.

To avoid ambiguity, we require a definition of limit not containing the word *close.* In Section 1.2, we present the traditional *$\epsilon - \delta$ definition of limit of a function.* This definition is precise and applicable to every situation we consider. The formal definition leads to theorems we use to determine the values of limits or to verify that a limit does not exist. The theorems give us techniques to find many limits, without applying the $\epsilon - \delta$ definition directly. Section 1.3 examines some of these techniques.

We consider in Section 1.4 limits of a function $f(x)$ where either $|x|$ or $|f(x)|$ grows unboundedly large. Finally, we use limits to define *continuous functions,* a concept used extensively throughout calculus.



*Limits provide a powerful tool for the mathematical analysis of objects, such as a river, that flow in a continuous manner.*

CHAPTER · 1

# Limits and Continuity

## 1.1 INTRODUCTION TO LIMITS

In calculus and its applications, we are often interested in function values $f(x)$ of a function $f$ when $x$ is *close* to a number $a$, *but not necessarily equal to* $a$. In fact, there are many instances where $a$ is not in the domain of $f$; that is, $f(a)$ is undefined. For example, consider

$$f(x) = \frac{x^3 - 2x^2}{3x - 6}$$

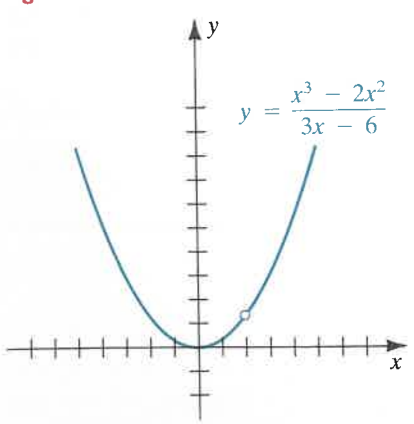
with $a = 2$. Note that 2 is not in the domain of $f$, since substituting $x = 2$ gives us the undefined expression 0/0. The following table, obtained with a calculator, lists some function values (to eight-decimal-place accuracy) for $x$ close to 2.

| $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|---|---|
| 1.97 | 1.293633333 | 1.9997 | 1.332933363 | 1.999997 | 1.333329333 |
| 1.98 | 1.306800000 | 1.9998 | 1.333066680 | 1.999998 | 1.333330667 |
| 1.99 | 1.320033333 | 1.9999 | 1.333200003 | 1.999999 | 1.333332000 |
| 2.01 | 1.346700000 | 2.0001 | 1.333466670 | 2.000001 | 1.333334667 |
| 2.02 | 1.360133333 | 2.0002 | 1.333600013 | 2.000002 | 1.333336000 |
| 2.03 | 1.373633333 | 2.0003 | 1.333733363 | 2.000003 | 1.333337333 |

It *appears* from the table that the closer $x$ is to 2, the closer $f(x)$ is to $\frac{4}{3}$; however, we cannot be certain of this because we have merely calculated several function values for $x$ near 2. To give a more convincing argument, let us factor the numerator and the denominator of $f(x)$ as follows:

$$f(x) = \frac{x^2(x-2)}{3(x-2)}$$

Figure 1.1




If $x \neq 2$, then we may cancel the common factor $x - 2$ and observe that $f(x)$ is given by $\frac{1}{3}x^2$. Thus the graph of $f$ is the parabola $y = \frac{1}{3}x^2$ with the point $(2, \frac{4}{3})$ deleted, as shown in Figure 1.1. It is geometrically evident that as $x$ gets closer to 2, $f(x)$ gets closer to $\frac{4}{3}$, as indicated in the preceding table.

In general, if a function $f$ is defined throughout an open interval containing a real number $a$, except possibly at $a$ itself, we may ask the following questions:

**1.** As $x$ gets closer to $a$ (but $x \neq a$), does the function value $f(x)$ get closer to some real number $L$?

**2.** Can we make the function value $f(x)$ *as close to* $L$ *as desired* by choosing $x$ sufficiently close to $a$ (but $x \neq a$)?



If the answers to these questions are *yes*, we use the notation

$$\lim_{x \to a} f(x) = L$$

and say that *the limit of* $f(x)$, *as* $x$ *approaches* $a$, *is* $L$, or that $f(x)$ *approaches* $L$ *as* $x$ *approaches* $a$. We may also write

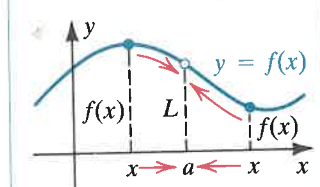
$$f(x) \to L \quad \text{as} \quad x \to a.$$

Thus the point $(x, f(x))$ on the graph of $f$ approaches the point $(a, L)$ as $x$ approaches $a$. Using the limit notation, we denote the result in our example as follows:

$$\lim_{x \to 2} \frac{x^3 - 2x^2}{3x - 6} = \frac{4}{3}$$

Note that in this section we define *limit* using the phrases *close to* and *approaches* in an intuitive manner; the next section contains a formal definition of limit that avoids this terminology. This discussion of the intuitive meaning of limit may be summarized as follows.

**Limit of a Function 1.1**

| Notation | Intuitive meaning | Graphical interpretation |
|---|---|---|
| $\lim_{x \to a} f(x) = L$ | We can make $f(x)$ as close to $L$ as desired by choosing $x$ sufficiently close to $a$, and $x \neq a$. |  |

If $f(x)$ approaches some number as $x$ approaches $a$, but we do not know what that number is, we use the phrase $\lim_{x \to a} f(x)$ *exists*.

The graph of $f$ shown in (1.1) illustrates only one way in which $f(x)$ might approach $L$ as $x$ approaches $a$. We have used an open circle, or "hole," in the graph rather than a point with $x$-coordinate $a$ because, when using the limit concept given in (1.1), we always assume that $x \neq a$; that is, *the function value* $f(a)$ *is completely irrelevant*. As we shall see, $f(a)$ may be different from $L$, may equal $L$, or may not exist, depending on the nature of the function $f$.

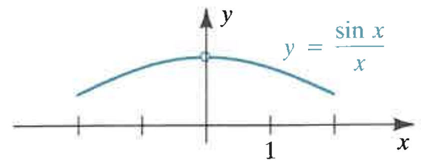
In our discussion of $f(x) = (x^3 - 2x^2)/(3x - 6)$, it was possible to simplify $f(x)$ by factoring the numerator and the denominator. In many cases, such algebraic simplifications are impossible. In particular, when we consider derivatives of trigonometric functions later in the text, it will be necessary to answer the following question.

$$\text{Question:} \quad \text{Does } \lim_{x \to 0} \frac{\sin x}{x} \text{ exist?}$$

Note that substituting 0 for $x$ gives us the undefined expression 0/0. The following table lists some approximations of $f(x) = (\sin x)/x$ for $x$ near

| $x$ | $f(x) = \dfrac{\sin x}{x}$ |
|---|---|
| ±2.0 | 0.454648713 |
| ±1.0 | 0.841470985 |
| ±0.5 | 0.958851077 |
| ±0.4 | 0.973545856 |
| ±0.3 | 0.985067356 |
| ±0.2 | 0.993346654 |
| ±0.1 | 0.998334166 |
| ±0.01 | 0.999983333 |
| ±0.001 | 0.999999833 |
| ±0.0001 | 0.999999998 |

Figure 1.2




0, *where* $x$ *is a real number or the radian measure of an angle*. The graph of $f$ is sketched in Figure 1.2 beside the table.

Referring to the table or the graph, we arrive at the following conjecture.

$$\text{Educated guess:}\quad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

As indicated, *we have merely guessed at the answer*. The table indicates that $(\sin x)/x$ gets closer to 1 as $x$ gets closer to 0; however, we cannot be absolutely sure of this fact. The function values could conceivably deviate from 1 if $x$ were closer to 0 than are those $x$-values listed in the table. Although a calculator may help us guess if a limit exists, it cannot be used in proofs. We will return to this limit in Section 2.4, where we will *prove* that our guess is correct.

It is easy to find $\lim_{x\to a} f(x)$ if $f(x)$ is a simple algebraic expression. For example, if $f(x) = 2x - 3$ and $a = 4$, it is evident that the closer $x$ is to 4, the closer $f(x)$ is to $2(4) - 3$, or 5. This example gives us the first limit in the following illustration. The remaining two limits may be obtained in the same intuitive manner.

**ILLUSTRATION**

- $\lim_{x\to 4}(2x-3) = 2(4) - 3 = 8 - 3 = 5$
- $\lim_{x\to -3}(x^2+1) = (-3)^2 + 1 = 9 + 1 = 10$
- $\lim_{x\to 7}\sqrt{x+2} = \sqrt{7+2} = \sqrt{9} = 3$

In the preceding illustrations, the limits as $x \to a$ can be found by merely substituting the number $a$ for $x$. For a special class of functions called *continuous functions,* which are discussed in Section 1.5, limits can always be found by such a substitution. The next illustration shows that we

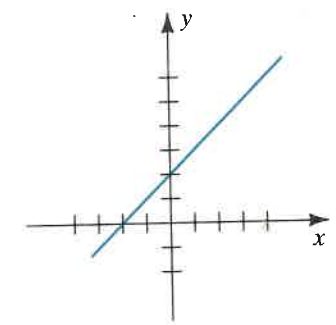
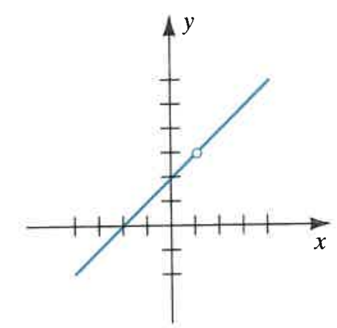
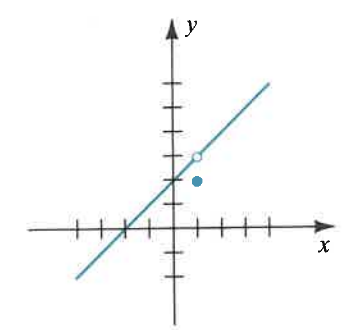


cannot use this substitution technique for *every* algebraic function $f$. In the next illustration, it is important to note that

$$\frac{x^2+x-2}{x-1} = \frac{(x-1)(x+2)}{x-1} = x+2, \quad \textit{provided } x \neq 1.$$

(If $x \neq 1$, then $x - 1 \neq 0$, and it is permissible to cancel the common factor $x - 1$ in the numerator and the denominator.) It follows that the graphs of the equations $y = (x^2+x-2)/(x-1)$ and $y = x+2$ are the same *except* for $x = 1$. Thus, the point $(1, 3)$ is on the graph of $y = x + 2$, but is not on the graph of $y = (x^2+x-2)/(x-1)$, as indicated in the illustration.

**ILLUSTRATION**

| Function value | Graph | Limit as $x \to 1$ |
|---|---|---|
| $f(x) = x + 2$ |  | $\lim_{x\to 1} f(x) = 3$ |
| $g(x) = \dfrac{x^2+x-2}{x-1}$ |  | $\lim_{x\to 1} g(x) = 3$ |
| $h(x) = \begin{cases} \dfrac{x^2+x-2}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ |  | $\lim_{x\to 1} h(x) = 3$ |

In the preceding illustration, the limit of each function as $x$ approaches 1 is 3; but in the first case, $f(1) = 3$; in the second, $g(1)$ does not exist; and in the third, $h(1) = 2 \neq 3$.

The following two examples illustrate how algebraic manipulations may be used to help find certain limits.

**EXAMPLE ■ 1** If $f(x) = \dfrac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$, find $\lim_{x \to 2} f(x)$.

**SOLUTION** The number 2 is not in the domain of $f$ since the meaningless expression 0/0 is obtained if 2 is substituted for $x$. Factoring the numerator and the denominator gives us

$$f(x) = \frac{(x-2)(2x-1)}{(x-2)(5x+3)}.$$

We cannot cancel the factor $x - 2$ at this stage; however, if we take the *limit* of $f(x)$ as $x \to 2$, this cancellation *is* allowed, because by (1.1), $x \neq 2$ and hence $x - 2 \neq 0$. Thus,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$$

$$= \lim_{x \to 2} \frac{(x-2)(2x-1)}{(x-2)(5x+3)}$$

$$= \lim_{x \to 2} \frac{2x-1}{5x+3} = \frac{3}{13}.$$

**EXAMPLE ■ 2** Let $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$.

**(a)** Find $\lim_{x \to 9} f(x)$.

**(b)** Sketch the graph of $f$ and illustrate the limit in part (a) graphically.

**SOLUTION**

**(a)** Note that the number 9 is not in the domain of $f$. To find the limit, we shall change the form of $f(x)$ by rationalizing the denominator as follows:

$$\lim_{x \to 9} f(x) = \lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$$

$$= \lim_{x \to 9} \left( \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \right)$$

$$= \lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9}$$

By (1.1), when investigating the limit as $x \to 9$, we assume that $x \neq 9$. Hence $x - 9 \neq 0$, and we can divide the numerator and the denominator by $x - 9$ (that is, we can *cancel* the expression $x - 9$) and thus obtain

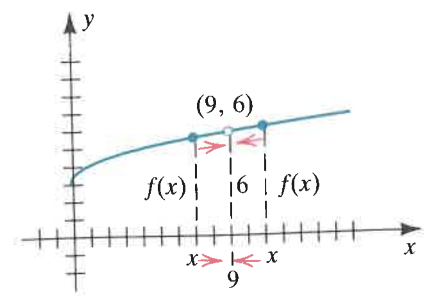
$$\lim_{x \to 9} f(x) = \lim_{x \to 9} (\sqrt{x} + 3) = \sqrt{9} + 3 = 6.$$

**(b)** If we rationalize the denominator of $f(x)$ as in part (a), we see that the graph of $f$ is the same as the graph of the equation $y = \sqrt{x} + 3$, *except for the point* (9, 6), as illustrated in Figure 1.3. As $x$ gets closer to 9, the point $(x, f(x))$ on the graph of $f$ gets closer to the point (9, 6). Note that $f(x)$ never actually attains the value 6; however, $f(x)$ can be made as close to 6 as desired by choosing $x$ sufficiently close to 9.



*Figure 1.3*

$f(x) = \dfrac{x-9}{\sqrt{x}-3}$




The first two examples show that we can often use algebraic manipulations to find limits. In many cases, however, there is no apparent algebraic simplification to try. In such instances, we may obtain numerical evidence for the value of a limit by constructing a table of values or by graphing the function. The next example illustrates both of these approaches.

**EXAMPLE ■ 3** Lend numerical support for the claim that

$$\lim_{x \to 1} f(x) \approx 0.4343, \quad \text{where} \quad f(x) = \frac{\log_{10} x}{x - 1}$$

**(a)** by creating a table of function values for $x$ close to 1

**(b)** by using a graphing utility to repeatedly zoom in on the graph of $f$ near $x = 1$

**SOLUTION**

**(a)** We construct the following table of function values for $x$ very close to but not equal to 1.

| $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|---|---|
| 0.97 | 0.440942191 | 0.997 | 0.434947229 | 0.9997 | 0.434359639 |
| 0.98 | 0.438696215 | 0.998 | 0.434729356 | 0.9998 | 0.434337917 |
| 0.99 | 0.436480540 | 0.999 | 0.434511774 | 0.9999 | 0.434316198 |
| 1.01 | 0.432137378 | 1.001 | 0.434077479 | 1.0001 | 0.434272769 |
| 1.02 | 0.430008588 | 1.002 | 0.433860766 | 1.0002 | 0.434251058 |
| 1.03 | 0.427907490 | 1.003 | 0.433644340 | 1.0003 | 0.434229351 |

*Figure 1.4*

$0.5 \le x \le 1.5, 0.3 \le y \le 0.6$

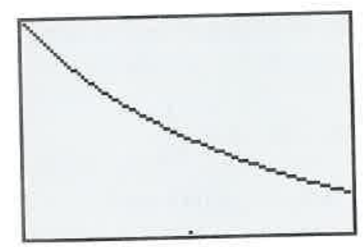
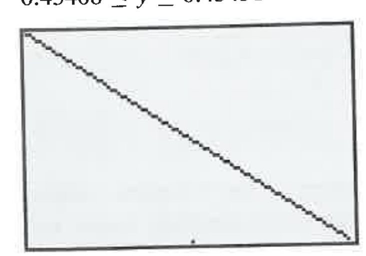


*Figure 1.5*

$0.999 \le x \le 1.001,$
$0.43408 \le y \le 0.43451$

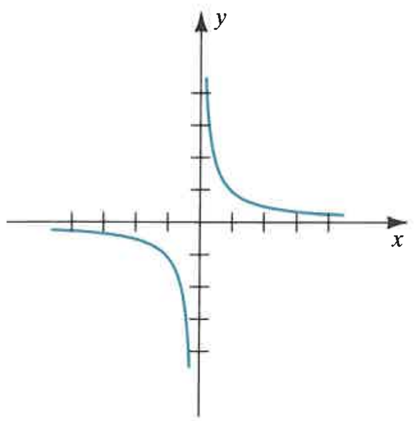


**(b)** We graph the function with $x$-interval $0.5 \le x \le 1.5$ to view the graph near $x = 1$, as in Figure 1.4, where we see that $y$ is close to 0.4. Repeatedly zooming in to obtain the graph in Figure 1.5 allows us to estimate the value of $y$ to be 0.4343 in the $x$-interval [0.999, 1.001].

Both procedures provide strong circumstantial evidence that the limit is approximately 0.4343. Neither procedure, however, gives an exact answer.

The finite limitations of calculators and graphing utilities allow us to get close to $x = 1$, but not *arbitrarily* close. For example, if $b$ is the smallest positive number available on our calculating device, then the function $f$ could behave very differently inside the interval $(1-b, 1+b)$ (where we cannot evaluate $f$) than it does outside that interval. In Chapter 6, we will introduce a new idea that will permit us to determine the *exact* value of limits like the one in this example.

Figure 1.6 $f(x) = \dfrac{1}{x}$




The next two examples involve functions that have no limit as $x$ approaches 0. The solutions are intuitive in nature. Rigorous proofs require the formal definition of limit discussed in the next section.

**EXAMPLE 4** Show that $\lim_{x\to 0} \dfrac{1}{x}$ does not exist.

**SOLUTION** The graph of $f(x) = 1/x$ is sketched in Figure 1.6. Note that we can make $|f(x)|$ as large as desired by choosing $x$ sufficiently close to 0 (but $x \neq 0$). For example, if we want $f(x) = -1{,}000{,}000$, we choose $x = -0.000001$. For $f(x) = 10^9$, we choose $x = 10^{-9}$. Since $f(x)$ does not approach a specific number $L$ as $x$ approaches 0, the limit does not exist.

**EXAMPLE 5** Show that $\lim_{x\to 0} \sin \dfrac{1}{x}$ does not exist.

**SOLUTION** Let us first determine some of the characteristics of the graph of $y = \sin(1/x)$. To find the $x$-intercepts, we note that the following statement is true for every integer $n$:

$$\sin\frac{1}{x} = 0 \quad \text{if and only if} \quad \frac{1}{x} = \pi n, \quad \text{or} \quad x = \frac{1}{\pi n}$$

Some specific $x$-intercepts (with $n = \pm1, \pm2, \pm3, \ldots, \pm100$) are

$$\pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \ldots, \pm\frac{1}{100\pi}.$$

If we let $x$ approach 0, then the distance between successive $x$-intercepts decreases and, in fact, approaches 0. Similarly,

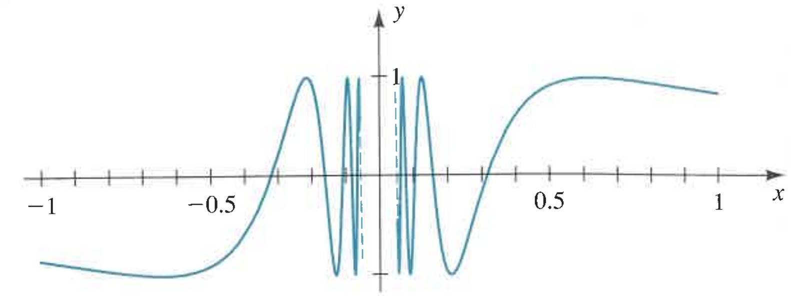
$$\sin\frac{1}{x} = 1 \quad \text{if and only if} \quad x = \frac{1}{(\pi/2) + 2\pi n}$$

and $$\sin\frac{1}{x} = -1 \quad \text{if and only if} \quad x = \frac{1}{(3\pi/2) + 2\pi n},$$

where $n$ is any integer. Thus, as $x$ approaches 0, the function values $\sin(1/x)$ oscillate between $-1$ and 1, and the corresponding waves on the graph become very compressed horizontally, as illustrated in Figure



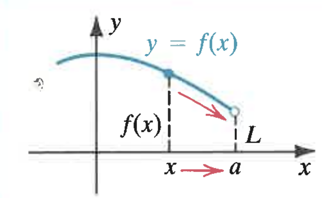
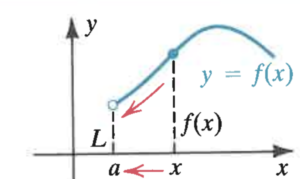
Figure 1.7 $f(x) = \sin \dfrac{1}{x}$




1.7. Hence $\lim_{x\to 0} \sin(1/x)$ does not exist, because the function values do not approach a specific number $L$ as $x$ approaches 0.

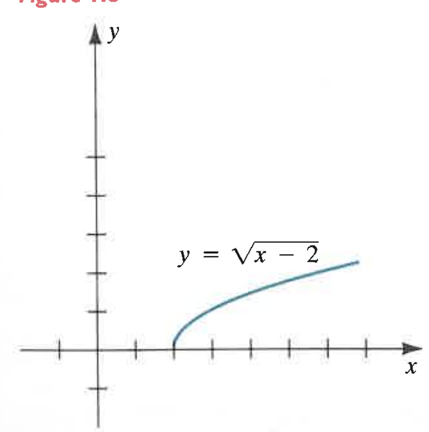
We sometimes use *one-sided* limits of the following types.

**One-Sided Limits 1.2**

| Notation | Intuitive meaning | Graphical interpretation |
| --- | --- | --- |
| $\lim_{x\to a^-} f(x) = L$ (left-hand limit) | We can make $f(x)$ as close to $L$ as desired by choosing $x$ sufficiently close to $a$, and $x < a$. |   |
| $\lim_{x\to a^+} f(x) = L$ (right-hand limit) | We can make $f(x)$ as close to $L$ as desired by choosing $x$ sufficiently close to $a$, and $x > a$. |   |

For a left-hand limit, the function $f$ must be defined in (at least) an open interval of the form $(c, a)$ for some real number $c$. For a right-hand limit, $f$ must be defined in $(a, c)$ for some $c$. The notation $x \to a^-$ is read *x approaches a from the left,* and $x \to a^+$ is read *x approaches a from the right.*

Figure 1.8




**EXAMPLE 6** If $f(x) = \sqrt{x-2}$, sketch the graph of $f$ and find, if possible,

**(a)** $\lim_{x\to 2^+} f(x)$ **(b)** $\lim_{x\to 2^-} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

**SOLUTION** The graph of $f$ is sketched in Figure 1.8.

**(a)** If $x > 2$, then $x - 2 > 0$ and hence $f(x) = \sqrt{x-2}$ *is* a real number;

that is, $f(x)$ is defined. Thus,

$$\lim_{x\to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0.$$

**(b)** If $x < 2$, then $x - 2 < 0$ and hence $f(x) = \sqrt{x-2}$ *is not* a real number. Thus, the left-hand limit does not exist.

