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Michael Olinick (molinick)
Department of Mathematic s
Warner Hall
Middlebury, VT 05753

27 $\displaystyle\int \frac{2x^3 - 5x^2 + 46x + 98}{(x^2 + x - 12)^2}\,dx$

28 $\displaystyle\int \frac{-2x^4 - 3x^3 - 3x^2 + 3x + 1}{x^2(x+1)^3}\,dx$

c **Exer. 29–32: Using a computer algebra system, determine the partial fraction decomposition of the integrand and then evaluate the integral.**

29 $\displaystyle\int \frac{282x^3 + 1021x^2 - 509x - 398}{36x^4 + 96x^3 - 131x^2 - 71x + 70}\,dx$

30 $\displaystyle\int \frac{-1302x^3 + 4075x^2 + 1742x - 13}{1980x^4 - 1641x^3 - 2684x^2 - 849x - 70}\,dx$

31 $\displaystyle\int \frac{-329x^2 - 440x + 4570}{30x^3 + 187x^2 - 555x - 250}\,dx$

32 $\displaystyle\int \frac{3244x^2 + 437x - 57}{6188x^3 + 1574x^2 - 420x + 18}\,dx$

Exer. 33–36: Use partial fractions to evaluate the integral (see Formulas 19, 49, 50, and 52 of the table of integrals in Appendix II).

33 $\displaystyle\int \frac{1}{a^2 - u^2}\,du$

34 $\displaystyle\int \frac{1}{u(a + bu)}\,du$

35 $\displaystyle\int \frac{1}{u^2(a + bu)}\,du$

36 $\displaystyle\int \frac{1}{u(a + bu)^2}\,du$

37 If $f(x) = x/(x^2 - 2x - 3)$, find the area of the region under the graph of f from $x = 0$ to $x = 2$.

38 The region bounded by the graphs of $y = 0$, $x = 2$, $x = 3$, and $y = 1/(x-1)(4-x)$ is revolved about the y-axis. Find the volume of the resulting solid.

39 If the region described in Exercise 38 is revolved about the x-axis, find the volume of the resulting solid.

40 Suppose $g(x) = (x - c_1)(x - c_2)\cdots(x - c_n)$ for a positive integer n and distinct real numbers $c_1, c_2, \ldots, c_n$. If $f(x)$ is a polynomial of degree less than n, show that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - c_1} + \frac{A_2}{x - c_2} + \cdots + \frac{A_n}{x - c_n}$$

with $A_k = f(c_k)/g'(c_k)$ for $k = 1, 2, \ldots, n$. (This is a method for finding the partial fraction decomposition if the denominator can be factored into distinct linear factors.)

41 Use Exercise 40 to find the partial fraction decomposition of

$$\frac{2x^4 - x^3 - 3x^2 + 5x + 7}{x^5 - 5x^3 + 4x}.$$

7.5 QUADRATIC EXPRESSIONS AND MISCELLANEOUS SUBSTITUTIONS

In this section, we study some additional techniques for finding antiderivatives. We first examine integrands that involve quadratic expressions, and then we consider a variety of integrals that can be handled by substitutions.

INTEGRALS INVOLVING QUADRATIC EXPRESSIONS

Partial fraction decompositions may lead to integrands containing an irreducible quadratic expression $ax^2 + bx + c$. If $b \neq 0$, it is sometimes necessary to complete the square as follows:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

The substitution $u = x + b/(2a)$ may then lead to an integrable form.

EXAMPLE ▪ 1 Evaluate $\displaystyle\int \frac{2x - 1}{x^2 - 6x + 13}\,dx$.

SOLUTION Note that the quadratic expression $x^2 - 6x + 13$ is irreducible, since $b^2 - 4ac = -16 < 0$. We complete the square as follows:

$$\begin{aligned} x^2 - 6x + 13 &= (x^2 - 6x \qquad) + 13 \\ &= (x^2 - 6x + 9) + 13 - 9 = (x - 3)^2 + 4 \end{aligned}$$

Thus,

$$\int \frac{2x - 1}{x^2 - 6x + 13}\,dx = \int \frac{2x - 1}{(x - 3)^2 + 4}\,dx.$$

We now make the substitution

$$u = x - 3, \quad x = u + 3, \quad dx = du.$$

Thus,

$$\begin{aligned} \int \frac{2x - 1}{x^2 - 6x + 13}\,dx &= \int \frac{2(u + 3) - 1}{u^2 + 4}\,du \\ &= \int \frac{2u + 5}{u^2 + 4}\,du \\ &= \int \frac{2u}{u^2 + 4}\,du + 5\int \frac{1}{u^2 + 4}\,du \\ &= \ln(u^2 + 4) + \frac{5}{2}\tan^{-1}\frac{u}{2} + C \\ &= \ln(x^2 - 6x + 13) + \frac{5}{2}\tan^{-1}\frac{x - 3}{2} + C. \end{aligned}$$

We may also use the technique of completing the square if a quadratic expression appears under a radical sign. In the next example, we make a trigonometric substitution after completing the square.

EXAMPLE ▪ 2 Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx$.