**(c)** The limit of $f$ as $x$ approaches 2 does not exist because $f(x) = \sqrt{x-2}$ is not defined throughout an open interval containing 2—that is, an interval containing numbers that are less than 2 *and* numbers that are greater than 2.

The relationship between one-sided limits and limits is described in the next theorem.

**Theorem 1.3**

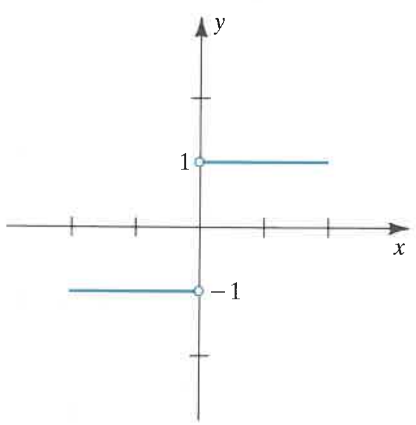
$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Theorem (1.3), which can be proved using definitions in Section 1.2, tells us that *the limit of $f(x)$ as $x$ approaches $a$ exists if and only if both the right-hand and left-hand limits exist and are equal to some real number $L$.*

**EXAMPLE 7** If $f(x) = \dfrac{|x|}{x}$, sketch the graph of $f$ and find, if possible,

**(a)** $\lim_{x\to 0^-} f(x)$ **(b)** $\lim_{x\to 0^+} f(x)$ **(c)** $\lim_{x\to 0} f(x)$

Figure 1.9 $f(x) = \dfrac{|x|}{x}$




**SOLUTION** The function $f$ is undefined at $x = 0$. If $x > 0$, then $|x| = x$ and $f(x) = x/x = 1$. Hence for $x > 0$, the graph of $f$ is the horizontal line $y = 1$. If $x < 0$, then $|x| = -x$ and $f(x) = -x/x = -1$. These results give us the sketch in Figure 1.9. Referring to the graph, we see that

**(a)** $\lim_{x\to 0^-} f(x) = -1$ the left-hand limit is $-1$

**(b)** $\lim_{x\to 0^+} f(x) = 1$ the right-hand limit is 1

**(c)** Since the left-hand and right-hand limits are not equal, it follows from Theorem (1.3) that $\lim_{x\to 0} f(x)$ does not exist.

In the next example, we consider a piecewise-defined function.

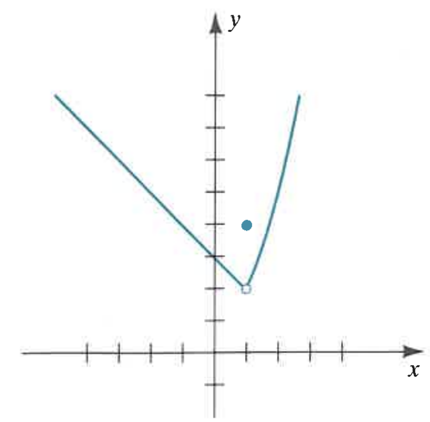
**EXAMPLE 8** Sketch the graph of the function $f$ defined as follows:

$$f(x) = \begin{cases} 3 - x & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^2 + 1 & \text{if } x > 1 \end{cases}$$

Find $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$.



Figure 1.10




**SOLUTION** • The graph is sketched in Figure 1.10. The one-sided limits are

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3 - x) = 2$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 + 1) = 2.$$

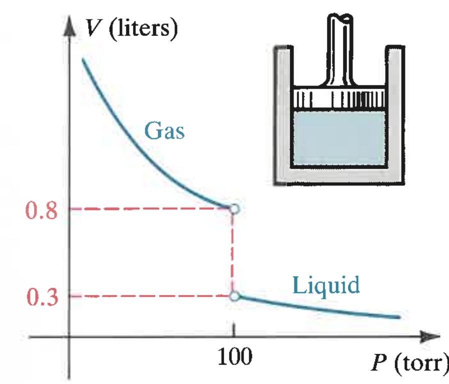
Since the left-hand and right-hand limits both equal 2, it follows from Theorem (1.3) that

$$\lim_{x\to 1} f(x) = 2.$$

Note that the function value $f(1) = 4$ is irrelevant in finding the limit.

The following application involves one-sided limits.

Figure 1.11




**EXAMPLE 9** A gas (such as water vapor or oxygen) is held at a constant temperature in the piston shown in Figure 1.11. As the gas is compressed, the volume $V$ decreases until a certain critical pressure is reached. Beyond this pressure, the gas assumes liquid form. Use the graph in Figure 1.11 to find and interpret

**(a)** $\lim_{P\to 100^-} V$ **(b)** $\lim_{P\to 100^+} V$ **(c)** $\lim_{P\to 100} V$

**SOLUTION**

**(a)** We see from Figure 1.11 that when the pressure $P$ (in torrs) is low, the substance is a gas and the volume $V$ (in liters) is large. (The definition of the unit of pressure, the *torr,* may be found in textbooks on physics.) If $P$ approaches 100 through values less than 100, $V$ decreases and approaches 0.8; that is,

$$\lim_{P\to 100^-} V = 0.8.$$

The limit 0.8 represents the volume at which the substance begins to change from a gas to a liquid.

**(b)** If $P > 100$, the substance is a liquid. If $P$ approaches 100 through values greater than 100, the volume $V$ increases very slowly (since liquids are nearly incompressible), and

$$\lim_{P\to 100^+} V = 0.3.$$

The limit 0.3 represents the volume at which the substance begins to change from a liquid to a gas.

**(c)** $\lim_{P\to 100} V$ does not exist since the left-hand and right-hand limits in parts (a) and (b) are not equal. (At $P = 100$, the gas and liquid forms exist together in equilibrium, and the substance cannot be classified as either a gas or a liquid.)

## EXERCISES 1.1

Exer. 1–10: Find the limit.

1 $\lim_{x\to -2}(3x-1)$

2 $\lim_{x\to 3}(x^2+2)$

3 $\lim_{x\to 4} x$

4 $\lim_{x\to -3}(-x)$

5 $\lim_{x\to 100} 7$

6 $\lim_{x\to 7} 100$

7 $\lim_{x\to -1} \pi$

8 $\lim_{x\to \pi}(-1)$

9 $\lim_{x\to -1}\dfrac{x+4}{2x+1}$

10 $\lim_{x\to 5}\dfrac{x+2}{x-4}$

Exer. 11–24: Use an algebraic simplification to help find the limit, if it exists.

11 $\lim_{x\to -3}\dfrac{(x+3)(x-4)}{(x+3)(x+1)}$

12 $\lim_{x\to -1}\dfrac{(x+1)(x^2+3)}{x+1}$

13 $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$

14 $\lim_{x\to 3}\dfrac{2x^3-6x^2+x-3}{x-3}$

15 $\lim_{r\to 1}\dfrac{r^2-r}{2r^2+5r-7}$

16 $\lim_{r\to -3}\dfrac{r^2+2r-3}{r^2+7r+12}$

17 $\lim_{k\to 4}\dfrac{k^2-16}{\sqrt{k}-2}$

18 $\lim_{x\to 25}\dfrac{\sqrt{x}-5}{x-25}$

19 $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

20 $\lim_{h\to 0}\dfrac{(x+h)^3-x^3}{h}$

21 $\lim_{h\to -2}\dfrac{h^3+8}{h+2}$

22 $\lim_{h\to 2}\dfrac{h^3-8}{h^2-4}$

23 $\lim_{z\to -2}\dfrac{z-4}{z^2-2z-8}$

24 $\lim_{z\to 5}\dfrac{z-5}{z^2-10z+25}$

Exer. 25–30: Find each limit, if it exists:

(a) $\lim_{x\to a^-} f(x)$ (b) $\lim_{x\to a^+} f(x)$ (c) $\lim_{x\to a} f(x)$

25 $f(x)=\dfrac{|x-4|}{x-4}$; $a=4$

26 $f(x)=\dfrac{x+5}{|x+5|}$; $a=-5$

27 $f(x)=\sqrt{x+6}+x$; $a=-6$

28 $f(x)=\sqrt{5-2x}-x^2$; $a=\frac{5}{2}$

29 $f(x)=\dfrac{1}{x^3}$; $a=0$

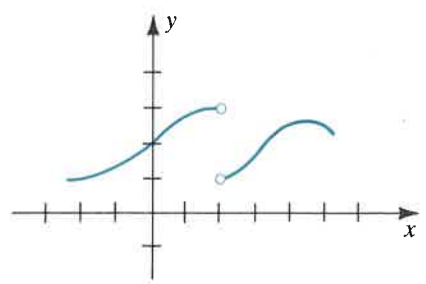
30 $f(x)=\dfrac{1}{x-8}$; $a=8$

Exer. 31–40: Refer to the graph to find each limit, if it exists:

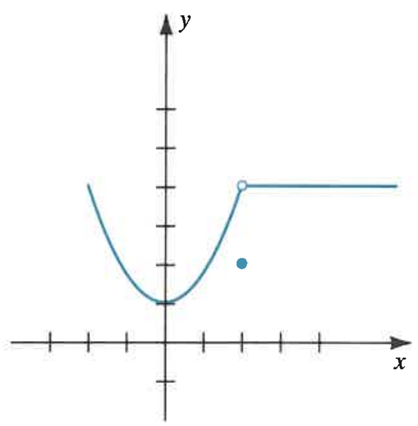
(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$

(d) $\lim_{x\to 0^-} f(x)$ (e) $\lim_{x\to 0^+} f(x)$ (f) $\lim_{x\to 0} f(x)$

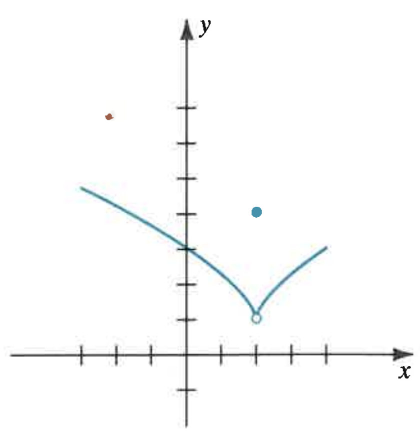
31




32

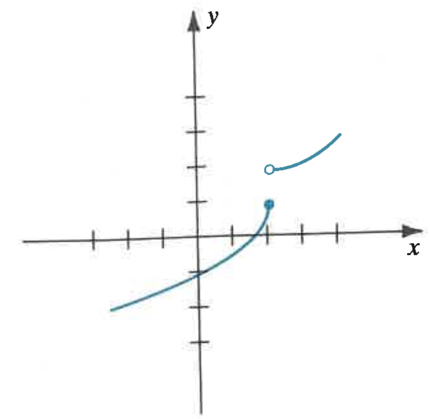



33

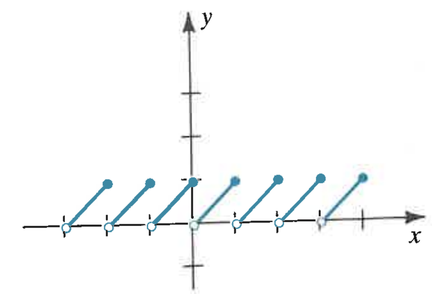




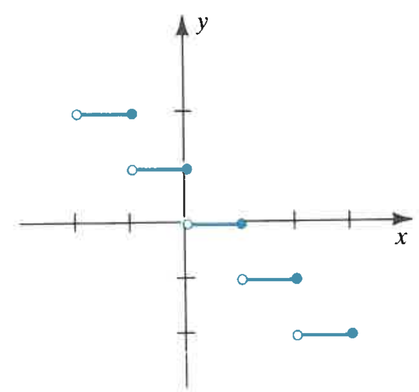

34



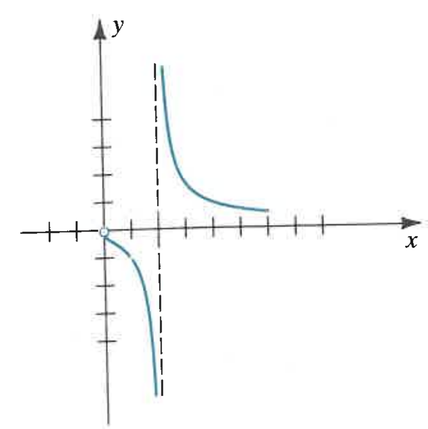

35



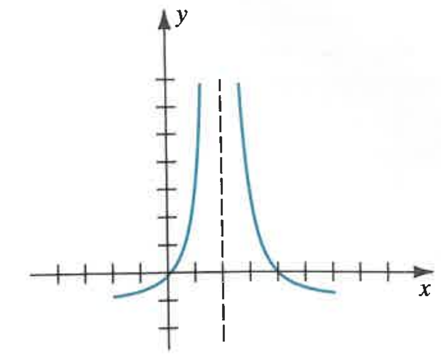

36



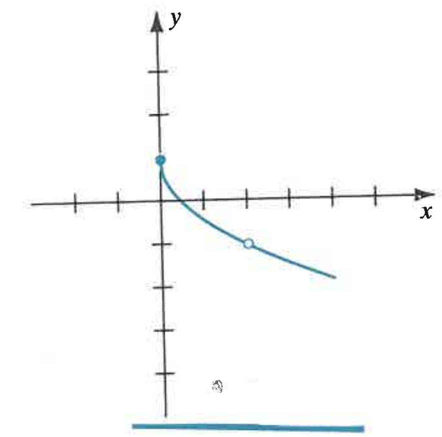

37



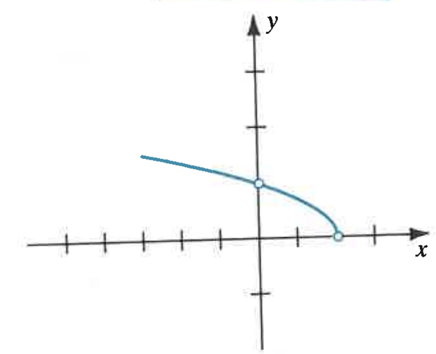

38




39




40




Exer. 41–46: Sketch the graph of $f$ and find each limit, if it exists:

(a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$

41 $f(x)=\begin{cases} x^2-1 & \text{if } x<1 \\ 4-x & \text{if } x\ge 1 \end{cases}$

42 $f(x)=\begin{cases} x^3 & \text{if } x\le 1 \\ 3-x & \text{if } x>1 \end{cases}$

43 $f(x)=\begin{cases} 3x-1 & \text{if } x\le 1 \\ 3-x & \text{if } x>1 \end{cases}$

44 $f(x)=\begin{cases} |x-1| & \text{if } x\ne 1 \\ 1 & \text{if } x=1 \end{cases}$

**45** $f(x) = \begin{cases} x^2+1 & \text{if } x<1 \\ 1 & \text{if } x=1 \\ x+1 & \text{if } x>1 \end{cases}$

**46** $f(x) = \begin{cases} -x^2 & \text{if } x<1 \\ 2 & \text{if } x=1 \\ x-2 & \text{if } x>1 \end{cases}$

**47** A country taxes the first \$20,000 of an individual's income at a rate of 15%, and all income over \$20,000 is taxed at 20%.

**(a)** Find a piecewise-defined function $T$ for the total tax on an income of $x$ dollars.

**(b)** Find

$$\lim_{x\to 20{,}000^-} T(x) \quad \text{and} \quad \lim_{x\to 20{,}000^+} T(x).$$

**48** A telephone company charges 25 cents for the first minute of a long-distance call and 15 cents for each additional minute.

**(a)** Find a piecewise-defined function $C$ for the total cost of a long-distance call of $x$ minutes.

**(b)** If $n$ is an integer greater than 1, find

$$\lim_{x\to n^-} C(x) \quad \text{and} \quad \lim_{x\to n^+} C(x).$$

**49** A mail-order company adds a shipping and handling fee of \$4 for any order that weighs up to 10 lb with an additional 40 cents for each pound over 10 lb.

**(a)** Find a piecewise-defined function $S$ for the shipping and handling fee on an order of $x$ pounds.

**(b)** If $a$ is an integer greater than 10, find

$$\lim_{x\to a^-} S(x) \quad \text{and} \quad \lim_{x\to a^+} S(x).$$

**50** The Campus Cinema charges \$3 admission for children (under age 12), \$6 for adults, and \$4.50 for senior citizens (over age 60).

**(a)** Find a piecewise-defined function $T$ for the ticket price for a person $x$ years old.

**(b)** For which values of $a$ are $\lim_{x\to a^-} T(x)$ and $\lim_{x\to a^+} T(x)$ equal and for which values of $a$ are they unequal?

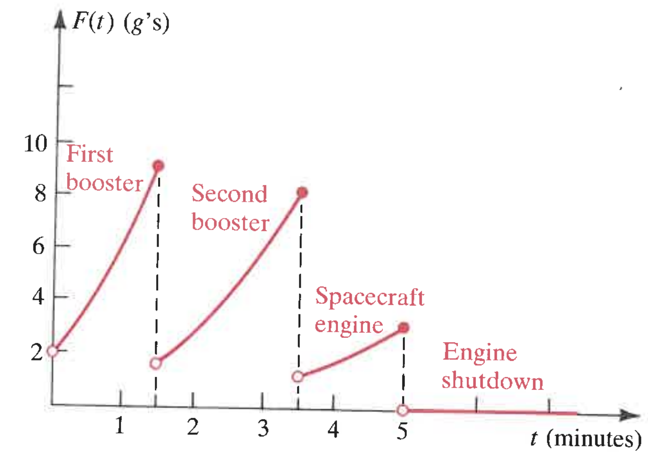
**51** The figure shows a graph of the $g$-forces experienced by astronauts during the takeoff of a spacecraft with two rocket boosters. (A force of $2g$'s is twice that of gravity, $3g$'s is three times that of gravity, etc.) If $F(t)$ denotes the $g$-force $t$ minutes into the flight, find and interpret

**(a)** $\lim_{t\to 0^+} F(t)$

**(b)** $\lim_{t\to 3.5^-} F(t)$ and $\lim_{t\to 3.5^+} F(t)$

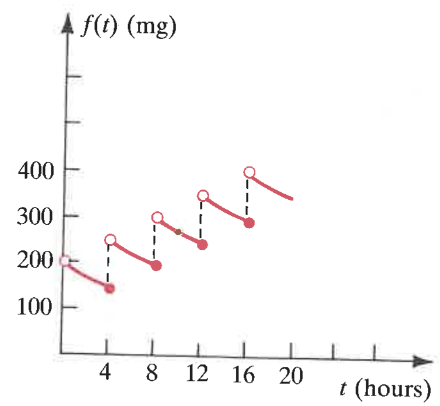
**(c)** $\lim_{t\to 5^-} F(t)$ and $\lim_{t\to 5^+} F(t)$

Exercise 51




**52** A hospital patient receives an initial 200-mg dose of a drug. Additional doses of 100 mg each are then administered every 4 hr. The amount $f(t)$ of the drug present in the bloodstream after $t$ hours is shown in the figure. Find and interpret $\lim_{t\to 8^-} f(t)$ and $\lim_{t\to 8^+} f(t)$.

Exercise 52




c **Exer. 53–60: The stated limit (of the form $\lim_{x\to a} f(x) = L$) may be verified by methods developed later in the text. Lend *numerical* support for the stated result by (a) creating a table of function values for $x$ close to $a$ and (b) using a graphing utility to repeatedly zoom in on the graph of $f$ near $x = a$. Give additional digits if the stated limit is an approximation.**

**53** $\lim_{x\to 0}(1+x)^{1/x} \approx 2.72$

**54** $\lim_{x\to 0}(1+2x)^{3/x} \approx 403.4$

**55** $\lim_{x\to 2}\dfrac{3^x-9}{x-2} \approx 9.89$

**56** $\lim_{x\to 1}\dfrac{2^x-2}{x-1} \approx 1.39$

**57** $\lim_{x\to 0}\left(\dfrac{4^{|x|}+9^{|x|}}{2}\right)^{1/|x|} = 6$



**58** $\lim_{x\to 0}|x|^x = 1$

**59** $\lim_{x\to 0}\dfrac{\sin x - 7x}{x\cos x} = -6$

**60** $\lim_{x\to 3}\dfrac{\sin(\pi x)}{x-3} = -\pi$

c **61** **(a)** Given that $f(x) = \cos(1/x) - \sin(1/x)$, investigate $\lim_{x\to 0} f(x)$ by first letting $x = 3.1830989 \times 10^{-n}$ for $n = 2, 3,$ and 4 and then letting $x = 3 \times 10^{-n}$ for $n = 2, 3,$ and 4.

**(b)** What is the limit in part (a)?

c **62** **(a)** Given that $f(x) = x^{1/100} - 0.933$, investigate $\lim_{x\to 0^+} f(x)$ by letting $x = 10^{-n}$ for $n = 20, 40, 60,$ and 80. (If your calculator allows, use $n = 200, 400, 600,$ and 800.)

**(b)** What *appears* to be the limit in part (a)?

## 1.2 DEFINITION OF LIMIT

We state the precise meaning of a limit of a function in Definition (1.4) of this section. Because the definition is rather abstract, let us begin with a physical illustration that may make it easier to understand.

Scientists often investigate the manner in which quantities vary and whether they approach specific values under certain conditions. Suppose that an industrial plant discharges waste water into a nearby river. The discharge contains a chemical that in large concentrations is toxic to humans. Several miles downstream from the plant, the river flows through a small town (Figure 1.12a on the following page). Since the townspeople use the water from the river for drinking, washing, and cooking, they are concerned about the possible dangers.

The operators of the plant have promised to keep the concentration of the chemical at the point of discharge small enough so that by the time the water reaches the town, its concentration will be low enough not to cause any harm. Meters are installed at the plant and at the water station in the town to measure the concentration of the chemical in the water at both points. We let $x$ denote the concentration indicated by the meter at the plant and $y$ represent the concentration indicated by the meter in town. The plant operators agree to regulate the discharge so that the concentration $y$ at the town's meter will be near a desired level of $L$, low enough so that the chemical poses no hazard to the town.

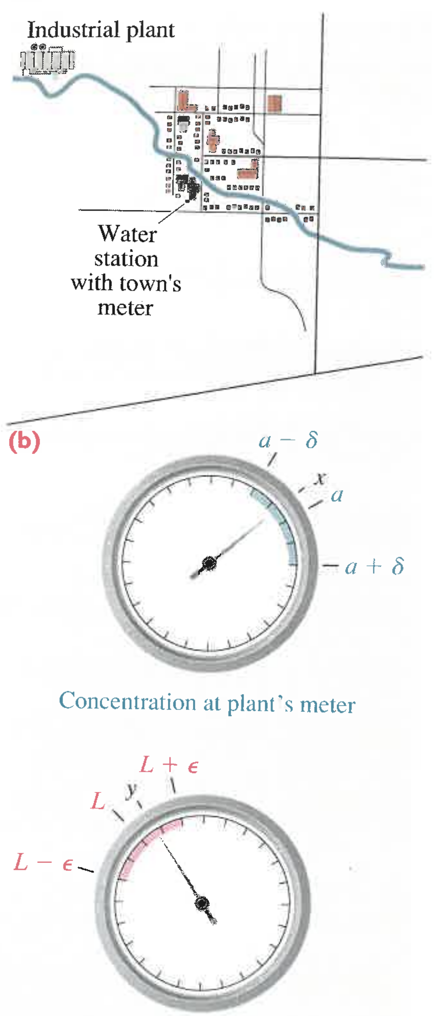
Workers at the plant and officials of the town observe that when the concentration $x$ of the chemical is near the level $a$ on the plant's meter, the measurement $y$ on the town's meter is close to $L$. They also note that the closer $x$ is to $a$, the closer $y$ is to $L$. Because of random fluctuations in the operation of the plant, it is not possible to maintain the concentration of the discharge exactly at $a$ for extended periods; there will always be fluctuations in the concentration.

We use these meters to give a precise meaning to the statement *$y$ approaches $L$ as $x$ approaches $a$*, or, symbolically,

$$\lim_{x\to a} y = L.$$

Figure 1.12
(a)

(b)




Concentration at plant's meter

Concentration at town's meter

When monitoring these meters, we would not expect the concentration $y$ in town to remain *exactly* at $L$ over a long period of time. Instead, our goal might be to force $y$ to remain very close to $L$ by restricting $x$ to values *near* $a$. In particular, if $\epsilon$ (epsilon) denotes a small positive real number, let us suppose it is sufficient that

$$L-\epsilon < y < L+\epsilon,$$

as indicated on the town's meter in Figure 1.12(b). An equivalent statement using absolute values is

$$|y-L| < \epsilon.$$

If these inequalities are true, we say that **$y$ has $\epsilon$-tolerance at $L$.** Thus, the statement $y$ *has* 0.01-*tolerance at* $L$ means that $|y-L| < 0.01$; that is, $y$ is within 0.01 unit of $L$. This tolerance may be sufficiently accurate for our purposes.

Similarly, we consider a small positive number $\delta$ (delta) and define $\delta$-tolerance at $a$ on the plant's meter in Figure 1.12(b). In our later work with functions, it will be important that $x \neq a$. Anticipating this restriction, we say that **$x$ has $\delta$-tolerance at $a$** if

$$0 < |x-a| < \delta$$

or, equivalently, if

$$a-\delta < x < a+\delta \quad \text{and} \quad x \neq a.$$

Let us now consider the following question.

Question: Given any $\epsilon > 0$, is there a $\delta > 0$ such that if $x$ has $\delta$-tolerance at $a$, then $y$ has $\epsilon$-tolerance at $L$?

If the answer to this question is *yes,* we write

$$\lim_{x\to a} y = L.$$

It is important to note that if $\lim_{x\to a} y = L$, then *no matter how small the number* $\epsilon$, *we can always find a* $\delta > 0$ such that if $x$ is restricted to the interval $(a-\delta, a+\delta)$ on the plant's meter (and $x \neq a$), then $y$ will lie in the interval $(L-\epsilon, L+\epsilon)$ on the town's meter.

This example has provided a more precise interpretation of the limit concept than that given in Section 1.1, where we used words such as *close to* and *approaches.* If we rephrase the last question and its answer in terms of inequalities, we obtain the following statement:

$$\lim_{x\to a} y = L$$

means that for every $\epsilon > 0$, there is a $\delta > 0$ such that

if $0 < |x-a| < \delta$, then $|y-L| < \epsilon$

It is now a small step to formulate the definition of a limit of a function $f$. Letting $y = f(x)$ in the preceding discussion gives us the following definition, which also states the conditions required for the function $f$.



**Definition of Limit of a Function 1.4**

Let a function $f$ be defined on an open interval containing $a$, except possibly at $a$ itself, and let $L$ be a real number. The statement

$$\lim_{x\to a} f(x) = L$$

means that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta, \quad \text{then} \quad |f(x)-L| < \epsilon.$$

We sometimes call the inequality $0 < |x-a| < \delta$ a $\delta$-*tolerance statement* and the inequality $|f(x)-L| < \epsilon$ an $\epsilon$-*tolerance statement.*

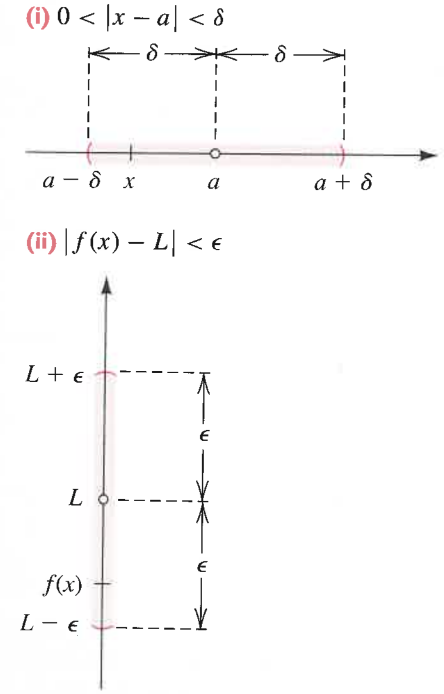
If we wish to use a form of Definition (1.4) that does not contain absolute value symbols, we note that

**(i)** $0 < |x-a| < \delta$ is equivalent to $a-\delta < x < a+\delta$ and $x \neq a$
**(ii)** $|f(x)-L| < \epsilon$ is equivalent to $L-\epsilon < f(x) < L+\epsilon$

The inequalities in (i) and (ii) are represented graphically on real lines in Figure 1.13. We may restate Definition (1.4) as follows.

Figure 1.13
(i) $0 < |x-a| < \delta$

(ii) $|f(x)-L| < \epsilon$




**Alternative Definition of Limit 1.5**

$$\lim_{x\to a} f(x) = L$$

means that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $x$ is in the open interval $(a-\delta, a+\delta)$ and $x \neq a$, then $f(x)$ is in the open interval $(L-\epsilon, L+\epsilon)$.

If $f(x)$ *has a limit as $x$ approaches $a$, then that limit is unique.* A proof of this fact is given at the beginning of Appendix I.

In using either Definition (1.4) or (1.5) to show that $\lim_{x\to a} f(x) = L$, *it is very important to remember the order in which we consider the numbers* $\epsilon$ *and* $\delta$. Always use the following steps:

**Step 1** Consider *any* $\epsilon > 0$.

**Step 2** Show that there is a $\delta > 0$ such that if $x$ has $\delta$-tolerance at $a$, then $f(x)$ has $\epsilon$-tolerance at $L$.

The number $\delta$ in the limit definitions is not unique, for if a specific $\delta$ can be found, then any *smaller* positive number $\delta_1$ will also satisfy the requirements.

Before considering examples, let us rephrase the preceding discussion in terms of the graph of the function $f$. In particular, for $\epsilon > 0$ and $\delta > 0$, we have the following graphical interpretations of tolerances, where $P(x, f(x))$ denotes a point on the graph of $f$.

| Tolerance statement | Graphical interpretation |
|---|---|
| $f(x)$ has $\epsilon$-tolerance at $L$. | $P(x, f(x))$ lies between the horizontal lines $y = L \pm \epsilon$. |
| $x$ has $\delta$-tolerance at $a$. | $x$ is in the interval $(a-\delta, a+\delta)$ on the $x$-axis, and $x \neq a$. |

The two steps in showing that $\lim_{x\to a} f(x) = L$ may now be interpreted graphically as follows:

**Step 1** For any $\epsilon > 0$, consider the horizontal lines $y = L \pm \epsilon$ (shown in Figure 1.14).

**Step 2** Show that there is a $\delta > 0$ such that if $x$ is in the open interval $(a-\delta, a+\delta)$ and $x \neq a$, then $P(x, f(x))$ lies between the horizontal lines $y = L \pm \epsilon$ (that is, inside the shaded rectangular region shown in Figure 1.15).

Figure 1.14

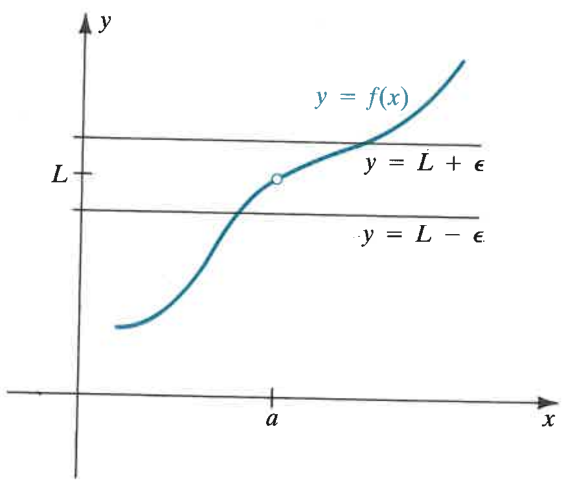
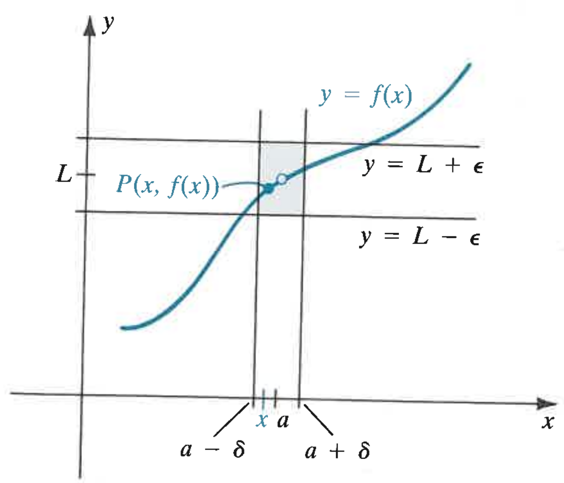



Figure 1.15




**EXAMPLE 1** Use Definition (1.4) to prove that

$$\lim_{x\to 4}(3x-5) = 7.$$

**SOLUTION** If, in Definition (1.4), we let $f(x) = 3x-5$, $a = 4$, and $L = 7$, then we must show that given any $\epsilon > 0$, we can find a $\delta > 0$ such that

(*) if $0 < |x-4| < \delta$, then $|(3x-5)-7| < \epsilon$.

To solve an inequality problem of this type, we can often obtain a clue to a proper choice for $\delta$ by first examining the $\epsilon$-tolerance statement. Doing so



leads to the following list of equivalent inequalities:

$$\begin{aligned} |(3x-5)-7| &< \epsilon && \epsilon\text{-tolerance statement}\\ |3x-12| &< \epsilon && \text{simplifying}\\ |3(x-4)| &< \epsilon && \text{common factor 3}\\ 3\,|x-4| &< \epsilon && \text{properties of absolute value}\\ |x-4| &< \tfrac{1}{3}\epsilon && \text{multiply by } \tfrac{1}{3} \end{aligned}$$

The final inequality in the list gives us the needed clue. Specifically, we choose $\delta$ such that $\delta \le \frac{1}{3}\epsilon$ and obtain the following equivalent inequalities:

$$\begin{aligned} 0 < |x-4| &< \delta && \delta\text{-tolerance statement}\\ 0 < |x-4| &< \tfrac{1}{3}\epsilon && \text{choice of } \delta \le \tfrac{1}{3}\epsilon\\ 0 < 3\,|x-4| &< \epsilon && \text{multiply by 3}\\ 0 < |3x-12| &< \epsilon && \text{properties of absolute value}\\ 0 < |(3x-5)-7| &< \epsilon && \text{equivalent form} \end{aligned}$$

These equivalent inequalities verify (*) and hence complete the proof.

Figure 1.16

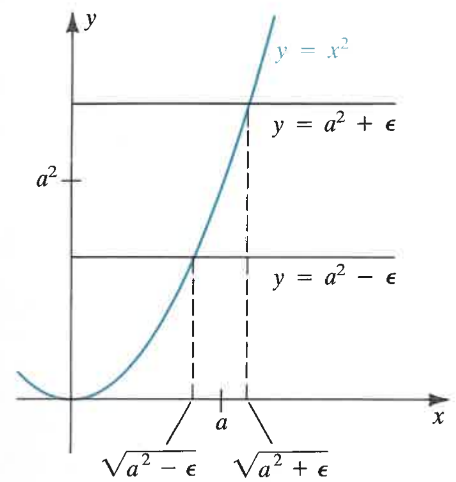
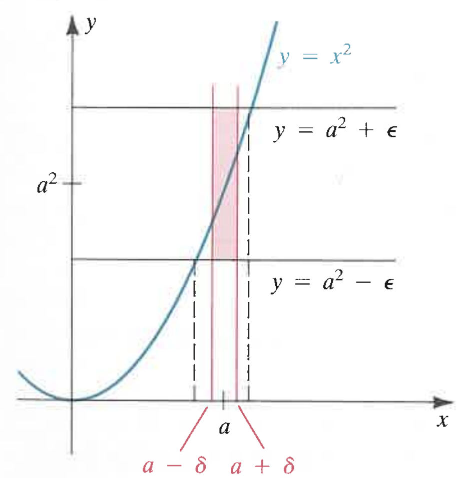



Figure 1.17




The next example illustrates how the geometric process shown in Figures 1.14 and 1.15 may be applied to a specific function.

**EXAMPLE 2** Prove that $\lim_{x\to a} x^2 = a^2$.

**SOLUTION** Let us consider the case $a > 0$. We shall apply the alternative definition (1.5) with $f(x) = x^2$ and $L = a^2$. Thus, given any $\epsilon > 0$, we must find a $\delta > 0$ such that

(*) if $x$ is in $(a-\delta,\ a+\delta)$ and $x \neq a$, then $x^2$ is in $(a^2-\epsilon,\ a^2+\epsilon)$.

We can obtain a clue to a proper choice for $\delta$ by examining graphical interpretations of tolerance statements. Thus, as in step (1) on page 100, consider the horizontal lines $y = a^2 \pm \epsilon$. As shown in Figure 1.16, these lines intersect the graph of $y = x^2$ at points with $x$-coordinates $\sqrt{a^2-\epsilon}$ and $\sqrt{a^2+\epsilon}$. Note that if $x$ is in the open interval $(\sqrt{a^2-\epsilon}, \sqrt{a^2+\epsilon})$, then the point $(x, x^2)$ on the graph of $f$ lies between the horizontal lines. If we choose a positive number $\delta$ *smaller* than both $\sqrt{a^2+\epsilon}-a$ and $a-\sqrt{a^2-\epsilon}$, with $a^2-\epsilon > 0$, as illustrated in Figure 1.17, then when $x$ has $\delta$-tolerance at $a$, the point $(x, x^2)$ lies between the horizontal lines $y = a^2 \pm \epsilon$ (that is, $x^2$ has $\epsilon$-tolerance at $a^2$). This geometric demonstration proves (*). Although we have considered only $a > 0$, a similar argument applies if $a \le 0$.

The next two examples, which were also discussed in Section 1.1, indicate how the geometric process illustrated in Figure 1.15 may be used to show that certain limits do *not* exist.

**EXAMPLE ■ 3** Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

**SOLUTION** Let us proceed in an indirect manner. Thus, *suppose* that

$$\lim_{x\to 0} \frac{1}{x} = L$$

for some number $L$. Consider any pair of horizontal lines $y = L \pm \epsilon$, as illustrated in Figure 1.18. Since we are assuming that the limit exists, it should be possible to find an open interval $(0-\delta, 0+\delta)$, or, equivalently, $(-\delta, \delta)$, such that if $-\delta < x < \delta$ and $x \neq 0$, then the point $(x, 1/x)$ on the graph lies between the horizontal lines. However, since $|1/x|$ can be made as large as desired by choosing $x$ close to 0, some points on the graph will lie either above or below the lines. Hence our supposition is false; that is, $\lim_{x\to 0}(1/x) \neq L$ for any real number $L$. Thus, the limit does not exist.

Figure 1.18 $f(x) = \frac{1}{x}$

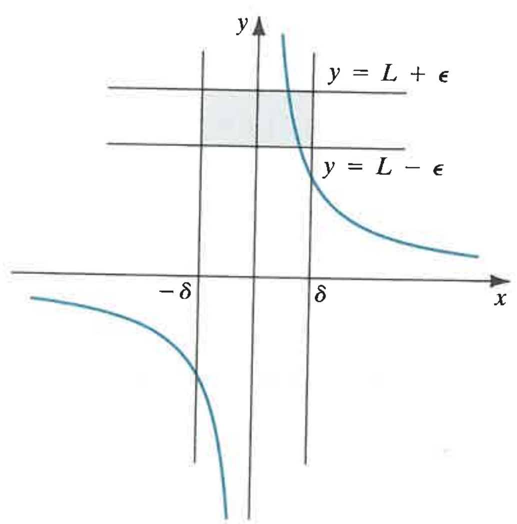
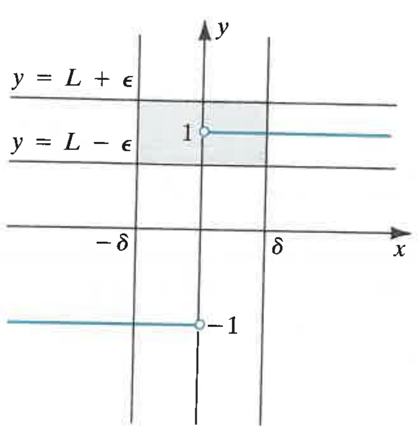



Figure 1.19
$f(x) = \frac{|x|}{x}$




**EXAMPLE ■ 4** If $f(x) = \frac{|x|}{x}$, show that $\lim_{x\to 0} f(x)$ does not exist.

**SOLUTION** The graph of $f$ is sketched in Figure 1.19. If we consider any pair of horizontal lines $y = L \pm \epsilon$, with $0 < \epsilon < 1$, then there are always some points on the graph that do not lie between these lines. In the figure, we have illustrated a case for $L = 1$; however, our proof is valid



for every $L$. Since we cannot find a $\delta > 0$ such that step (2) on page 100 is true, the limit does not exist.

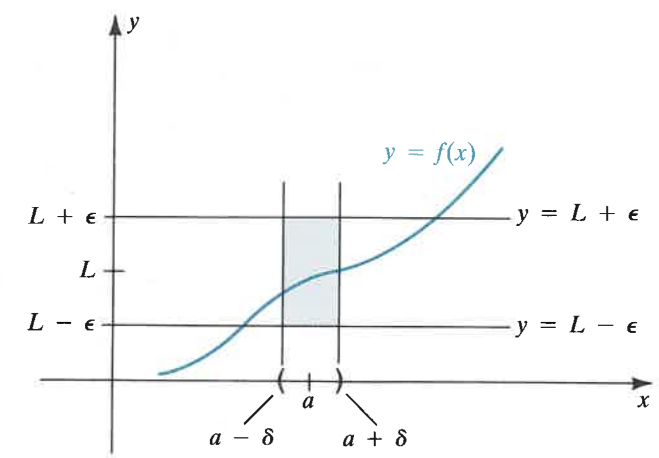
The following theorem states that *if a function $f$ has a positive limit as $x$ approaches $a$, then $f(x)$ is positive throughout some open interval containing $a$, with the possible exception of $a$.*

**Theorem 1.6**

If $\lim_{x\to a} f(x) = L$ and $L > 0$, then there is an open interval $(a-\delta, a+\delta)$ containing $a$ such that $f(x) > 0$ for every $x$ in $(a-\delta, a+\delta)$, except possibly $x = a$.

**PROOF** If $L > 0$ and we let $\epsilon = \frac{1}{2}L$, then the horizontal lines $y = L \pm \epsilon$ are above the $x$-axis, as illustrated in Figure 1.20. By Definition (1.5), there is a $\delta > 0$ such that if $a - \delta < x < a + \delta$ and $x \neq a$, then $L - \epsilon < f(x) < L + \epsilon$. Since $f(x) > L - \epsilon$ and $L - \epsilon > 0$, it follows that $f(x) > 0$ for these values of $x$. ■

Figure 1.20




We can also prove that *if $f$ has a negative limit as $x$ approaches $a$, then there is an open interval containing $a$ such that $f(x) < 0$ for every $x$ in the interval, with the possible exception of $x = a$.*

Formal definitions can be given for one-sided limits. For the right-hand limit $x \to a^+$, we replace the condition $0 < |x - a| < \delta$ in Definition (1.4) by $a < x < a + \delta$. In terms of the alternative definition (1.5), we restrict $x$ to the *right* half $(a, a+\delta)$ of the interval $(a-\delta, a+\delta)$. Similarly, for the left-hand limit $x \to a^-$, we replace $0 < |x - a| < \delta$ in (1.4) by $a - \delta < x < a$. This is equivalent to restricting $x$ to the *left* half $(a-\delta, a)$ of the interval $(a-\delta, a+\delta)$ in (1.5).

## EXERCISES 1.2

**Exer. 1–2: Express the limit statement in the form of (a) Definition (1.4) and (b) Alternative Definition (1.5).**

1 $\lim_{t \to c} v(t) = K$

2 $\lim_{t \to b} f(t) = M$

**Exer. 3–6: Express the one-sided limit statement in a form similar to (a) Definition (1.4) and (b) Alternative Definition (1.5).**

3 $\lim_{x \to p^-} g(x) = C$

4 $\lim_{z \to a^-} h(z) = L$

5 $\lim_{z \to t^+} f(z) = N$

6 $\lim_{x \to c^+} s(x) = D$

**Exer. 7–14: For the given $\lim_{x \to a} f(x) = L$ and $\epsilon$, use the graph of the function $f$ to find the largest value of $\delta$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.**

7 $\lim_{x \to 3/2} \dfrac{4x^2 - 9}{2x - 3} = 6;$ $\quad \epsilon = 0.01$

8 $\lim_{x \to -2/3} \dfrac{9x^2 - 4}{3x + 2} = -4;$ $\quad \epsilon = 0.1$

9 $\lim_{x \to 4} x^2 = 16;$ $\quad \epsilon = 0.1$

10 $\lim_{x \to 3} x^3 = 27;$ $\quad \epsilon = 0.01$

11 $\lim_{x \to 16} \sqrt{x} = 4;$ $\quad \epsilon = 0.1$

12 $\lim_{x \to 27} \sqrt[3]{x} = 3;$ $\quad \epsilon = 0.1$

c 13 $\lim_{x \to \pi/3} \tan x = \sqrt{3};$ $\quad \epsilon = 0.1$

c 14 $\lim_{x \to \pi/3} \cos x = \frac{1}{2};$ $\quad \epsilon = 0.1$

**Exer. 15–24: Use Definition (1.4) to prove that the limit exists.**

15 $\lim_{x \to 3} 5x = 15$

16 $\lim_{x \to 5} (-4x) = -20$

17 $\lim_{x \to -3} (2x + 1) = -5$

18 $\lim_{x \to 2} (5x - 3) = 7$

19 $\lim_{x \to -6} (10 - 9x) = 64$

20 $\lim_{x \to 4} (15 - 8x) = -17$

21 $\lim_{x \to 3} 5 = 5$

22 $\lim_{x \to 5} 3 = 3$

23 $\lim_{x \to a} c = c$ for all real numbers $a$ and $c$

24 $\lim_{x \to a} (mx + b) = ma + b$ for all real numbers $m$, $b$, and $a$

**Exer. 25–30: Use the graphical method illustrated in Example 2 to verify the limit for $a > 0$.**

25 $\lim_{x \to -a} x^2 = a^2$

26 $\lim_{x \to a} (x^2 + 1) = a^2 + 1$

27 $\lim_{x \to a} x^3 = a^3$

28 $\lim_{x \to a} x^4 = a^4$

29 $\lim_{x \to a} \sqrt{x} = \sqrt{a}$

30 $\lim_{x \to a} \sqrt[3]{x} = \sqrt[3]{a}$

**Exer. 31–38: Use the method illustrated in Examples 3 and 4 to show that the limit does not exist.**

31 $\lim_{x \to 3} \dfrac{|x - 3|}{x - 3}$

32 $\lim_{x \to -2} \dfrac{x + 2}{|x + 2|}$

33 $\lim_{x \to -1} \dfrac{3x + 3}{|x + 1|}$

34 $\lim_{x \to 5} \dfrac{2x - 10}{|x - 5|}$

35 $\lim_{x \to 0} \dfrac{1}{x^2}$

36 $\lim_{x \to 4} \dfrac{7}{x - 4}$

37 $\lim_{x \to -5} \dfrac{1}{x + 5}$

38 $\lim_{x \to 1} \dfrac{1}{(x - 1)^2}$

39 A country taxes the first \$20,000 of an individual's income at a rate of 15%, and all income over \$20,000 is taxed at 20%. For an income of $x$ dollars, let $T(x)$ be the total tax and $P(x)$ the percentage owed in taxes on the next dollar earned. Explain why $\lim_{x \to 20{,}000} T(x)$ exists but $\lim_{x \to 20{,}000} P(x)$ does not exist.

40 Prove that if $f$ has a negative limit as $x$ approaches $a$, then there is an open interval containing $a$ such that $f(x) < 0$ for every $x$ in the interval, with the possible exception of $x = a$.

41 Give an example of a function $f$ that is defined at $a$ such that $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) \neq f(a)$.

42 If $f$ is the greatest integer function (see Example 5 of Section B in the Precalculus Review) and $a$ is any integer, show that $\lim_{x \to a} f(x)$ does not exist.

43 Let $f$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that for every real number $a$, $\lim_{x \to a} f(x)$ does not exist.



## Mathematicians and Their Times

### RENÉ DESCARTES

**DURING THE NIGHT** of November 10, 1619, a twenty-three-year-old Frenchman, René Descartes, had three vivid dreams that changed the course of intellectual history. In these dreams, filled with whirlwinds, terrifying phantoms, thunderclaps, and sparks, Descartes felt a supernatural force pointing to the unification and illumination of all knowledge by a single method, the method of *reason*. These nightmares and revelations marked, in the words of mathematicians Philip Davis and Reuben Hersh, the beginning of "the modern world, our world of triumphant rationality."*

The middle of the seventeenth century was one of the most critical periods in the history of mathematics. In this period, France was the undisputed mathematical center of the world, boasting such great thinkers as Pierre de Fermat, Blaise Pascal, Gilles Persone de Roberval, and Girard Desargues. But few men before or since have achieved as much distinction in philosophy and mathematics as Descartes (1596–1650). Regarded by many historians as the "father of modern philosophy," Descartes is perhaps best known to us through his dictum "Cogito, ergo sum"—that is, "I think; therefore I am."

Descartes' most important contribution to mathematics was the creation of *analytic geometry*, the linking of algebra and geometry. Beginning with the representation of a point in the plane by a pair of numbers (now called *Cartesian coordinates* in his honor), Descartes set out to show, in his own words, "how the calculations of arithmetic are related to the operations of geometry." He demonstrated how the familiar figures of Euclidean geometry—lines, polygons, circles, ellipses, and the other conic sections—corresponded to algebraic equations. Geometric questions could be translated into algebraic equations and algebraic operations could be represented in the language of geometry. Thus two of the main branches of mathematics, which had previously

*Philip J. Davis and Reuben Hersh, *Descartes' Dream: The World According to Mathematics*. New York: Harcourt Brace Jovanovich, 1986.

been treated as distinct and independent fields, were now seen to be unified. If the classic geometric techniques failed to solve a problem in geometry, there was now hope of recasting it into algebra where a new set of tools was available. Similarly, one could often gain great insight into difficult algebra problems by interpreting them geometrically.

At age 8, Descartes went to a Jesuit school, where he spent the mornings in bed because of his delicate health. He used these periods productively for study and contemplation, following the custom of remaining in bed until late morning for the rest of his life. In 1649, Sweden's Queen Christina invited him to become her tutor. Perhaps tempted by the glamour of royalty, Descartes accepted, but the cruel winter proved too much for him; he died of pneumonia in 1650.

## 1.3 TECHNIQUES FOR FINDING LIMITS

It would be an excruciating task to verify every limit by means of Definition (1.4) or (1.5). The purpose of this section is to introduce theorems that may be used to simplify problems involving limits. Before stating the first theorem, let us consider the limits of two very simple functions:

**(i)** the constant function $f$ given by $f(x) = c$

**(ii)** the linear function $g$ given by $g(x) = x$

**Figure 1.21** $f(x) = c$

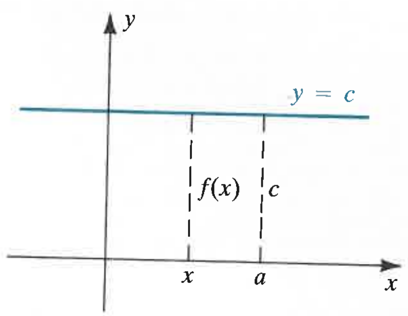
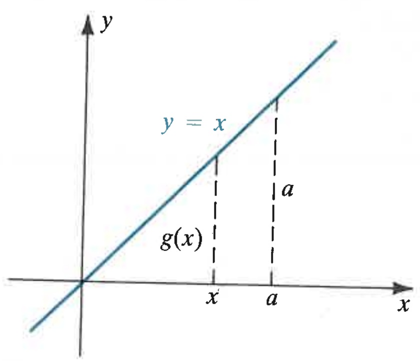



**Figure 1.22** $g(x) = x$




The graph of $f$ is the horizontal line $y = c$ shown in Figure 1.21 for the case $c > 0$. Since

$$|f(x) - c| = |c - c| = 0 \quad \text{for every } x$$

and since 0 is less than any $\epsilon > 0$, it follows from Definition (1.4) that $f(x)$ has the limit $c$ as $x$ approaches $a$. Thus,

$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c.$$

This limit is often described by the phrase *the limit of a constant is the constant.*

The graph of the linear function $g$ given in (ii) is shown in Figure 1.22, and the limit can also be established by means of Definition (1.4). As $x$ approaches $a$, $g(x)$ approaches $a$; that is,

$$\lim_{x \to a} g(x) = \lim_{x \to a} x = a.$$





The facts demonstrated by these examples are given for reference in the following theorem.

**Theorem 1.7**

**(i)** $\lim_{x \to a} c = c$

**(ii)** $\lim_{x \to a} x = a$

**ILLUSTRATION**

- $\lim_{x \to 3} 8 = 8$
- $\lim_{x \to 8} 3 = 3$
- $\lim_{x \to \sqrt{2}} x = \sqrt{2}$
- $\lim_{x \to -4} x = -4$

The preceding illustration gives simple examples of the limits in Theorem (1.7), but as we shall see, the limits in (1.7) can be used as building blocks for finding limits of very complicated expressions.

Many functions can be expressed as sums, differences, products, and quotients of other functions. Suppose $f$ and $g$ are functions and $L$ and $M$ are real numbers. If

$$f(x) \to L \quad \text{and} \quad g(x) \to M \quad \text{as} \quad x \to a,$$

we would expect that

$$f(x) + g(x) \to L + M \quad \text{as} \quad x \to a.$$

The next theorem states that this expectation is true and gives analogous results for products and quotients.

**Theorem 1.8**

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then

**(i)** $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

**(ii)** $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

**(iii)** $\lim_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

**(iv)** $\lim_{x \to a} [cf(x)] = c\left[\lim_{x \to a} f(x)\right]$, for any number $c$

**(v)** $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

We may state the properties in Theorem (1.8) as follows:

**(i)** The limit of a sum is the sum of the limits.

**(ii)** The limit of a product is the product of the limits.

**(iii)** The limit of a quotient is the quotient of the limits, provided the denominator has a nonzero limit.

**(iv)** The limit of a constant times a function is the constant times the limit of the function.

**(v)** The limit of a difference is the difference of the limits.

Proofs for (i)–(iii), based on Definition (1.4), are given in Appendix I. Part (iv) of the theorem follows readily from part (ii) and from Theorem (1.7)(i):

$$\lim_{x\to a}[cf(x)] = \left[\lim_{x\to a} c\right]\left[\lim_{x\to a} f(x)\right] = c\left[\lim_{x\to a} f(x)\right]$$

To prove (v), we may write

$$f(x) - g(x) = f(x) + (-1)g(x)$$

and then use parts (i) and (iv) (with $c = -1$).

We now use the preceding theorems to establish the following.

**Theorem 1.9**

If $m$, $b$, and $a$ are real numbers, then

$$\lim_{x\to a}(mx + b) = ma + b.$$

**PROOF** By Theorem (1.7),

$$\lim_{x\to a} x = a \quad\text{and}\quad \lim_{x\to a} b = b.$$

We next use (i) and (iv) of Theorem (1.8) to obtain

$$\lim_{x\to a}(mx + b) = \lim_{x\to a}(mx) + \lim_{x\to a} b = m\left(\lim_{x\to a} x\right) + b = ma + b. \blacksquare$$

This result can also be proved directly from Definition (1.4).

**ILLUSTRATION**

- $\lim_{x\to -2}(5x + 2) = 5(-2) + 2 = -10 + 2 = -8$
- $\lim_{x\to 6}(4x - 11) = 4(6) - 11 = 24 - 11 = 13$

It is easy to find the limit in the next two examples by means of Theorems (1.8) and (1.9). To obtain a better appreciation of the power of these theorems, you could try to verify the limits in each by using only Definition (1.4).



**EXAMPLE 1** Find $\lim_{x\to 2} \frac{3x + 4}{5x + 7}$.

**SOLUTION** From Theorem (1.9), we know that the limits of the numerator and the denominator exist. Moreover, the limit of the denominator is not 0. Hence, by Theorem (1.8)(iii) and Theorem (1.9),

$$\lim_{x\to 2} \frac{3x + 4}{5x + 7} = \frac{\lim_{x\to 2}(3x + 4)}{\lim_{x\to 2}(5x + 7)} = \frac{3(2) + 4}{5(2) + 7} = \frac{10}{17}.$$

**EXAMPLE 2** If Achilles runs at a rate of 600 ft/min while the tortoise pokes along at 100 ft/min,

**(a)** find an expression for the distance between them as a function of time $x$ if the tortoise is given a head start of 2000 ft

**(b)** determine the limit of this distance as $x \to 4$

**SOLUTION** Let $A(x)$ and $T(x)$ denote the position along the race course (in feet) for Achilles and the tortoise, respectively, where $x$ is the time (in minutes).

**(a)** Since each racer runs at a constant speed, we have $A(x) = 600x$ and $T(x) = 2000 + 100x$. The distance between them is given by

$$T(x) - A(x) = (2000 + 100x) - 600x = 2000 - 500x.$$

**(b)** By Theorem (1.9), with $m = -500$ and $b = 2000$, we have

$$\lim_{x\to 4}[T(x) - A(x)] = \lim_{x\to 4}[2000 - 500x] = 0,$$

so it appears that Achilles will catch up to the tortoise at time $x = 4$ min.

Theorem (1.8) can be extended to limits of sums, differences, products, and quotients that involve any number of functions. In the next example, we use part (ii) for a product of three (equal) functions.

**EXAMPLE 3** Prove that $\lim_{x\to a} x^3 = a^3$.

**SOLUTION** Since $\lim_{x\to a} x = a$,

$$\lim_{x\to a} x^3 = \lim_{x\to a}(x \cdot x \cdot x) = \left(\lim_{x\to a} x\right)\cdot\left(\lim_{x\to a} x\right)\cdot\left(\lim_{x\to a} x\right) = a \cdot a \cdot a = a^3.$$

The method used in Example 3 can be extended to $x^n$ for any positive integer $n$. We merely write $x^n$ as a product $x \cdot x \cdot x \cdot \cdots \cdot x$ of $n$ factors and then take the limit of each factor. Thus, we obtain (i) of the next theorem. Part (ii) may be proved in similar fashion by using Theorem (1.8)(ii). Another method of proof is to use mathematical induction.

**Theorem 1.10**

If $n$ is a positive integer, then

**(i)** $\lim_{x\to a} x^n = a^n$

**(ii)** $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, provided $\lim_{x\to a} f(x)$ exists

**EXAMPLE 4** Find $\lim_{x\to 2} (3x+4)^5$.

**SOLUTION** Applying Theorems (1.10)(ii) and (1.9), we have

$$\begin{aligned}\lim_{x\to 2}(3x+4)^5 &= \left[\lim_{x\to 2}(3x+4)\right]^5\\ &= [3(2)+4]^5\\ &= 10^5 = 100{,}000.\end{aligned}$$

**EXAMPLE 5** Find $\lim_{x\to -2} (5x^3+3x^2-6)$.

**SOLUTION** We may proceed as follows, with the reasons justifying each step as indicated:

$\lim_{x\to -2}(5x^3+3x^2-6)$

| | |
|---|---|
| $= \lim_{x\to -2}(5x^3) + \lim_{x\to -2}(3x^2) + \lim_{x\to -2}(-6)$ | Theorem (1.8)(i) |
| $= \lim_{x\to -2}(5x^3) + \lim_{x\to -2}(3x^2) - 6$ | Theorem (1.7)(i) |
| $= 5\lim_{x\to -2}(x^3) + 3\lim_{x\to -2}(x^2) - 6$ | Theorem (1.8)(iv) |
| $= 5(-2)^3 + 3(-2)^2 - 6$ | Theorem (1.10)(i) |
| $= 5(-8) + 3(4) - 6 = -34$ | simplify |

The limit in Example 5 is the number obtained by substituting $-2$ for $x$ in $5x^3+3x^2-6$. The next theorem states that the same is true for the limit of *every* polynomial.

**Theorem 1.11**

If $f$ is a polynomial function and $a$ is a real number, then

$$\lim_{x\to a} f(x) = f(a).$$

**PROOF** Since $f$ is a polynomial function,

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$$



for real numbers $b_n, b_{n-1}, \ldots, b_0$. As in Example 5,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}(b_n x^n) + \lim_{x\to a}(b_{n-1}x^{n-1}) + \cdots + \lim_{x\to a} b_0\\ &= b_n \lim_{x\to a}(x^n) + b_{n-1}\lim_{x\to a}(x^{n-1}) + \cdots + \lim_{x\to a} b_0\\ &= b_n a^n + b_{n-1}a^{n-1} + \cdots + b_0 = f(a).\end{aligned}$$

**Corollary 1.12**

If $q$ is a rational function and $a$ is in the domain of $q$, then

$$\lim_{x\to a} q(x) = q(a).$$

**PROOF** Since $q$ is a rational function, $q(x) = f(x)/h(x)$, where $f$ and $h$ are polynomial functions. If $a$ is in the domain of $q$, then $h(a) \neq 0$. Using Theorems (1.8)(iii) and (1.11) gives us

$$\lim_{x\to a} q(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} h(x)} = \frac{f(a)}{h(a)} = q(a).$$

**NOTE** Corollary (1.12) also remains true if $q$ is a trigonometric, exponential, or logarithmic function. We will examine proofs and examples for limits of such transcendental functions in Chapters 2 and 6.

**EXAMPLE 6** Find $\lim_{x\to 3} \dfrac{5x^2-2x+1}{4x^3-7}$.

**SOLUTION** Applying Corollary (1.12) yields

$$\lim_{x\to 3}\frac{5x^2-2x+1}{4x^3-7} = \frac{5(3)^2-2(3)+1}{4(3)^3-7} = \frac{45-6+1}{108-7} = \frac{40}{101}.$$

The next theorem states that for positive integral roots of $x$, we may determine a limit by substitution. A proof, using Definition (1.4), may be found in Appendix I.

**Theorem 1.13**

If $a > 0$ and $n$ is a positive integer, or if $a \leq 0$ and $n$ is an odd positive integer, then

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

If $m$ and $n$ are positive integers and $a > 0$, then using Theorems (1.10)(ii) and (1.13) gives us

$$\lim_{x\to a} (\sqrt[n]{x})^m = \left(\lim_{x\to a} \sqrt[n]{x}\right)^m = (\sqrt[n]{a})^m.$$

In terms of rational exponents,

$$\lim_{x \to a} x^{m/n} = a^{m/n}.$$

This limit formula may be extended to negative exponents by writing $x^{-r} = 1/x^r$ and then using Theorem (1.8)(iii).

**EXAMPLE ■ 7** Find $\displaystyle\lim_{x \to 8} \frac{x^{2/3} + 3\sqrt{x}}{4 - (16/x)}$.

**SOLUTION** We may proceed as follows (supply reasons):

$$\begin{aligned}
\lim_{x \to 8} \frac{x^{2/3} + 3\sqrt{x}}{4 - (16/x)} &= \frac{\lim_{x \to 8}(x^{2/3} + 3\sqrt{x})}{\lim_{x \to 8}[4 - (16/x)]} \\
&= \frac{\lim_{x \to 8} x^{2/3} + \lim_{x \to 8} 3\sqrt{x}}{\lim_{x \to 8} 4 - \lim_{x \to 8}(16/x)} \\
&= \frac{8^{2/3} + 3\sqrt{8}}{4 - (16/8)} = \frac{4 + 6\sqrt{2}}{4 - 2} = 2 + 3\sqrt{2}
\end{aligned}$$

**Theorem 1.14**

If a function $f$ has a limit as $x$ approaches $a$, then

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)},$$

provided either $n$ is an odd positive integer or $n$ is an even positive integer and $\lim_{x \to a} f(x) > 0$.

The preceding theorem will be proved in Section 1.5. In the meantime, we shall use it whenever applicable to gain experience in finding limits that involve roots of algebraic expressions.

**EXAMPLE ■ 8** Find $\displaystyle\lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9}$.

**SOLUTION** Using Theorems (1.14) and (1.11), we obtain

$$\begin{aligned}
\lim_{x \to 5} \sqrt[3]{3x^2 - 4x + 9} &= \sqrt[3]{\lim_{x \to 5}(3x^2 - 4x + 9)} \\
&= \sqrt[3]{75 - 20 + 9} = \sqrt[3]{64} = 4.
\end{aligned}$$

The next theorem concerns three functions $f$, $h$, and $g$ such that $h(x)$ is "sandwiched" between $f(x)$ and $g(x)$. If $f$ and $g$ have a common limit $L$ as $x$ approaches $a$, then, as stated in the theorem, $h$ must have the same limit.

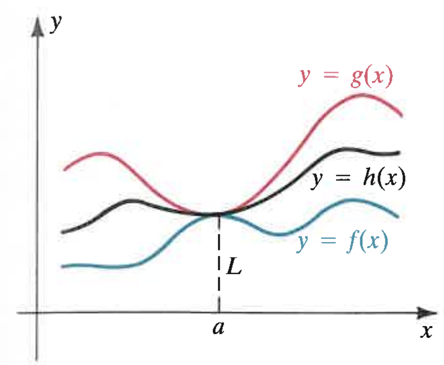


**Sandwich Theorem 1.15**

Suppose $f(x) \le h(x) \le g(x)$ for every $x$ in an open interval containing $a$, except possibly at $a$.

If $\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x)$, then $\lim_{x \to a} h(x) = L$.

Figure 1.23




If $f(x) \le h(x) \le g(x)$ for every $x$ in an open interval containing $x$, then the graph of $h$ lies between the graphs of $f$ and $g$ in that interval, as illustrated in Figure 1.23. If $f$ and $g$ have the same limit $L$ as $x$ approaches $a$, then it appears from the graphs that $h$ also has the limit $L$. A proof of the sandwich theorem based on the definition of limit may be found in Appendix I.

**EXAMPLE ■ 9** Use the sandwich theorem (1.15) to prove that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x^2} = 0.$$

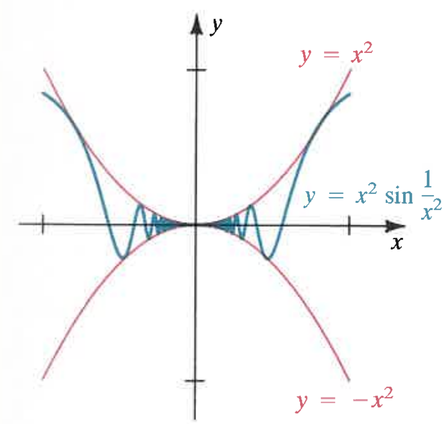
**SOLUTION** Since $-1 \le \sin t \le 1$ for every real number $t$,

$$-1 \le \sin \frac{1}{x^2} \le 1$$

for every $x \ne 0$. Multiplying by $x^2$ (which is positive if $x \ne 0$), we obtain

$$-x^2 \le x^2 \sin \frac{1}{x^2} \le x^2.$$

Figure 1.24




This inequality implies that the graph of $y = x^2 \sin(1/x^2)$ lies between the parabolas $y = -x^2$ and $y = x^2$ (see Figure 1.24). Since

$$\lim_{x \to 0}(-x^2) = 0 \quad \text{and} \quad \lim_{x \to 0} x^2 = 0,$$

it follows from the sandwich theorem, with $f(x) = -x^2$ and $g(x) = x^2$, that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x^2} = 0.$$

Theorems similar to the limit theorems given in this section can be proved for one-sided limits. For example,

$$\lim_{x \to a^+} [f(x) + g(x)] = \lim_{x \to a^+} f(x) + \lim_{x \to a^+} g(x)$$

and

$$\lim_{x \to a^+} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a^+} f(x)}$$

with the usual restrictions on the existence of limits and $n$th roots. Analogous results are true for left-hand limits.

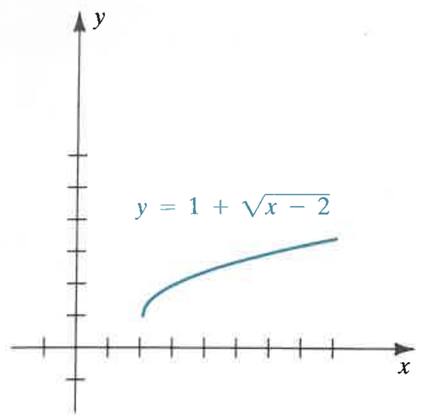
**EXAMPLE ■ 10** Find $\lim_{x\to 2^+}(1+\sqrt{x-2})$.

**SOLUTION** The graph of $f(x) = 1+\sqrt{x-2}$ is sketched in Figure 1.25. Using (one-sided) limit theorems, we obtain

$$\begin{aligned}\lim_{x\to 2^+}(1+\sqrt{x-2}) &= \lim_{x\to 2^+}1 + \lim_{x\to 2^+}\sqrt{x-2}\\ &= 1+\sqrt{\lim_{x\to 2^+}(x-2)}\\ &= 1+0 = 1.\end{aligned}$$

Note that since $\sqrt{x-2}$ is not a real number if $x<2$, there is no left-hand limit, nor is there a limit of $f$ as $x$ approaches 2.

*Figure 1.25*

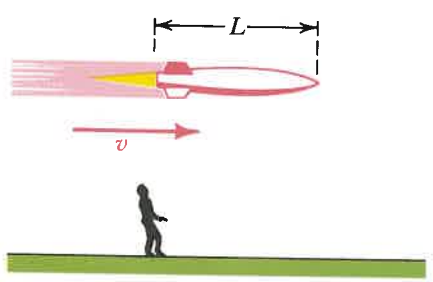



**EXAMPLE ■ 11** Let $c$ denote the speed of light (approximately $3.0\times 10^8$ m/sec, or 186,000 mi/sec). In Einstein's theory of relativity, the *Lorentz contraction formula*

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

specifies the relationship between (1) the length $L$ of an object that is moving at a velocity $v$ with respect to an observer and (2) its length $L_0$ at rest (see Figure 1.26). The formula implies that the length of the object measured by the observer is shorter when it is moving than when it is at rest. Find and interpret $\lim_{v\to c^-} L$, and explain why a left-hand limit is necessary.

*Figure 1.26*




**SOLUTION** Using (one-sided) limit theorems yields

$$\begin{aligned}\lim_{v\to c^-} L &= \lim_{v\to c^-} L_0\sqrt{1-\frac{v^2}{c^2}}\\ &= L_0\lim_{v\to c^-}\sqrt{1-\frac{v^2}{c^2}}\\ &= L_0\sqrt{\lim_{v\to c^-}\left(1-\frac{v^2}{c^2}\right)}\\ &= L_0\sqrt{0} = 0.\end{aligned}$$

Thus, if the velocity of an object could approach the speed of light, then its length, as measured by an observer at rest, would approach 0. This result is sometimes used to help justify the theory that the speed of light is the ultimate speed in the universe; that is, no object can have a velocity that is greater than or equal to $c$.

A left-hand limit is necessary because if $v > c$, then $\sqrt{1-(v^2/c^2)}$ is not a real number.



## EXERCISES 1.3

**Exer. 1–48: Use theorems on limits to find the limit, if it exists.**

1 $\lim_{x\to\sqrt{2}} 15$

2 $\lim_{x\to 15}\sqrt{2}$

3 $\lim_{x\to -2} x$

4 $\lim_{x\to 3} x$

5 $\lim_{x\to 4}(3x-4)$

6 $\lim_{x\to -2}(-3x+1)$

7 $\lim_{x\to -2}\dfrac{x-5}{4x+3}$

8 $\lim_{x\to 4}\dfrac{2x-1}{3x+1}$

9 $\lim_{x\to 1}(-2x+5)^4$

10 $\lim_{x\to -2}(3x-1)^5$

11 $\lim_{x\to 3}(3x-9)^{100}$

12 $\lim_{x\to 1/2}(4x-1)^{50}$

13 $\lim_{x\to -2}(3x^3-2x+7)$

14 $\lim_{x\to 4}(5x^2-9x-8)$

15 $\lim_{x\to\sqrt{2}}(x^2+3)(x-4)$

16 $\lim_{t\to -3}(3t+4)(7t-9)$

17 $\lim_{x\to\pi}(x-3.1416)$

18 $\lim_{x\to\pi}(\frac{1}{2}x-\frac{11}{7})$

19 $\lim_{s\to 4}\dfrac{6s-1}{2s-9}$

20 $\lim_{x\to 1/2}\dfrac{4x^2-6x+3}{16x^3+8x-7}$

21 $\lim_{x\to 1/2}\dfrac{2x^2+5x-3}{6x^2-7x+2}$

22 $\lim_{x\to 2}\dfrac{x-2}{x^3-8}$

23 $\lim_{x\to 2}\dfrac{x^2-x-2}{(x-2)^2}$

24 $\lim_{x\to -2}\dfrac{x^2+2x-3}{x^2+5x+6}$

25 $\lim_{x\to -2}\dfrac{x^3+8}{x^4-16}$

26 $\lim_{x\to 16}\dfrac{x-16}{\sqrt{x}-4}$

27 $\lim_{x\to 2}\dfrac{(1/x)-(1/2)}{x-2}$

28 $\lim_{x\to -3}\dfrac{x+3}{(1/x)+(1/3)}$

29 $\lim_{x\to 1}\left(\dfrac{x^2}{x-1}-\dfrac{1}{x-1}\right)$

30 $\lim_{x\to 1}\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^6$

31 $\lim_{x\to 16}\dfrac{2\sqrt{x}+x^{3/2}}{\sqrt[4]{x}+5}$

32 $\lim_{x\to -8}\dfrac{16x^{2/3}}{4-x^{4/3}}$

33 $\lim_{x\to 4}\sqrt[3]{x^2-5x-4}$

34 $\lim_{x\to -2}\sqrt{x^4-4x+1}$

35 $\lim_{x\to 3}\sqrt[3]{\dfrac{2+5x-3x^3}{x^2-1}}$

36 $\lim_{x\to\pi}\sqrt[5]{\dfrac{x-\pi}{x+\pi}}$

37 $\lim_{h\to 0}\dfrac{4-\sqrt{16+h}}{h}$

38 $\lim_{h\to 0}\left(\dfrac{1}{h}\right)\left(\dfrac{1}{\sqrt{1+h}}-1\right)$

39 $\lim_{x\to 1}\dfrac{x^2+x-2}{x^5-1}$

40 $\lim_{x\to 2}\dfrac{x^2-7x+10}{x^6-64}$

41 $\lim_{v\to 3}v^2(3v-4)(9-v^3)$

42 $\lim_{k\to 2}\sqrt{3k^2+4}\,\sqrt[3]{3k+2}$

43 $\lim_{x\to 5^+}\left(\sqrt{x^2-25}+3\right)$

44 $\lim_{x\to 3^-}x\sqrt{9-x^2}$

45 $\lim_{x\to 3^+}\dfrac{\sqrt{(x-3)^2}}{x-3}$

46 $\lim_{x\to -10^-}\dfrac{x+10}{\sqrt{(x+10)^2}}$

47 $\lim_{x\to 5^+}\dfrac{1+\sqrt{2x-10}}{x+3}$

48 $\lim_{x\to 4^+}\dfrac{\sqrt[4]{x^2-16}}{x+4}$

**Exer. 49–52: Find each limit, if it exists:**

(a) $\lim_{x\to a^-}f(x)$ (b) $\lim_{x\to a^+}f(x)$ (c) $\lim_{x\to a}f(x)$

49 $f(x)=\sqrt{5-x}$; $a=5$

50 $f(x)=\sqrt{8-x^3}$; $a=2$

51 $f(x)=\sqrt[3]{x^3-1}$; $a=1$

52 $f(x)=x^{2/3}$; $a=-8$

**Exer. 53–56: Let $n$ denote an arbitrary integer. Sketch the graph of $f$ and find $\lim_{x\to n^-}f(x)$ and $\lim_{x\to n^+}f(x)$.**

53 $f(x)=(-1)^n$ if $n\le x<n+1$

54 $f(x)=n$ if $n\le x<n+1$

55 $f(x)=\begin{cases}x & \text{if } x=n\\ 0 & \text{if } x\ne n\end{cases}$

56 $f(x)=\begin{cases}0 & \text{if } x=n\\ 1 & \text{if } x\ne n\end{cases}$

**Exer. 57–60: Let $[\![\ ]\!]$ denote the greatest integer function and $n$ an arbitrary integer. Find**

(a) $\lim_{x\to n^-}f(x)$ (b) $\lim_{x\to n^+}f(x)$

57 $f(x)=[\![x]\!]$

58 $f(x)=x-[\![x]\!]$

59 $f(x)=-[\![-x]\!]$

60 $f(x)=[\![x]\!]-x^2$

**Exer. 61–64: Use the sandwich theorem to verify the limit.**

61 $\lim_{x\to 0}(x^2+1)=1$ (*Hint:* Use $\lim_{x\to 0}(|x|+1)=1$.)

62 $\lim_{x\to 0}\dfrac{|x|}{\sqrt{x^4+4x^2+7}}=0$

(*Hint:* Use $f(x)=0$ and $g(x)=|x|$.)

63 $\lim_{x\to 0}x\sin(1/x)=0$

(*Hint:* Use $f(x)=-|x|$ and $g(x)=|x|$.)

64 $\lim_{x\to 0}x^4\sin(1/\sqrt[3]{x})=0$ (*Hint:* See Example 9.)

65 If $0\le f(x)\le c$ for some real number $c$, prove that $\lim_{x\to 0}x^2 f(x)=0$.

**66** If $\lim_{x\to a} f(x) = L \neq 0$ and $\lim_{x\to a} g(x) = 0$, prove that $\lim_{x\to a}[f(x)/g(x)]$ does not exist. (*Hint:* Assume there is a number $M$ such that $\lim_{x\to a}[f(x)/g(x)] = M$ and consider $\lim_{x\to a} f(x) = \lim_{x\to a}[g(x)\cdot f(x)/g(x)]$.)

**67** Explain why $\lim_{x\to 0}\left(x \sin \frac{1}{x}\right) \neq \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \sin \frac{1}{x}\right)$.

**68** Explain why $\lim_{x\to 0}\left(\frac{1}{x} + x\right) \neq \lim_{x\to 0} \frac{1}{x} + \lim_{x\to 0} x$.

**69** Charles's law for gases states that if the pressure remains constant, then the relationship between the volume $V$ that a gas occupies and its temperature $T$ (in °C) is given by $V = V_0(1 + \frac{1}{273}T)$. The temperature $T = -273\,°\mathrm{C}$ is *absolute zero.*

**(a)** Find $\lim_{T\to -273^+} V$.

**(b)** Why is a right-hand limit necessary?

**70** According to the theory of relativity, the length of an object depends on its velocity $v$ (see Example 11). Einstein also proved that the mass $m$ of an object is related to $v$ by the formula

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}},$$

where $m_0$ is the mass of the object at rest.

**(a)** Investigate $\lim_{v\to c^-} m$.

**(b)** Why is a left-hand limit necessary?

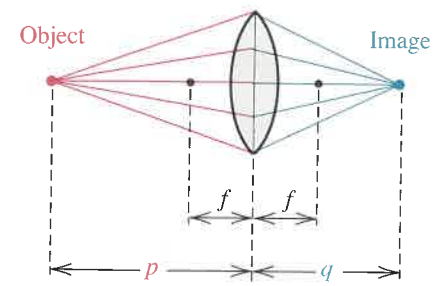
**71** A convex lens has focal length $f$ centimeters. If an object is placed a distance $p$ centimeters from the lens, then the distance $q$ centimeters of the image from the lens is related to $p$ and $f$ by the *lens equation*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

As shown in the figure, $p$ must be greater than $f$ for the rays to converge.

**(a)** Investigate $\lim_{p\to f^+} q$.

Exercise 71



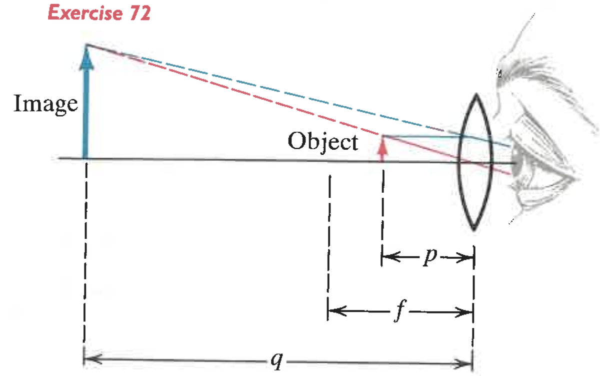

**(b)** What is happening to the image as $p \to f^+$?

**72** Shown in the figure is a simple magnifier consisting of a convex lens. The object to be magnified is positioned so that its distance $p$ from the lens is less than the focal length $f$. The linear magnification $M$ is the ratio of the image size to the object size. Using similar triangles, we obtain $M = q/p$, where $q$ is the distance between the image and the lens.

**(a)** Find $\lim_{p\to 0^+} M$ and explain why a right-hand limit is necessary.

**(b)** Investigate $\lim_{p\to f^-} M$ and explain what is happening to the image size as $p \to f^-$.

Exercise 72




## 1.4 LIMITS INVOLVING INFINITY

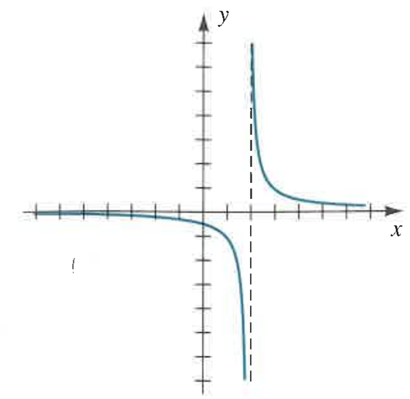
When investigating $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, we may find that as $x$ approaches $a$, the function value $f(x)$ either increases without bound or decreases without bound. To illustrate, let us consider

$$f(x) = \frac{1}{x-2}.$$



Figure 1.27

$f(x) = \dfrac{1}{x-2}$




The graph of $f$ is sketched in Figure 1.27. We can show, as in Example 4 of Section 1.1 and Example 3 of Section 1.2, that

$$\lim_{x\to 2} \frac{1}{x-2} \quad \text{does not exist.}$$

Some function values for $x$ near 2, with $x > 2$, are listed in the following table.

| $x$ | 2.1 | 2.01 | 2.001 | 2.0001 | 2.00001 | 2.000001 |
|---|---|---|---|---|---|---|
| $f(x)$ | 10 | 100 | 1000 | 10,000 | 100,000 | 1,000,000 |

As $x$ approaches 2 from the right, $f(x)$ *increases without bound* in the sense that *we can make $f(x)$ as large as desired by choosing $x$ sufficiently close to* 2 *and* $x > 2$. We denote this by writing

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty, \quad \text{or} \quad \frac{1}{x-2} \to \infty \quad \text{as} \quad x \to 2^+.$$

*The symbol* $\infty$ (**infinity**) *does not represent a real number.* It is a notation we use to denote how certain functions behave. Thus, although we may state that as $x$ *approaches* 2 *from the right,* $1/(x-2)$ *approaches* $\infty$ (*or tends to* $\infty$), or that *the limit of* $1/(x-2)$ *equals* $\infty$, we do *not* mean that $1/(x-2)$ gets closer to some specific real number nor do we mean that $\lim_{x\to 2^+}[1/(x-2)]$ *exists.*

The symbol $-\infty$ (**minus infinity**) is used in similar fashion to denote that $f(x)$ *decreases without bound* (takes on very large *negative* values) as $x$ approaches a real number. Thus, for $f(x) = 1/(x-2)$ (see Figure 1.27), we write

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty, \quad \text{or} \quad \frac{1}{x-2} \to -\infty \quad \text{as} \quad x \to 2^-.$$

Figure 1.28 shows typical (partial) graphs of arbitrary functions that approach $\infty$ or $-\infty$ in various ways. We have pictured $a$ as positive; however, we can also have $a \leq 0$.

Figure 1.28

$\lim_{x\to a^-} f(x) = \infty$, or $f(x) \to \infty$ as $x \to a^-$

$\lim_{x\to a^+} f(x) = \infty$, or $f(x) \to \infty$ as $x \to a^+$

$\lim_{x\to a^-} f(x) = -\infty$, or $f(x) \to -\infty$ as $x \to a^-$

$\lim_{x\to a^+} f(x) = -\infty$, or $f(x) \to -\infty$ as $x \to a^+$

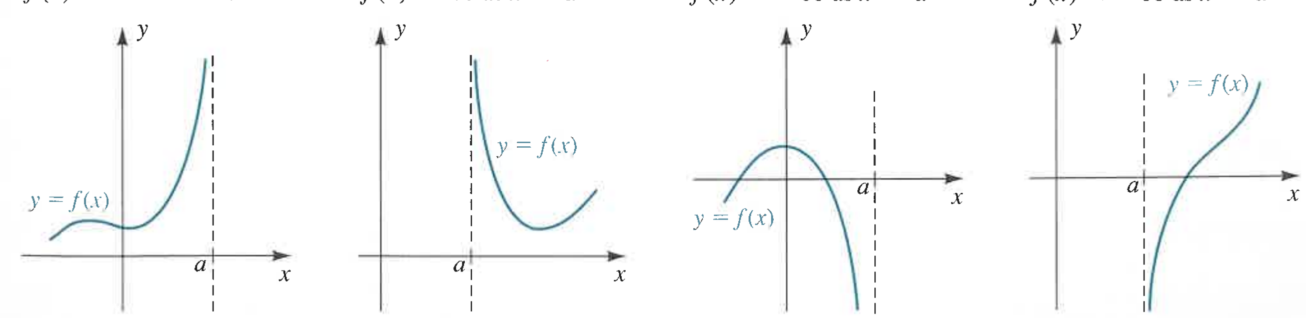
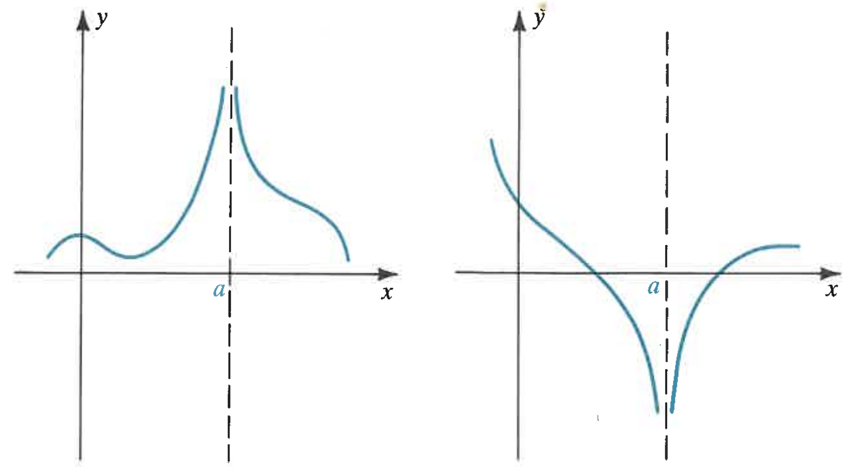


Figure 1.29

$\lim_{x \to a} f(x) = \infty$, or $f(x) \to \infty$ as $x \to a$ $\quad$ $\lim_{x \to a} f(x) = -\infty$, or $f(x) \to -\infty$ as $x \to a$



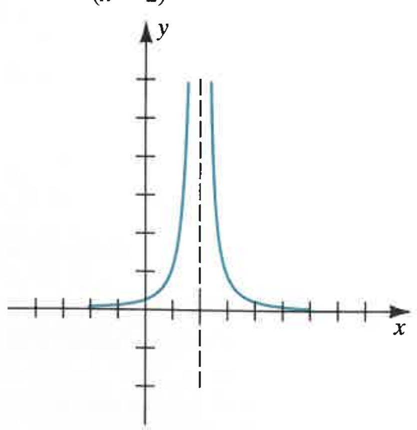

We now consider the two-sided limits illustrated in Figure 1.29. The line $x = a$ in Figures 1.28 and 1.29 is called a **vertical asymptote** for the graph of $f$.

Note that for $f(x)$ to approach $\infty$ as $x$ approaches $a$, *both* the right-hand and left-hand limits must be $\infty$. For $f(x)$ to approach $-\infty$, *both* one-sided limits must be $-\infty$. If the limit of $f(x)$ from one side of $a$ is $\infty$ and from the other side of $a$ is $-\infty$, as in Figure 1.27, we say that $\lim_{x \to a} f(x)$ *does not exist.*

It is possible to investigate many algebraic functions that approach $\infty$ or $-\infty$ by reasoning intuitively, as in the following examples. A formal definition that can be used for rigorous proofs is stated at the end of this section.

Figure 1.30

$f(x) = \dfrac{1}{(x-2)^2}$




**EXAMPLE 1** Find $\displaystyle\lim_{x \to 2} \frac{1}{(x-2)^2}$, if it exists.

**SOLUTION** If $x$ is close to 2 and $x \neq 2$, then $(x-2)^2$ is positive and close to 0. Hence, the reciprocal of $(x-2)^2$, $1/(x-2)^2$, is positive and large. There is no real number $L$ that is the limit of $1/(x-2)^2$ as $x$ approaches 0. The limit does not exist, because we can make $1/(x-2)^2$ as large as desired by choosing $x$ sufficiently close to 2. Since $1/(x-2)^2$ increases without bound, we may write

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty.$$

The graph of $y = 1/(x-2)^2$ is sketched in Figure 1.30. The line $x = 2$ is a vertical asymptote for the graph.

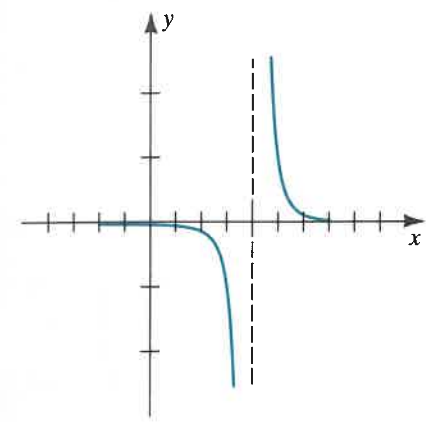


**EXAMPLE 2** Find each limit, if it exists.

**(a)** $\displaystyle\lim_{x \to 4^-} \frac{1}{(x-4)^3}$ $\quad$ **(b)** $\displaystyle\lim_{x \to 4^+} \frac{1}{(x-4)^3}$ $\quad$ **(c)** $\displaystyle\lim_{x \to 4} \frac{1}{(x-4)^3}$

Figure 1.31

$f(x) = \dfrac{1}{(x-4)^3}$




**SOLUTION** In all three cases, the limit does not exist, because the denominator approaches 0 as $x$ approaches 4 and hence the fraction has an unbounded absolute value.

**(a)** If $x$ is close to 4 and $x < 4$, then $x - 4$ is close to 0 and *negative,* and

$$\lim_{x \to 4^-} \frac{1}{(x-4)^3} = -\infty.$$

**(b)** If $x$ is close to 4 and $x > 4$, then $x - 4$ is close to 0 and *positive,* and

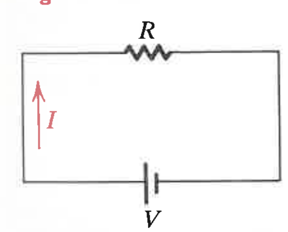
$$\lim_{x \to 4^+} \frac{1}{(x-4)^3} = \infty.$$

**(c)** Since the one-sided limits are not both $\infty$ or both $-\infty$, we can only conclude that

$$\lim_{x \to 4} \frac{1}{(x-4)^3} \quad \text{does not exist.}$$

The graph of $y = 1/(x-4)^3$ is sketched in Figure 1.31. The line $x = 4$ is a vertical asymptote for the graph.

Figure 1.32

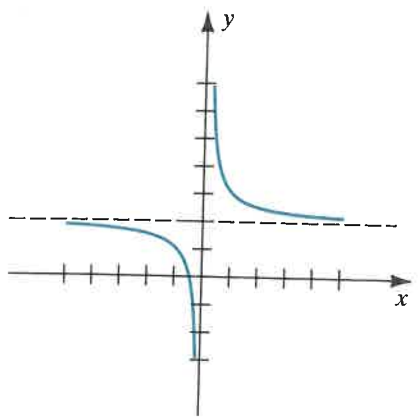



Formulas that represent physical quantities may lead to limits involving infinity. Obviously, a physical quantity cannot approach infinity, but an analysis of a hypothetical situation in which that *could* occur may suggest uses for other related quantities. For example, consider Ohm's law in electrical theory, which states that $I = V/R$, where $R$ is the resistance (in ohms) of a conductor, $V$ is the potential difference (in volts) across the conductor, and $I$ is the current (in amperes) that flows through the conductor (see Figure 1.32). The resistance of certain alloys approaches zero as the temperature approaches absolute zero (approximately $-273$ °C), and the alloy becomes a *superconductor* of electricity. If the voltage $V$ is fixed, then, for such a superconductor,

$$\lim_{R \to 0^+} I = \lim_{R \to 0^+} \frac{V}{R} = \infty;$$

that is, the current increases without bound. Superconductors allow very large currents to be used in generating plants or motors. They also have applications in experimental high-speed ground transportation, where the strong magnetic fields produced by superconducting magnets enable trains to levitate so that there is essentially no friction between the wheels and the track. Perhaps the most important use for superconductors is in circuits for computers, because such circuits produce very little heat.

Figure 1.33
$f(x) = 2 + \dfrac{1}{x}$




Let us next discuss functions whose values approach some number $L$ as $|x|$ becomes very large. Consider

$$f(x) = 2 + \frac{1}{x},$$

the graph of which is sketched in Figure 1.33. Some values of $f(x)$ if $x$ is large are listed in the following table.

| $x$ | 100 | 1000 | 10,000 | 100,000 | 1,000,000 |
|---|---|---|---|---|---|
| $f(x)$ | 2.01 | 2.001 | 2.0001 | 2.00001 | 2.000001 |

We can make $f(x)$ as close to 2 as desired by choosing $x$ sufficiently large. We denote this fact by

$$\lim_{x\to\infty}\left(2 + \frac{1}{x}\right) = 2,$$

which may be read *the limit of* $2 + (1/x)$ *as* $x$ *approaches* $\infty$ *is* 2.

Once again, remember that $\infty$ is not a real number, and hence $\infty$ *should never be substituted for the variable* $x$. Note that the terminology $x$ *approaches* $\infty$ does not mean that $x$ gets close to some real number. Intuitively, we think of $x$ as increasing without bound or being assigned arbitrarily large values.

If we let $x$ *decrease* without bound—that is, if we let $x$ take on very large *negative* values—then, as indicated by the second-quadrant portion of the graph shown in Figure 1.33, $2 + (1/x)$ again approaches 2, and we write

$$\lim_{x\to-\infty}\left(2 + \frac{1}{x}\right) = 2.$$

Before considering additional examples, let us state definitions for such limits involving infinity, using $\epsilon$-tolerances for $f(x)$ at $L$. When we considered $\lim_{x\to a} f(x) = L$ in Section 1.2, we wanted $|f(x) - L| < \epsilon$ whenever $x$ was close to $a$ and $x \neq a$. In the present situation, we want $|f(x) - L| < \epsilon$ whenever $x$ is *sufficiently large*—say, larger than any given positive number $M$. The precise definition for the limit of a function as $x$ increases without bound follows next.

**Definition 1.16**

Let a function $f$ be defined on an infinite interval $(c, \infty)$ for a real number $c$, and let $L$ be a real number. The statement

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\epsilon > 0$, there is a number $M > 0$ such that

$$\text{if} \quad x > M, \quad \text{then} \quad |f(x) - L| < \epsilon.$$



If $\lim_{x\to\infty} f(x) = L$, we say that *the limit of* $f(x)$ *as* $x$ *approaches* $\infty$ *is* $L$, or that $f(x)$ *approaches* $L$ *as* $x$ *approaches* $\infty$. We sometimes write

$$f(x) \to L \quad \text{as} \quad x \to \infty.$$

We may give a graphical interpretation of $\lim_{x\to\infty} f(x) = L$ as follows. Consider any horizontal lines $y = L \pm \epsilon$, as in Figure 1.34. According to Definition (1.16), if $x$ is larger than some positive number $M$, the point $P(x, f(x))$ on the graph lies between these horizontal lines. Intuitively, we know that the graph of $f$ gets closer to the line $y = L$ as $x$ gets larger. We call the line $y = L$ a **horizontal asymptote** for the graph of $f$. As illustrated in Figure 1.34, *a graph may cross a horizontal asymptote.* The line $y = 2$ in Figure 1.33 is a horizontal asymptote for the graph of $f(x) = 2 + (1/x)$.

Figure 1.34 $\lim_{x\to\infty} f(x) = L$

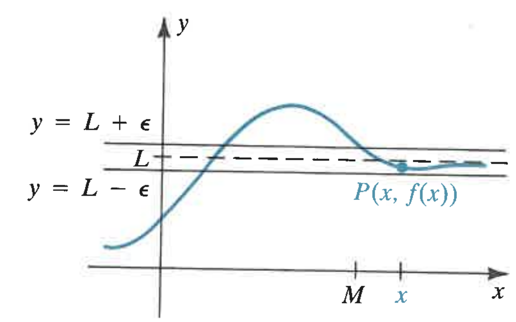
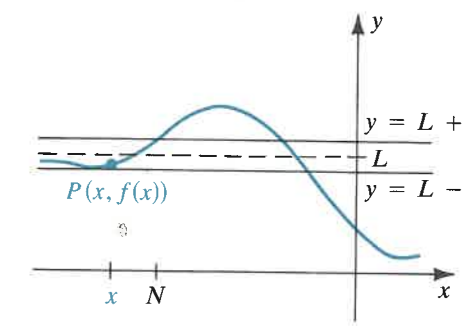



Figure 1.35 $\lim_{x\to-\infty} f(x) = L$




In Figure 1.34, the graph of $f$ approaches the asymptote $y = L$ from below—that is, with $f(x) < L$. A graph can also approach $y = L$ from above—that is, with $f(x) > L$—or in other ways, such as with $f(x)$ alternately greater than and less than $L$ as $x \to \infty$.

The next definition covers the case in which $x$ is a *large negative* number.

**Definition 1.17**

Let a function $f$ be defined on an infinite interval $(-\infty, c)$ for a real number $c$, and let $L$ be a real number. The statement

$$\lim_{x\to-\infty} f(x) = L$$

means that for every $\epsilon > 0$, there is a number $N < 0$ such that

$$\text{if} \quad x < N, \quad \text{then} \quad |f(x) - L| < \epsilon.$$

If $\lim_{x\to-\infty} f(x) = L$, we say *the limit of* $f(x)$ *as* $x$ *approaches* $-\infty$ *is* $L$, or that $f(x)$ *approaches* $L$ *as* $x$ *approaches* $-\infty$.

Definition (1.17) is illustrated in Figure 1.35. If we consider any horizontal lines $y = L \pm \epsilon$, then every point $P(x, f(x))$ on the graph lies between these lines if $x$ is *less* than some negative number $N$. The line $y = L$ is a horizontal asymptote for the graph of $f$.

Limit theorems that are analogous to those in Section 1.3 may be established for limits involving infinity. In particular, Theorem (1.8) concerning limits of sums, products, and quotients is true for $x \to \infty$ or $x \to -\infty$. Similarly, Theorem (1.14) on the limit of $\sqrt[n]{f(x)}$ holds if $x \to \infty$ or $x \to -\infty$. We can also show that

$$\lim_{x\to\infty} c = c \quad \text{and} \quad \lim_{x\to-\infty} c = c.$$

A proof of the next theorem, using Definition (1.16), is given in Appendix I.

**Theorem 1.18**

If $k$ is a positive rational number and $c$ is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^k} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{c}{x^k} = 0,$$

provided $x^k$ is always defined.

Theorem 1.18 is useful for investigating limits of rational functions. Specifically, *to find* $\lim_{x\to\infty} f(x)$ *or* $\lim_{x\to-\infty} f(x)$ *for a rational function $f$, first divide the numerator and the denominator of $f(x)$ by $x^n$, where $n$ is the highest power of $x$ that appears in the denominator, and then use limit theorems.* This technique is illustrated in the next examples.

**EXAMPLE 3** Find $\displaystyle\lim_{x\to-\infty} \frac{2x^2-5}{3x^2+x+2}$.

**SOLUTION** The highest power of $x$ in the denominator is 2. Hence, by the rule stated in the preceding paragraph, we divide the numerator and the denominator by $x^2$ and then use limit theorems. Thus,

$$\lim_{x\to-\infty} \frac{2x^2-5}{3x^2+x+2} = \lim_{x\to-\infty} \frac{\dfrac{2x^2-5}{x^2}}{\dfrac{3x^2+x+2}{x^2}} = \lim_{x\to-\infty} \frac{2-\dfrac{5}{x^2}}{3+\dfrac{1}{x}+\dfrac{2}{x^2}}$$

$$= \frac{\lim_{x\to-\infty}\left(2-\dfrac{5}{x^2}\right)}{\lim_{x\to-\infty}\left(3+\dfrac{1}{x}+\dfrac{2}{x^2}\right)}$$

$$= \frac{\lim_{x\to-\infty} 2 - \lim_{x\to-\infty} \dfrac{5}{x^2}}{\lim_{x\to-\infty} 3 + \lim_{x\to-\infty} \dfrac{1}{x} + \lim_{x\to-\infty} \dfrac{2}{x^2}}$$

$$= \frac{2-0}{3+0+0} = \frac{2}{3}.$$



It follows that the line $y = \frac{2}{3}$ is a horizontal asymptote for the graph of $f$.

**EXAMPLE 4** Find $\displaystyle\lim_{x\to\infty} \frac{2x^3-5}{3x^2+x+2}$.

**SOLUTION** The highest power of $x$ in the denominator is 2, so we first divide the numerator and the denominator by $x^2$, obtaining

$$\lim_{x\to\infty} \frac{2x^3-5}{3x^2+x+2} = \lim_{x\to\infty} \frac{2x-\dfrac{5}{x^2}}{3+\dfrac{1}{x}+\dfrac{2}{x^2}}.$$

Since each term of the form $c/x^k$ approaches 0 as $x \to \infty$, we see that

$$\lim_{x\to\infty}\left(2x-\frac{5}{x^2}\right) = \infty$$

and

$$\lim_{x\to\infty}\left(3+\frac{1}{x}+\frac{2}{x^2}\right) = 3.$$

It follows that

$$\lim_{x\to\infty} \frac{2x^3-5}{3x^2+x+2} = \infty.$$

**EXAMPLE 5** If $f(x) = \dfrac{\sqrt{9x^2+2}}{4x+3}$, find $\lim_{x\to\infty} f(x)$.

**SOLUTION** If $x$ is large and positive, then

$$\sqrt{9x^2+2} \approx \sqrt{9x^2} = 3x$$

and

$$4x+3 \approx 4x$$

and hence

$$f(x) = \frac{\sqrt{9x^2+2}}{4x+3} \approx \frac{3x}{4x} = \frac{3}{4}.$$

This approximation *suggests* that $\lim_{x\to\infty} f(x) = \frac{3}{4}$. To give a rigorous proof, we may write

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(9+\dfrac{2}{x^2}\right)}}{4x+3}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{9+\dfrac{2}{x^2}}}{4x+3}.$$

If $x$ is positive, then $\sqrt{x^2} = x$, and dividing the numerator and the denominator of the last fraction by $x$ gives us

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x\to\infty} \frac{\sqrt{9+\dfrac{2}{x^2}}}{4+\dfrac{3}{x}}$$

$$= \frac{\sqrt{9+0}}{4+0} = \frac{3}{4}.$$

**EXAMPLE ■ 6** Glucose is transported within an enzyme-glucose complex through the placenta from the mother to the fetus. The enzyme acts as a catalyst, accelerating the transportation process. The *Michaelis–Menten law,*

$$C(x) = \frac{ax}{x+b},$$

approximates the relationship between the concentration $C$ of the enzyme-glucose complex and the concentration $x$ of glucose ($a$ and $b$ are positive constants).

**(a)** Determine $\lim_{x\to\infty} C(x)$.

**(b)** Sketch the graph of $C$ for $a = 8$ and $b = 3$.

**SOLUTION**

**(a)** Since the highest power of $x$ in the denominator is 1, we first divide the numerator and the denominator by $x$, obtaining

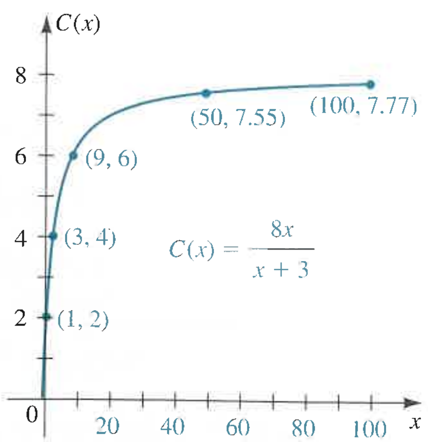
$$\lim_{x\to\infty} C(x) = \lim_{x\to\infty} \frac{ax}{x+b} = \lim_{x\to\infty} \frac{a}{1+\dfrac{b}{x}} = \frac{a}{1+0} = a.$$

**(b)** From $C(x) = 8x/(x+3)$, we have $C(0) = 0$ and $C(x) > 0$ when $x > 0$. Writing $C(x)$ as $8/[1+(3/x)]$, we see that $C(x)$ increases as $x$ increases (the numerator is constant and the denominator decreases in value). In the following table, we show values of $C(x)$ for several values of $x$.

| $x$ | 0 | 1 | 3 | 9 | 21 | 50 | 100 | 200 |
|---|---|---|---|---|---|---|---|---|
| $C(x)$ | 0 | 2 | 4 | 6 | 7 | 7.55 | 7.77 | 7.88 |

Plotting these points and using the fact that the limit of $C(x)$ as $x \to \infty$ is $a = 8$, we obtain the graph shown in Figure 1.36.

Figure 1.36
$\lim_{x\to\infty} \frac{8x}{x+3} = 8$






We may also consider cases in which *both* $x$ and $f(x)$ approach $\infty$ or $-\infty$. For example, the limit statement

$$\lim_{x\to-\infty} f(x) = \infty$$

means that $f(x)$ increases without bound as $x$ decreases without bound, as would be the case for $f(x) = x^2$.

The preceding types of limits involving $\infty$ occur in applications. To illustrate, Newton's law of universal gravitation may be stated: *Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses and inversely proportional to the square of the distance between them.* In symbols, this statement may be represented by

$$F = G\,\frac{m_1 m_2}{r^2},$$

where $F$ is the force on each particle, $m_1$ and $m_2$ are their masses, $r$ is the distance between them, and $G$ is a gravitational constant. Assuming that $m_1$ and $m_2$ are constant, we obtain

$$\lim_{r\to\infty} F = \lim_{r\to\infty} G\,\frac{m_1 m_2}{r^2} = 0,$$

which tells us that as the distance between the particles increases without bound, the force of attraction approaches 0. Theoretically, there is always *some* attraction; however, if $r$ is very large, the attraction cannot be measured with conventional laboratory equipment.

We shall conclude this section by stating a formal definition of $\lim_{x\to a} f(x) = \infty$. The main difference from our work in Section 1.2 is that instead of showing that $|f(x) - L| < \epsilon$ whenever $x$ is near $a$, we consider any (large) positive number $M$ and show that $f(x) > M$ whenever $x$ is near $a$.

**Definition 1.19**

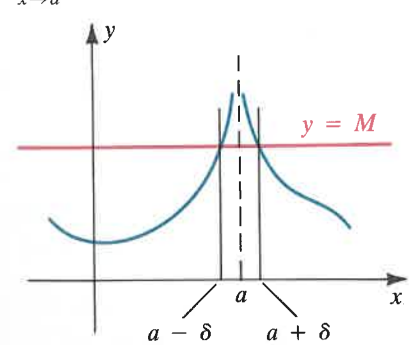
Let a function $f$ be defined on an open interval containing $a$, except possibly at $a$ itself. The statement

$$\lim_{x\to a} f(x) = \infty$$

means that for every $M > 0$, there is a $\delta > 0$ such that

if $0 < |x - a| < \delta$, then $f(x) > M$.

Figure 1.37
$\lim_{x\to a} f(x) = \infty$




For a graphical interpretation of Definition (1.19), consider any horizontal line $y = M$, as in Figure 1.37. If $\lim_{x\to a} f(x) = \infty$, then whenever $x$ is in a suitable interval $(a-\delta, a+\delta)$ and $x \neq a$, the points on the graph of $f$ lie *above* the horizontal line.

To define $\lim_{x\to a} f(x) = -\infty$, we may alter Definition (1.19), replacing $M > 0$ by $N < 0$ and $f(x) > M$ by $f(x) < N$. Then if we consider any horizontal line $y = N$ (with $N$ negative), the graph of $f$ lies *below* this line whenever $x$ is in a suitable interval $(a-\delta, a+\delta)$ and $x \neq a$.

## EXERCISES 1.4

**Exer. 1–10: For the given $f(x)$, express each of the following limits as $\infty$, $-\infty$, or DNE (Does Not Exist):**

(a) $\lim_{x\to a^-} f(x)$ (b) $\lim_{x\to a^+} f(x)$ (c) $\lim_{x\to a} f(x)$

1 $f(x)=\dfrac{5}{x-4}$; $a=4$

2 $f(x)=\dfrac{5}{4-x}$; $a=4$

3 $f(x)=\dfrac{8}{(2x+5)^3}$; $a=-\dfrac{5}{2}$

4 $f(x)=\dfrac{-4}{7x+3}$; $a=-\dfrac{3}{7}$

5 $f(x)=\dfrac{3x}{(x+8)^2}$; $a=-8$

6 $f(x)=\dfrac{3x^2}{(2x-9)^2}$; $a=\dfrac{9}{2}$

7 $f(x)=\dfrac{2x^2}{x^2-x-2}$; $a=-1$

8 $f(x)=\dfrac{4x}{x^2-4x+3}$; $a=1$

9 $f(x)=\dfrac{1}{x(x-3)^2}$; $a=3$

10 $f(x)=\dfrac{-1}{(x+1)^2}$; $a=-1$

**Exer. 11–24: Find the limit, if it exists.**

11 $\lim_{x\to\infty}\dfrac{5x^2-3x+1}{2x^2+4x-7}$

12 $\lim_{x\to\infty}\dfrac{3x^3-x+1}{6x^3+2x^2-7}$

13 $\lim_{x\to-\infty}\dfrac{4-7x}{2+3x}$

14 $\lim_{x\to-\infty}\dfrac{(3x+4)(x-1)}{(2x+7)(x+2)}$

15 $\lim_{x\to-\infty}\dfrac{2x^2-3}{4x^3+5x}$

16 $\lim_{x\to\infty}\dfrac{2x^2-x+3}{x^3+1}$

17 $\lim_{x\to\infty}\dfrac{-x^3+2x}{2x^2-3}$

18 $\lim_{x\to-\infty}\dfrac{x^2+2}{x-1}$

19 $\lim_{x\to-\infty}\dfrac{2-x^2}{x+3}$

20 $\lim_{x\to\infty}\dfrac{3x^4+x+1}{x^2-5}$

21 $\lim_{x\to\infty}\sqrt[3]{\dfrac{8+x^2}{x(x+1)}}$

22 $\lim_{x\to-\infty}\dfrac{4x-3}{\sqrt{x^2+1}}$

23 $\lim_{x\to\infty}\sin x$

24 $\lim_{x\to\infty}\cos x$

c **Exer. 25–26: Investigate the limit by letting $x=10^n$ for $n=1, 2, 3,$ and 4.**

25 $\lim_{x\to\infty}\dfrac{1}{x}\tan\left(\dfrac{\pi}{2}-\dfrac{1}{x}\right)$

26 $\lim_{x\to\infty}\sqrt{x}\sin\dfrac{1}{x}$

**Exer. 27–36: Find the vertical and horizontal asymptotes for the graph of $f$.**

27 $f(x)=\dfrac{1}{x^2-4}$

28 $f(x)=\dfrac{5x}{4-x^2}$

29 $f(x)=\dfrac{2x^2}{x^2+1}$

30 $f(x)=\dfrac{3x}{x^2+1}$

31 $f(x)=\dfrac{1}{x^3+x^2-6x}$

32 $f(x)=\dfrac{x^2-x}{16-x^2}$

33 $f(x)=\dfrac{x^2+3x+2}{x^2+2x-3}$

34 $f(x)=\dfrac{x^2-5x}{x^2-25}$

35 $f(x)=\dfrac{x+4}{x^2-16}$

36 $f(x)=\dfrac{\sqrt[3]{16-x^2}}{4-x}$

**Exer. 37–40: A function $f$ satisfies the given conditions. Sketch a possible graph for $f$, assuming that it does not cross a horizontal asymptote.**

37 $\lim_{x\to-\infty} f(x)=1$; $\lim_{x\to\infty} f(x)=1$;
$\lim_{x\to3^-} f(x)=-\infty$; $\lim_{x\to3^+} f(x)=\infty$

38 $\lim_{x\to-\infty} f(x)=-1$; $\lim_{x\to\infty} f(x)=-1$;
$\lim_{x\to2^-} f(x)=\infty$; $\lim_{x\to2^+} f(x)=-\infty$

39 $\lim_{x\to-\infty} f(x)=-2$; $\lim_{x\to\infty} f(x)=-2$;
$\lim_{x\to3^-} f(x)=\infty$; $\lim_{x\to3^+} f(x)=-\infty$;
$\lim_{x\to-1^-} f(x)=-\infty$; $\lim_{x\to-1^+} f(x)=\infty$

40 $\lim_{x\to-\infty} f(x)=3$; $\lim_{x\to\infty} f(x)=3$;
$\lim_{x\to1^-} f(x)=\infty$; $\lim_{x\to1^+} f(x)=-\infty$;
$\lim_{x\to-2^-} f(x)=-\infty$; $\lim_{x\to-2^+} f(x)=\infty$

41 Salt water of concentration 0.1 lb of salt per gallon flows into a large tank that initially contains 50 gal of pure water.

(a) If the flow rate of salt water into the tank is 5 gal/min, find the volume $V(t)$ of water and the amount $A(t)$ of salt in the tank after $t$ minutes.



(b) Find a formula for the salt concentration $c(t)$ (in pounds per gallon) after $t$ minutes.

(c) What happens to $c(t)$ over a long period of time?

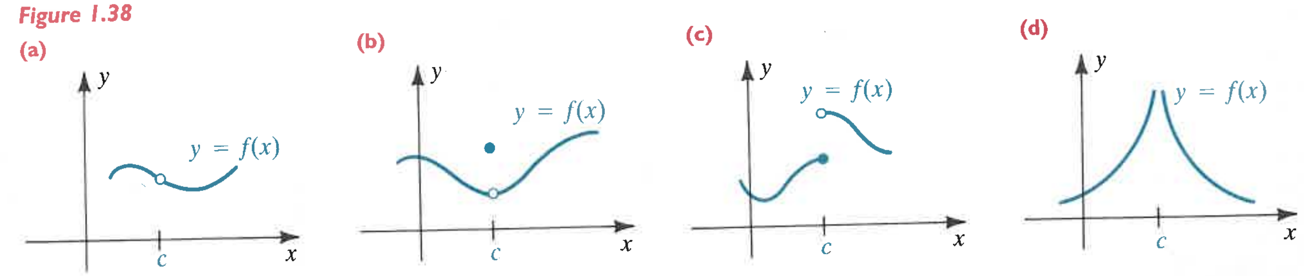
42 An important problem in fishery science is predicting next year's adult breeding population $R$ (the recruits) from the number $S$ that are presently spawning. For some species (such as North Sea herring), the relationship between $R$ and $S$ is given by $R=aS/(S+b)$, where $a$ and $b$ are positive constants. What happens as the number of spawners increases?

## 1.5 CONTINUOUS FUNCTIONS

In everyday usage, we say that time is continuous, since it proceeds in an uninterrupted manner. On any given day, time does not jump from 1:00 P.M. to 1:01 P.M., leaving a gap of one minute. If an object is dropped from a hot air balloon, we regard its subsequent motion as continuous. If the initial altitude is 500 ft above ground, the object passes through every altitude between 500 and 0 ft before it hits the ground. The concentration of a chemical at a particular spot along a river may vary continuously, increasing at some times and decreasing at others. In this section, we will use our knowledge of limits to define *continuous functions* and examine their behavior.

Intuitively, we regard a continuous function as a function whose graph has no breaks, holes, or vertical asymptotes. To illustrate, the graph of each function in Figure 1.38 is *not continuous* at the number $c$.

Figure 1.38




Note that in part (a) of the figure, $f(c)$ is not defined. In part (b), $f(c)$ is defined; however, $\lim_{x\to c} f(x)\neq f(c)$. In part (c), $\lim_{x\to c} f(x)$ does not exist. In part (d), $f(c)$ is undefined and, in addition, $\lim_{x\to c} f(x)=\infty$. The graph of a function $f$ is *not* one of these types if $f$ satisfies the three conditions listed in the next definition.

**Definition 1.20**

A function $f$ is **continuous** at a number $c$ if the following conditions are satisfied:

(i) $f(c)$ is defined
(ii) $\lim_{x\to c} f(x)$ exists
(iii) $\lim_{x\to c} f(x)=f(c)$

Whenever this definition is used to show that a function $f$ is continuous at $c$, it is sufficient to verify only the third condition, because if $\lim_{x\to c} f(x) = f(c)$, then $f(c)$ must be defined and also $\lim_{x\to c} f(x)$ must exist; that is, the first two conditions are satisfied automatically.

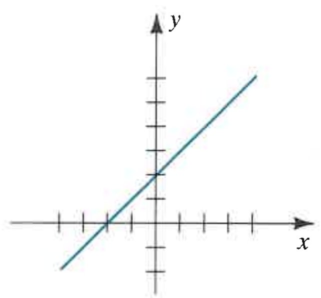
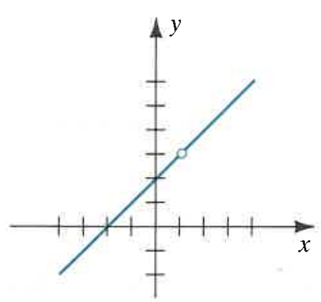
Intuitively, we know that condition (iii) implies that as $x$ gets closer to $c$, the function value $f(x)$ gets closer to $f(c)$. More precisely, we can make *$f(x)$ as close to $f(c)$ as desired* by choosing $x$ sufficiently close to $c$.

If one (or more) of the three conditions in Definition (1.20) is not satisfied, we say that $f$ is **discontinuous** at $c$, or that $f$ has a **discontinuity** at $c$. Certain types of discontinuities are given special names. The discontinuities in parts (a) and (b) of Figure 1.38 are **removable discontinuities,** because we could remove each discontinuity by defining the function value $f(c)$ appropriately. The discontinuity in part (c) is a **jump discontinuity,** so named because of the appearance of the graph. If $f(x)$ approaches $\infty$ or $-\infty$ as $x$ approaches $c$ from either side, as, for example, in part (d), we say that $f$ has an **infinite discontinuity** at $c$.

In general, if a function $f$ is not continuous at $c$, then it has a removable discontinuity at $c$ if the right-hand and left-hand limits exist at $c$ and are equal; a jump discontinuity at $c$ if they are not equal; and an infinite discontinuity if $|f(x)|$ can be made arbitrarily large near $c$.

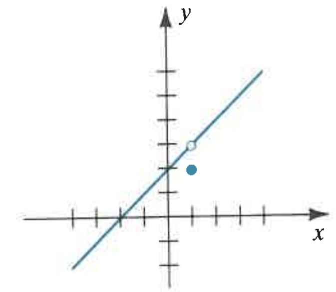
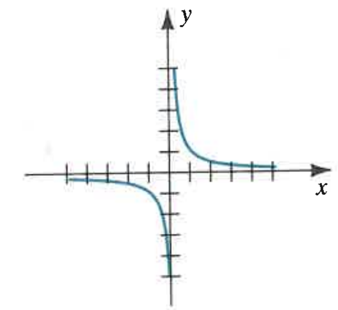
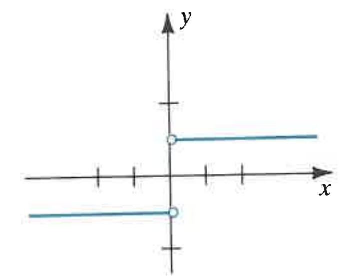
In the following illustration, we reconsider some specific functions that were discussed in Sections 1.1 and 1.2.

## ILLUSTRATION

| Function value | Graph | Discontinuity |
|---|---|---|
| $f(x) = x + 2$ |  | None, since for every $c$, $\lim_{x\to c} f(x) = c + 2 = f(c)$. |
| $g(x) = \dfrac{x^2 + x - 2}{x - 1}$ |  | Removable discontinuity at $c = 1$ since $\lim_{x\to 1^-} g(x) = 3 = \lim_{x\to 1^+} g(x)$ |

*(continued)*



| Function value | Graph | Discontinuity |
|---|---|---|
| $h(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ |  | Removable discontinuity at $c = 1$ since $\lim_{x\to 1^-} h(x) = 3 = \lim_{x\to 1^+} h(x)$ |
| $h(x) = \dfrac{1}{x}$ |  | Infinite discontinuity at $c = 0$ since $\lvert h(x)\rvert$ can be arbitrarily large if $x$ is arbitrarily close to 0. |
| $p(x) = \dfrac{\lvert x\rvert}{x}$ |  | Jump discontinuity at $c = 0$ since $\lim_{x\to 0^-} p(x) = -1$, $\lim_{x\to 0^+} p(x) = 1$, but $-1 \neq 1$. |

The next theorem states that polynomial functions and rational functions (quotients of polynomial functions) are continuous at every number in their domains.

**Theorem 1.21**

**(i)** A polynomial function $f$ is continuous at every real number $c$.
**(ii)** A rational function $q = f/g$ is continuous at every number except the numbers $c$ such that $g(c) = 0$.

## PROOF

**(i)** If $f$ is a polynomial function and $c$ is a real number, then, by Theorem (1.11), $\lim_{x\to c} f(x) = f(c)$. Hence, $f$ is continuous at every real number $c$.

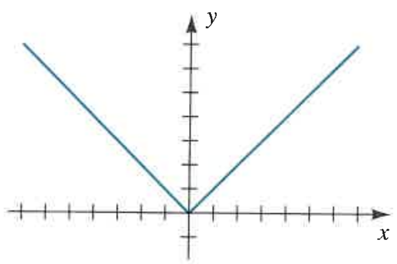
**(ii)** If $g(c) \neq 0$, then $c$ is in the domain of $q = f/g$ and, by Theorem (1.12), $\lim_{x\to c} q(x) = q(c)$; that is, $q$ is continuous at $c$. ■

**NOTE** A similar version of this theorem is true for the trigonometric, exponential, and logarithmic functions: If $q$ is one of these functions and $a$ is in the domain of $q$, then $q$ is continuous at $a$. We will present proofs in Chapters 2 and 6.

**EXAMPLE 1** If $f(x) = |x|$, show that $f$ is continuous at every real number $c$.

**SOLUTION** The graph of $f$ is sketched in Figure 1.39. If $x > 0$, then $f(x) = x$. If $x < 0$, then $f(x) = -x$. Since $x$ and $-x$ are polynomials, it follows from Theorem (1.21)(i) that $f$ is continuous at every nonzero real number. It remains to be shown that $f$ is continuous at 0. The one-sided limits of $f(x)$ at 0 are

Figure 1.39
$f(x) = |x|$




$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

and

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0.$$

Since the right-hand and left-hand limits exist and are equal, it follows from Theorem (1.3) that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0 = |0| = f(0).$$

Hence, $f$ is continuous at 0, and therefore continuous at every real number.

**EXAMPLE 2** If $f(x) = \dfrac{x^2-1}{x^3+x^2-2x}$, find the discontinuities of $f$.

**SOLUTION** Since $f$ is a rational function, it follows from Theorem (1.21) that the only discontinuities occur at the zeros of the denominator, $x^3 + x^2 - 2x$. By factoring, we obtain

$$x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1).$$

Setting each factor equal to zero, we see that the discontinuities of $f$ are at 0, $-2$, and 1.

If a function $f$ is continuous at every number in an open interval $(a, b)$, we say that **$f$ is continuous on the interval $(a, b)$**. Similarly, a function is continuous on an infinite interval of the form $(a, \infty)$ or $(-\infty, b)$ if it is continuous at every number in the interval. The next definition covers the case of a closed interval.

**Definition 1.22**

Let a function $f$ be defined on a closed interval $[a, b]$. The **function $f$ is continuous on $[a, b]$** if it is continuous on $(a, b)$ and if, in addition,

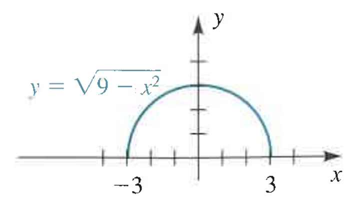
$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$



If a function $f$ has either a right-hand or a left-hand limit of the type indicated in Definition (1.22), we say that **$f$ is continuous from the right at $a$** or that **$f$ is continuous from the left at $b$,** respectively.

**EXAMPLE 3** If $f(x) = \sqrt{9 - x^2}$, sketch the graph of $f$ and prove that $f$ is continuous on the closed interval $[-3, 3]$.

Figure 1.40




**SOLUTION** The graph of $x^2 + y^2 = 9$ is a circle with center at the origin and radius 3. Solving for $y$ gives us $y = \pm\sqrt{9 - x^2}$, and hence the graph of $y = \sqrt{9 - x^2}$ is the upper half of that circle (see Figure 1.40).

If $-3 < c < 3$, then, using Theorem (1.14), we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{9 - x^2} = \sqrt{9 - c^2} = f(c).$$

Hence $f$ is continuous at $c$ by Definition (1.20). All that remains is to check the endpoints of the interval $[-3, 3]$ using one-sided limits as follows:

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9 - x^2} = \sqrt{9 - 9} = 0 = f(-3)$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9 - x^2} = \sqrt{9 - 9} = 0 = f(3)$$

Thus, $f$ is continuous from the right at $-3$ and from the left at 3. By Definition (1.22), $f$ is continuous on $[-3, 3]$.

Strictly speaking, the function $f$ in Example 3 is discontinuous at every number $c$ *outside* of the interval $[-3, 3]$, because $f(c)$ is not a real number if $x < -3$ or $x > 3$. However, it is *not* customary to use the phrase *discontinuous at c* if $c$ is in an open interval throughout which $f$ is undefined.

We may also define continuity on other types of intervals. For example, a function $f$ is continuous on $[a, b)$ or $[a, \infty)$ if it is continuous at every number greater than $a$ in the interval and if, in addition, $f$ is continuous from the right at $a$. For intervals of the form $(a, b]$ or $(-\infty, b]$, we require continuity at every number less than $b$ in the interval and also continuity from the left at $b$.

Using facts stated in Theorem (1.8), we can prove the following.

**Theorem 1.23**

If two functions $f$ and $g$ are continuous at a real number $c$, then the following are also continuous at $c$:

**(i)** the sum $f + g$
**(ii)** the difference $f - g$
**(iii)** the product $fg$
**(iv)** the quotient $f/g$, provided $g(c) \neq 0$

**PROOF** If $f$ and $g$ are continuous at $c$, then

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c).$$

By definition of the sum of two functions,

$$(f+g)(x) = f(x) + g(x).$$

Consequently,

$$\begin{aligned}\lim_{x\to c}(f+g)(x) &= \lim_{x\to c}[f(x)+g(x)]\\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x)\\ &= f(c) + g(c)\\ &= (f+g)(c).\end{aligned}$$

We have thus proved that $f+g$ is continuous at $c$. Parts (ii)–(iv) are proved in similar fashion. ■

If $f$ and $g$ are continuous on an interval, then $f+g$, $f-g$, and $fg$ are continuous on the interval. If, in addition, $g(c) \neq 0$ for every $c$ in the interval, then $f/g$ is continuous on the interval. These results may be extended to more than two functions; that is, sums, differences, products, or quotients involving any number of continuous functions are continuous (provided zero denominators do not occur).

**EXAMPLE ■ 4** If $k(x) = \dfrac{\sqrt{9-x^2}}{3x^4+5x^2+1}$, prove that $k$ is continuous on the closed interval $[-3, 3]$.

**SOLUTION** Let $f(x) = \sqrt{9-x^2}$ and $g(x) = 3x^4+5x^2+1$. From Example 3, $f$ is continuous on $[-3, 3]$, and from Theorem (1.21), $g$ is continuous at every real number. Moreover, $g(c) \neq 0$ for every number $c$ in $[-3, 3]$. Hence, by Theorem (1.23)(iv), the quotient $k = f/g$ is continuous on $[-3, 3]$.

A proof of the next result on the limit of a composite function $f \circ g$ is given in Appendix I.

**Theorem 1.24**

If $\lim_{x\to c} g(x) = b$ and if $f$ is continuous at $b$, then

$$\lim_{x\to c} f(g(x)) = f(b) = f\left(\lim_{x\to c} g(x)\right).$$

The principal use of Theorem (1.24) is to prove other theorems. To illustrate, let us use Theorem (1.24) to prove Theorem (1.14) from Section 1.3, in which we assumed that $\lim_{x\to c} g(x)$ and the indicated $n$th roots exist.



**Conclusion of Theorem 1.14**

$$\lim_{x\to c} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to c} g(x)}$$

**PROOF** Let $f(x) = \sqrt[n]{x}$. Applying Theorem (1.24), which states that

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right),$$

we obtain

$$\lim_{x\to c} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to c} g(x)}.$$ ■

The next theorem follows from Theorem (1.24) and the definitions of a continuous function and of the composite function $f \circ g$.

**Theorem 1.25**

If $g$ is continuous at $c$ and if $f$ is continuous at $g(c)$, then the composite function $f \circ g$ is continuous at $c$; that is,

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(g(c)).$$

**EXAMPLE ■ 5** If $k(x) = |3x^2 - 7x - 12|$, show that $k$ is continuous at every real number.

**SOLUTION** If we let

$$f(x) = |x| \quad \text{and} \quad g(x) = 3x^2 - 7x - 12,$$

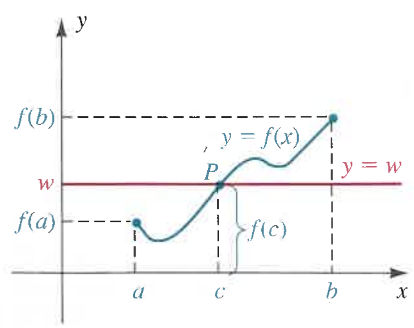
then $k(x) = f(g(x)) = (f \circ g)(x)$. Since both $f$ and $g$ are continuous functions (see Example 1 and (i) of Theorem (1.21)), it follows from Theorem (1.25) that the composite function $k = f \circ g$ is continuous at $c$.

A proof of the following property of continuous functions may be found in more advanced texts on calculus.

**Intermediate Value Theorem 1.26**

If $f$ is continuous on a closed interval $[a, b]$ and if $w$ is any number between $f(a)$ and $f(b)$, then there is at least one number $c$ in $[a, b]$ such that $f(c) = w$.

Figure 1.41




The intermediate value theorem states that *as $x$ varies from $a$ to $b$, the continuous function $f$ takes on every value between $f(a)$ and $f(b)$.* If the graph of the continuous function $f$ is regarded as extending in an unbroken manner from the point $(a, f(a))$ to the point $(b, f(b))$, as illustrated in Figure 1.41, then for any number $w$ between $f(a)$ and $f(b)$, the horizontal line with $y$-intercept $w$ intersects the graph in at least one point $P$. The $x$-coordinate $c$ of $P$ is a number such that $f(c) = w$.

A consequence of the intermediate value theorem is that *if $f(a)$ and $f(b)$ have opposite signs, then there is a number $c$ between $a$ and $b$ such that $f(c) = 0$; that is, $f$ has a zero at $c$.* Thus, if the point $(a, f(a))$ on the graph of a continuous function lies below the $x$-axis and the point $(b, f(b))$ lies above the $x$-axis, or vice versa, then the graph crosses the $x$-axis at some point $(c, 0)$ for $a < c < b$.

We can use this consequence of the intermediate value theorem to help locate zeros of a function, as in the next example.

**EXAMPLE ▪ 6** Let $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$.

**(a)** Use the intermediate value theorem (1.26) to show that $f$ has three zeros in the interval $[-4, 2]$.

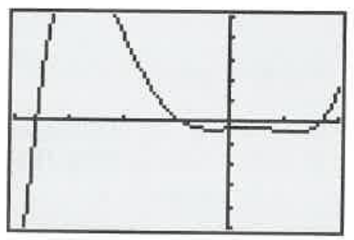
**(b)** Use a graphing utility to approximate these zeros to two decimal places.

**SOLUTION**

**(a)** We compute the value of $f$ at the integers from $-4$ to 2, as shown.

| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $-139$ | 72 | 41 | 2 | $-3$ | $-4$ | 17 |

Figure 1.42
$-4 \le x \le 2, -50 \le y \le 50$



Since $f$ is a polynomial, it is continuous at all values of $x$. By the intermediate value theorem, $f$ has a zero between $-4$ and $-3$ since $f(-4)$ and $f(-3)$ are of opposite sign. Similarly, $f$ has a zero between $-1$ and 0 and another zero between 1 and 2.

**(b)** With the aid of a graphing utility, we can look for other possible zeros. Figure 1.42 shows the graph of $f$, which indicates that only these three zeros exist on the $x$-interval $[-4, 2]$. Using the *trace* and *zoom* features, or a *solve* feature, we determine that the zeros are approximately $-3.60$, $-0.88$, and 1.63.

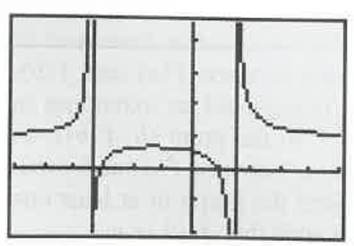
---

Analytic and algebraic techniques can be combined with the effective use of a graphing utility to locate the discontinuities of a given function, as demonstrated in the next example.

**EXAMPLE ▪ 7** Approximate the discontinuities of the function

$$f(x) = 1 + \frac{\sqrt{x+6}}{x^2 + 2x - 5}.$$

Figure 1.43
$-6 \le x \le 5, -2 \le y \le 4.5$



**SOLUTION** The term $\sqrt{x+6}$ restricts the domain of $f$ to those values for which $x + 6 \ge 0$; that is, $x \ge -6$. We select the viewing window $-6 \le x \le 5$ and $-2 \le y \le 4.5$ to obtain Figure 1.43. Note that the graph of $f$ begins at $x = -6$.

You may obtain a graph slightly different from Figure 1.43, depending on the size and the resolution of your screen and the graphing utility you use. In Figure 1.43, the exact behavior of the graph of $f$ may not be entirely



Figure 1.44
$g(x) = x^2 + 2x - 5$
$-5 \le x \le 5, -6 \le y \le 30$

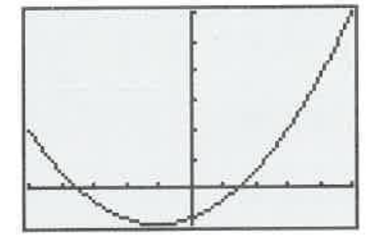
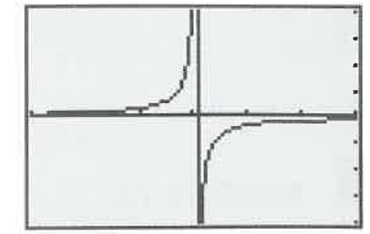


Figure 1.45
$f(x) = 1 + \dfrac{\sqrt{x+6}}{x^2 + 2x - 5}$
$-3.48 \le x \le -3.42, -400 \le y \le 400$



clear, but we *can* see that something unusual occurs at two different values of $x$. In such cases, before zooming in on a segment of the graph, it is useful to do some preliminary analysis of the function.

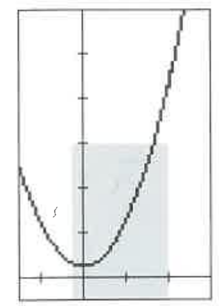
Because the quantity $x^2 + 2x - 5$ appears in the denominator of $f(x)$, we cannot have values of $x$ where $x^2 + 2x - 5 = 0$. To find these values, we either solve this quadratic equation or estimate the values from the graph of the function $g(x) = x^2 + 2x - 5$. Figure 1.44 shows the graph of $g$, where it appears that there is a zero of $g$ between $-4$ and $-3$ and another zero between 1 and 2.

Repeatedly zooming in gives zeros for $g$ at $x_1 \approx -3.449$ and $x_2 \approx 1.449$. By Theorem (1.23)(iv) on the continuity of the quotient of continuous functions, the function $f$ will be continuous except possibly at the zeros of $g$. We can examine the behavior of $f$ near $x_1$ by viewing the graph of $f$ in a window with $-3.48 \le x \le -3.42$, as shown in Figure 1.45.

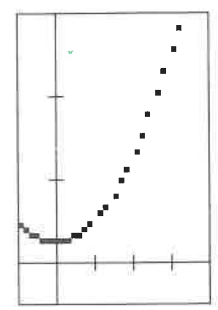
From Figure 1.45, we can easily infer what is occurring near $x_1$: As $x \to x_1^-$, we have $f(x) \to \infty$, but as $x \to x_1^+$, we have $f(x) \to -\infty$. Since neither the right-hand nor the left-hand limit of $f$ exists as $x \to x_1$, we have a discontinuity of $f$ at $x_1$. A similar analysis at $x_2$ shows that $\lim_{x \to x_2^-} f(x) = -\infty$ and $\lim_{x \to x_2^+} f(x) = \infty$, so there is a discontinuity at $x_2$ as well.

---

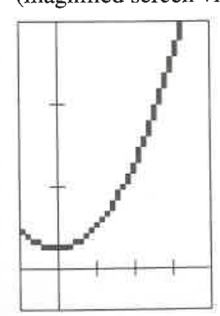
Figure 1.46
(a) Graph of $f$
(normal screen view)



(b) DOT or POINT mode
(magnified screen view)



(c) LINE or CONNECTED mode
(magnified screen view)



In Example 7, the graph of $f$ near $x_1$ has two branches, one for values less than $x_1$ and a second for values greater than $x_1$. The graphing utility, however, shows a nearly vertical line connecting these two branches. Although this line is not really part of the graph of $f$, it is displayed because the underlying software of the graphing utility assumes that functions are continuous.

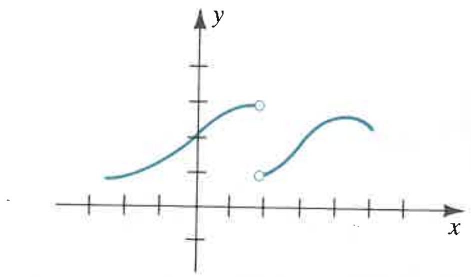
A graphing utility can evaluate a function only at a finite number of points. For each of these computed values, it darkens a picture element (called a *pixel*) on the screen. Assuming continuity, it draws a line segment between adjacent plotted pixels to represent the intermediate values that a continuous function assumes. If sufficiently small segments are used, the human eye perceives a smooth curve.

Some graphing utilities have a DOT or POINT mode which displays only the pixels for computed function values. In LINE or CONNECTED mode, the default mode for graphing utilities, the connecting line segments are also displayed. Figure 1.46 shows magnified views of the graph of a function in both DOT mode and CONNECTED mode. In generating Figure 1.46, the graphing utility calculated a large positive value to the left of $x_1$ and a large negative value to the right of $x_1$, and then darkened the corresponding pixels as well as the line segment between them, thereby producing the unusual view of the function.

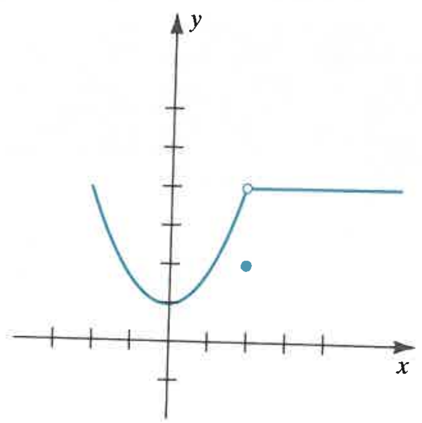
## EXERCISES 1.5

**Exer. 1–10: The graph of a function $f$ is given. Classify the discontinuities of $f$ as removable, jump, or infinite.**

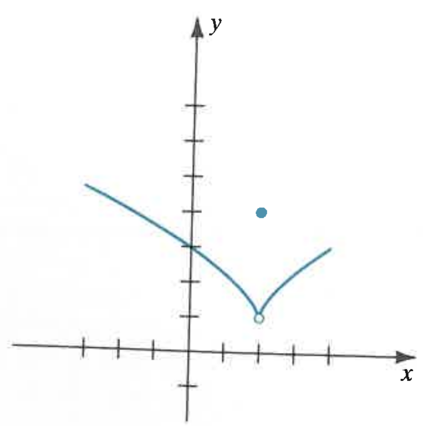
1



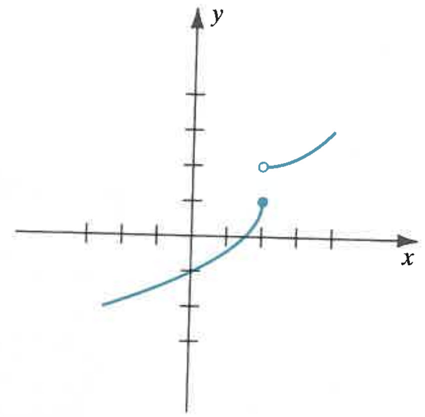

2



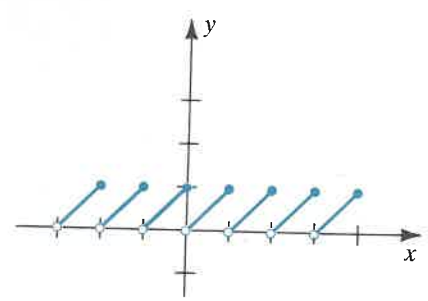

3



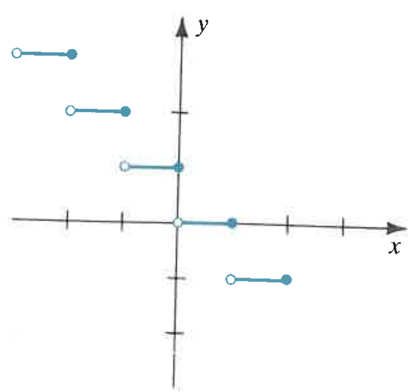

4




5

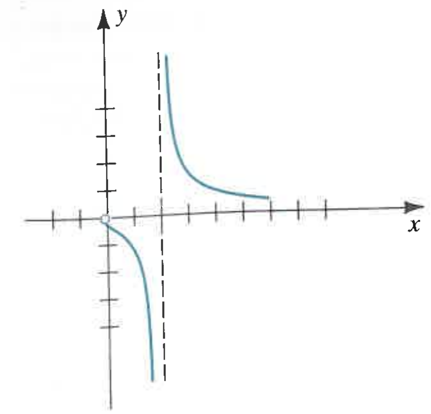



6

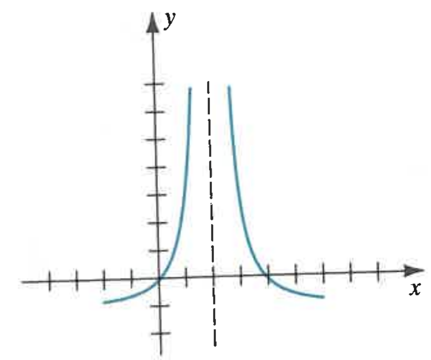




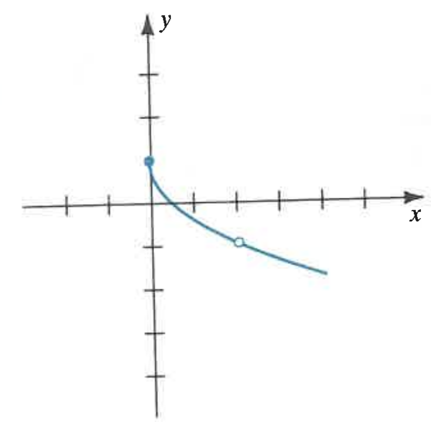

7



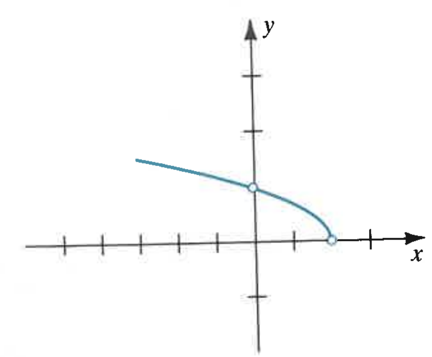

8




9




10




**Exer. 11–18: Classify the discontinuities of $f$ as removable, jump, or infinite.**

11 $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$

12 $f(x) = \begin{cases} x^3 & \text{if } x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$

13 $f(x) = \begin{cases} |x + 3| & \text{if } x \neq -2 \\ 2 & \text{if } x = -2 \end{cases}$

14 $f(x) = \begin{cases} |x - 1| & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

15 $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

16 $f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ x - 2 & \text{if } x > 1 \end{cases}$

c 17 $f(x) = x^{-1/3} \sin\left[\cos\left(\frac{\pi}{2} - x^2\right)\right]$

c 18 $f(x) = \dfrac{\sin(x^2 - 1)}{(x - 1)^2}$

**Exer. 19–22: Show that $f$ is continuous at $a$.**

19 $f(x) = \sqrt{2x - 5} + 3x;$ $\quad a = 4$

20 $f(x) = \sqrt[3]{x^2 + 2};$ $\quad a = -5$

21 $f(x) = 3x^2 + 7 - \dfrac{1}{\sqrt{-x}};$ $\quad a = -2$

22 $f(x) = \dfrac{\sqrt[3]{x}}{2x + 1};$ $\quad a = 8$

**Exer. 23–30: Explain why $f$ is not continuous at $a$.**

23 $f(x) = \dfrac{3}{x + 2};$ $\quad a = -2$

24 $f(x) = \dfrac{1}{x - 1};$ $\quad a = 1$

25 $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$ $\quad a = 3$

26 $f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3} & \text{if } x \neq -3 \\ 2 & \text{if } x = -3 \end{cases}$ $\quad a = -3$

27 $f(x) = \begin{cases} 1 & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$ $\quad a = 3$

28 $f(x) = \begin{cases} \dfrac{|x - 3|}{x - 3} & \text{if } x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$ $\quad a = 3$

c **29** $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ $\quad a = 0$

c **30** $f(x) = \begin{cases} \dfrac{1 - \cos x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ $\quad a = 0$

**Exer. 31–34: Find all numbers at which $f$ is discontinuous.**

**31** $f(x) = \dfrac{3}{x^2 + x - 6}$ **32** $f(x) = \dfrac{5}{x^2 - 4x - 12}$

**33** $f(x) = \dfrac{x - 1}{x^2 + x - 2}$ **34** $f(x) = \dfrac{x - 4}{x^2 - x - 12}$

**Exer. 35–38: Show that $f$ is continuous on the given interval.**

**35** $f(x) = \sqrt{x - 4}$; $[4, 8]$

**36** $f(x) = \sqrt{16 - x}$; $(-\infty, 16]$

**37** $f(x) = \dfrac{1}{x^2}$; $(0, \infty)$

**38** $f(x) = \dfrac{1}{x - 1}$; $(1, 3)$

**Exer. 39–54: Find all numbers at which $f$ is continuous.**

**39** $f(x) = \dfrac{3x - 5}{2x^2 - x - 3}$

**40** $f(x) = \dfrac{x^2 - 9}{x - 3}$

**41** $f(x) = \sqrt{2x - 3} + x^2$

**42** $f(x) = \dfrac{x}{\sqrt[3]{x - 4}}$

**43** $f(x) = \dfrac{x - 1}{\sqrt{x^2 - 1}}$ **44** $f(x) = \dfrac{x}{\sqrt{1 - x^2}}$

**45** $f(x) = \dfrac{|x + 9|}{x + 9}$ **46** $f(x) = \dfrac{x}{x^2 + 1}$

**47** $f(x) = \dfrac{5}{x^3 - x^2}$

**48** $f(x) = \dfrac{4x - 7}{(x + 3)(x^2 + 2x - 8)}$

**49** $f(x) = \dfrac{\sqrt{x^2 - 9}\sqrt{25 - x^2}}{x - 4}$

**50** $f(x) = \dfrac{\sqrt{9 - x}}{\sqrt{x - 6}}$

**51** $f(x) = \tan 2x$ **52** $f(x) = \cot \frac{1}{3}x$

**53** $f(x) = \csc \frac{1}{2}x$ **54** $f(x) = \sec 3x$

**55** Suppose that

$$f(x) = \begin{cases} cx^2 - 3 & \text{if } x \leq 2 \\ cx + 2 & \text{if } x > 2 \end{cases}$$

Find a value of $c$ such that $f$ is continuous on $\mathbb{R}$.

**56** Suppose that

$$f(x) = \begin{cases} c^2 x & \text{if } x < 1 \\ 3cx - 2 & \text{if } x \geq 1 \end{cases}$$

Determine all values of $c$ such that $f$ is continuous on $\mathbb{R}$.

**57** Suppose that

$$f(x) = \begin{cases} c & \text{if } x = -3 \\ \dfrac{9 - x^2}{4 - \sqrt{x^2 + 7}} & \text{if } |x| < 3 \\ d & \text{if } x = 3 \end{cases}$$

Find values of $c$ and $d$ such that $f$ is continuous on $[-3, 3]$.

**58** Suppose that

$$f(x) = \begin{cases} 4x & \text{if } x \leq -1 \\ cx + d & \text{if } -1 < x < 2 \\ -5x & \text{if } x \geq 2 \end{cases}$$

Find values of $c$ and $d$ such that $f$ is continuous on $\mathbb{R}$.

**Exer. 59–62: Verify the intermediate value theorem (1.26) for $f$ on the stated interval $[a, b]$ by showing that if $f(a) \leq w \leq f(b)$, then $f(c) = w$ for some $c$ in $[a, b]$.**

**59** $f(x) = x^3 + 1$; $[-1, 2]$

**60** $f(x) = -x^3$; $[0, 2]$

**61** $f(x) = x^2 - x$; $[1, 3]$

**62** $f(x) = 2x - x^2$; $[-2, -1]$

**63** If $f(x) = x^3 - 5x^2 + 7x - 9$, use the intermediate value theorem (1.26) to prove that there is a real number $a$ such that $f(a) = 100$.

**64** Prove that the equation $x^5 - 3x^4 - 2x^3 - x + 1 = 0$ has a solution between 0 and 1.

**65** Use the intermediate value theorem (1.26) to show that the graphs of the functions $f(x) = x^4 - 5x^2$ and $g(x) = 2x^3 - 4x + 6$ intersect between $x = 3$ and $x = 4$. (*Hint:* Consider $h = f - g$.)

**66** Show that if a function $f$ is continuous and has no zeros on an interval, then either $f(x) > 0$ or $f(x) < 0$ for every $x$ in the interval.



c **67** In models for free fall, it is generally assumed that the gravitational acceleration $g$ is the constant 9.8 m/sec$^2$ (or 32 ft/sec$^2$). Actually, $g$ varies with latitude. If $\theta$ is the latitude (in degrees), then a formula that approximates $g$ is

$$g(\theta) = 9.78049(1 + 0.005264 \sin^2 \theta + 0.000024 \sin^4 \theta).$$

Use the intermediate value theorem (1.26) to show that $g = 9.8$ somewhere between latitudes 35° and 40°.

c **68** The temperature $T$ (in °C) at which water boils may be approximated by the formula

$$T(h) = 100.862 - 0.0415\sqrt{h + 431.03},$$

where $h$ is the elevation (in meters above sea level). Use the intermediate value theorem (1.26) to show that water boils at 98 °C at an elevation somewhere between 4000 and 4500 m.

c **Exer. 69–74: Approximate the zeros of $f$ to three decimal places.**

**69** $f(x) = x^5 - x + 3$

**70** $f(x) = x^3 - \sin x + 0.5$

**71** $f(x) = 8x^4 - 14x^3 - 9x^2 + 12x + 2$

**72** $f(x) = 3x^5 - 10x^4 + 10x^3 + 3x + 7$

**73** $f(x) = \ln(1 + x^2) - 5$

**74** $f(x) = 3^x - x^3 - 3x - 3$

c **Exer. 75–77: Approximate the discontinuities of $f$ to three decimal places.**

**75** $f(x) = \dfrac{\sqrt{x + 4}}{x^2 - 14x + 47}$

**76** $f(x) = \dfrac{x + 3}{|2 \cos x - 1|}$

**77** $f(x) = \dfrac{1}{x^3 - x + 2}$

## CHAPTER 1 REVIEW EXERCISES

**Exer. 1–26: Find the limit, if it exists.**

**1** $\lim\limits_{x \to 3} \dfrac{5x + 11}{\sqrt{x + 1}}$ **2** $\lim\limits_{x \to -2} \dfrac{6 - 7x}{(3 + 2x)^4}$

**3** $\lim\limits_{x \to -2} \left(2x - \sqrt{4x^2 + x}\right)$ **4** $\lim\limits_{x \to 4^-} \left(x - \sqrt{16 - x^2}\right)$

**5** $\lim\limits_{x \to 3/2} \dfrac{2x^2 + x - 6}{4x^2 - 4x - 3}$ **6** $\lim\limits_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 - x - 2}$

**7** $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x^2 - x - 2}$ **8** $\lim\limits_{x \to 3^+} \dfrac{1}{x - 3}$

**9** $\lim\limits_{x \to 0^+} \dfrac{1}{\sqrt{x}}$ **10** $\lim\limits_{x \to 5} \dfrac{(1/x) - (1/5)}{x - 5}$

**11** $\lim\limits_{x \to 1/2} \dfrac{8x^3 - 1}{2x - 1}$ **12** $\lim\limits_{x \to 2} 5$

**13** $\lim\limits_{x \to 3^+} \dfrac{3 - x}{|3 - x|}$ **14** $\lim\limits_{x \to 2} \dfrac{\sqrt{x} - \sqrt{2}}{x - 2}$

**15** $\lim\limits_{h \to 0} \dfrac{(a + h)^4 - a^4}{h}$ **16** $\lim\limits_{h \to 0} \dfrac{(2 + h)^{-3} - 2^{-3}}{h}$

**17** $\lim\limits_{x \to -3} \sqrt[3]{\dfrac{x + 3}{x^3 + 27}}$ **18** $\lim\limits_{x \to 5/2^-} \left(\sqrt{5 - 2x} - x^2\right)$

**19** $\lim\limits_{x \to -\infty} \dfrac{(2x - 5)(3x + 1)}{(x + 7)(4x - 9)}$ **20** $\lim\limits_{x \to \infty} \dfrac{2x + 11}{\sqrt{x + 1}}$

**21** $\lim\limits_{x \to -\infty} \dfrac{6 - 7x}{(3 + 2x)^4}$ **22** $\lim\limits_{x \to \infty} \dfrac{x - 100}{\sqrt{x^2 + 100}}$

**23** $\lim\limits_{x \to 2/3^+} \dfrac{x^2}{4 - 9x^2}$ **24** $\lim\limits_{x \to 3/5^-} \dfrac{1}{5x - 3}$

**25** $\lim\limits_{x \to 0^+} \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)$ **26** $\lim\limits_{x \to 1} \dfrac{x - 1}{\sqrt{(x - 1)^2}}$

**Exer. 27–32: Sketch the graph of the piecewise-defined function $f$ and, for the indicated value of $a$, find each limit, if it exists:**

**(a)** $\lim\limits_{x \to a^-} f(x)$ **(b)** $\lim\limits_{x \to a^+} f(x)$ **(c)** $\lim\limits_{x \to a} f(x)$

**27** $f(x) = \begin{cases} 3x & \text{if } x \leq 2 \\ x^2 & \text{if } x > 2 \end{cases}$ $\quad a = 2$

**28** $f(x) = \begin{cases} x^3 & \text{if } x \leq 2 \\ 4 - 2x & \text{if } x > 2 \end{cases}$ $\quad a = 2$

**29** $f(x) = \begin{cases} \dfrac{1}{2-3x} & \text{if } x < -3 \\ \sqrt[3]{x+2} & \text{if } x \geq -3 \end{cases}$ $\quad a = -3$

**30** $f(x) = \begin{cases} \dfrac{9}{x^2} & \text{if } x \leq -3 \\ 4+x & \text{if } x > -3 \end{cases}$ $\quad a = -3$

**31** $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 4-x^2 & \text{if } x > 1 \end{cases}$ $\quad a = 1$

**32** $f(x) = \begin{cases} \dfrac{x^4+x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$ $\quad a = 0$

**33** Use Definition (1.4) to prove that $\lim_{x\to 6}(5x - 21) = 9$.

**34** Let $f$ be defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Prove that for every real number $a$, $\lim_{x\to a} f(x)$ does not exist.

**Exer. 35–38: Find all numbers at which $f$ is discontinuous.**

**35** $f(x) = \dfrac{|x^2-16|}{x^2-16}$ **36** $f(x) = \dfrac{1}{x^2-16}$

**37** $f(x) = \dfrac{x^2-x-2}{x^2-2x}$ **38** $f(x) = \dfrac{x+2}{x^3-8}$

**Exer. 39–42: Find all numbers at which $f$ is continuous.**

**39** $f(x) = 2x^4 - \sqrt[3]{x} + 1$ **40** $f(x) = \sqrt{(2+x)(3-x)}$

**41** $f(x) = \dfrac{\sqrt{9-x^2}}{x^4-16}$ **42** $f(x) = \dfrac{\sqrt{x}}{x^2-1}$

**Exer. 43–44: Show that $f$ is continuous at the number $a$.**

**43** $f(x) = \sqrt{5x+9}$; $a = 8$

**44** $f(x) = \sqrt[3]{x^2} - 4$; $a = 27$

c **Exer. 45–48: Lend numerical support for the stated result (of the form $\lim_{x\to a} f(x) = L$) by (a) creating a table of function values for $x$ close to $a$ and (b) using a graphing utility to repeatedly zoom in on the graph of $f$ near $x = a$.**

**45** $\displaystyle\lim_{x\to 3} \frac{x^3+2x^2-9x-18}{x-3} = 30$

**46** $\displaystyle\lim_{x\to 0} \frac{x}{\tan x} = 1$

**47** $\displaystyle\lim_{x\to 3/2} \frac{\cos(\pi x)}{x-(3/2)} = \pi$

**48** $\displaystyle\lim_{x\to 0^+} (\sin x)^x = 1$

c **Exer. 49–50: Approximate the zeros of $f$ to three decimal places.**

**49** $f(x) = x^4 - x^3 - 2x - 3$

**50** $f(x) = 2\sin x - x^2 + 1$

c **Exer. 51–52: Approximate the discontinuities of $f$ to three decimal places.**

**51** $f(x) = \dfrac{\sqrt{x+3}}{x^2+x-1}$ **52** $f(x) = \dfrac{x+5}{|2\sin x - x|}$

## EXTENDED PROBLEMS AND GROUP PROJECTS

**1** If $f$ is continuous on the closed interval [0, 1] with $0 \leq f(x) \leq 1$ for all $x$, then

(a) Show that the graph of $f$ lies inside the square with vertices at (0, 0), (0, 1), (1, 1), and (1, 0),

(b) Prove that $f$ has a *fixed point*; that is, there is a number $c$ in [0, 1] with $f(c) = c$. (*Hint*: Apply the intermediate value theorem (1.26) to the function $g(x) = f(x) - x$, examining the signs of $g(0)$ and $g(1)$.)

(c) Part (b) shows that the graph of $f$ must hit the diagonal that runs from (0, 0) to (1, 1). Must the graph of $f$ also hit the diagonal between (0, 1) and (1, 0)?

(d) Suppose that $f$ is a continuous function on a closed interval $I$ and that for each $x$ in $I$, $f(x)$ also belongs to $I$. Does the function $f$ necessarily have at least one fixed point?

(e) Investigate the question posed in part (d) if the interval $I$ is not closed.

**2** Show that a function $f$ has a removable discontinuity at $a$ if $\lim_{x\to a} f(x)$ exists but the limit does not equal $f(a)$. We say that $f$ has an **essential discontinuity** at $a$ if $\lim_{x\to a} f(x)$ does not exist. Investigate the distinction between these types of discontinuity. Construct an example of a function that has an essential discontinuity at every real number. Construct a function that has an infinite number of removable discontinuities on an interval $I$. Can you construct a function that has a removable discontinuity at every point of an interval $I$? Discuss your attempts to build such an example or to prove that it cannot be done.

**3** The intermediate value theorem (1.26) suggests a procedure for obtaining approximate solutions to equations of the form $f(x) = 0$, where $f$ is a given continuous function.

**Step 1** Find numbers $a$ and $b$ with $a < b$ so that $f(a)$ and $f(b)$ have opposite signs (one is positive and the other is negative). Let $I$ be the closed interval $[a, b]$.

Repeat Step 2 until the length of $I$ is less than $10^{-3}$.

**Step 2** Let $m$ be the midpoint of the interval $I$ and determine the sign of $f(m)$. If $f(a)$ and $f(m)$ have opposite signs, then let $I^*$ be the closed left-hand half of $I$; otherwise, let $I^*$ be the closed right-hand half of $I$. Let $I = I^*$.

(a) Show that at each step, there is a solution to $f(x) = 0$, which lies in $I$.

(b) Show that each new interval $I$ is half the length of the preceding interval.

(c) If the length of $[a, b]$ is less than 1, then show that the procedure will stop in 10 or fewer repetitions of step (2).

(d) Show that if we repeat step (2) 20 times, then we will have located a solution of $f(x) = 0$ inside an interval of length less than $(b-a)/10^6$.

c (e) Use this procedure with $f(x) = x^2 - 2$ to approximate $\sqrt{2}$, starting with $a = 1$ and $b = 2$.

c (f) Use this procedure to determine the positive cube root of 100 to three decimal places.

c (g) Use this procedure to determine a positive-number solution of the equation $\sin x = x$ to two decimal places.