SOLUTION We complete the square for the quadratic expression as follows:

$$\begin{aligned} x^2 + 8x + 25 &= (x^2 + 8x \qquad) + 25 \\ &= (x^2 + 8x + 16) + 25 - 16 \\ &= (x + 4)^2 + 9 \end{aligned}$$

Thus, $\displaystyle\int \frac{1}{\sqrt{x^2 + 8x + 25}}\,dx = \int \frac{1}{\sqrt{(x + 4)^2 + 9}}\,dx.$

If we make the trigonometric substitution

$$x + 4 = 3\tan\theta, \qquad dx = 3\sec^2\theta\,d\theta,$$

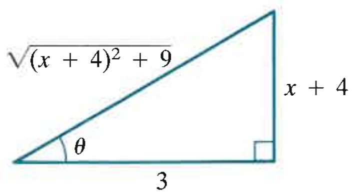
then

$$\sqrt{(x+4)^2+9} = \sqrt{9\tan^2\theta+9} = 3\sqrt{\tan^2\theta+1} = 3\sec\theta$$

and
$$\begin{aligned}\int \frac{1}{\sqrt{x^2+8x+25}}\,dx &= \int \frac{1}{3\sec\theta}3\sec^2\theta\,d\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln|\sec\theta+\tan\theta|+C.\end{aligned}$$

Figure 7.7

$\tan\theta = \dfrac{x+4}{3}$



To return to the variable x, we use the triangle in Figure 7.7, obtaining

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+8x+25}}\,dx &= \ln\left|\frac{\sqrt{x^2+8x+25}}{3}+\frac{x+4}{3}\right|+C\\ &= \ln\left|\sqrt{x^2+8x+25}+x+4\right| - \ln|3| + C\\ &= \ln\left|\sqrt{x^2+8x+25}+x+4\right| + K\end{aligned}$$

with $K = C - \ln 3$.

MISCELLANEOUS SUBSTITUTIONS

We now consider substitutions that are useful for evaluating certain types of integrals. The first example illustrates that if an integral contains an expression of the form $\sqrt[n]{f(x)}$, then one of the substitutions $u = \sqrt[n]{f(x)}$ or $u = f(x)$ may simplify the evaluation.

EXAMPLE 3 Evaluate $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx.$

SOLUTION 1 The substitution $u = \sqrt[3]{x^2+4}$ leads to the following equivalent equations:

$$u = \sqrt[3]{x^2+4},\qquad u^3 = x^2+4,\qquad x^2 = u^3-4$$

Taking the differential of each side of the last equation, we obtain

$$2x\,dx = 3u^2\,du,\quad\text{or}\quad x\,dx = \tfrac{3}{2}u^2\,du.$$

We now substitute as follows:

$$\begin{aligned}\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx &= \int \frac{x^2}{\sqrt[3]{x^2+4}}\cdot x\,dx\\ &= \int \frac{u^3-4}{u}\cdot\frac{3}{2}u^2\,du = \frac{3}{2}\int (u^4-4u)\,du\\ &= \tfrac{3}{2}\left(\tfrac{1}{5}u^5-2u^2\right)+C = \tfrac{3}{10}u^2(u^3-10)+C\\ &= \tfrac{3}{10}(x^2+4)^{2/3}(x^2-6)+C\end{aligned}$$

SOLUTION 2 If we substitute u for the expression *underneath* the radical, then

$$u = x^2+4,\quad\text{or}\quad x^2 = u-4$$

and
$$2x\,dx = du,\quad\text{or}\quad x\,dx = \tfrac{1}{2}\,du.$$

In this case, we may write

$$\begin{aligned}\int \frac{x^3}{\sqrt[3]{x^2+4}}\,dx &= \int \frac{x^2}{\sqrt[3]{x^2+4}}\cdot x\,dx\\ &= \int \frac{u-4}{u^{1/3}}\cdot\frac{1}{2}\,du = \frac{1}{2}\int \left(u^{2/3}-4u^{-1/3}\right)du\\ &= \tfrac{1}{2}\left(\tfrac{3}{5}u^{5/3}-6u^{2/3}\right)+C = \tfrac{3}{10}u^{2/3}(u-10)+C\\ &= \tfrac{3}{10}(x^2+4)^{2/3}(x^2-6)+C.\end{aligned}$$

EXAMPLE 4 Evaluate $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx.$

SOLUTION To obtain a substitution that will eliminate the two radicals $\sqrt{x} = x^{1/2}$ and $\sqrt[3]{x} = x^{1/3}$, we use $u = x^{1/n}$, where n is the least common denominator of $\frac{1}{2}$ and $\frac{1}{3}$. Thus, we let

$$u = x^{1/6},\quad\text{or, equivalently,}\quad x = u^6.$$

Hence,

$$dx = 6u^5\,du,\qquad x^{1/2} = (u^6)^{1/2} = u^3,\qquad x^{1/3} = (u^6)^{1/3} = u^2$$

and, therefore,

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int \frac{1}{u^3+u^2}6u^5\,du = 6\int \frac{u^3}{u+1}\,du.$$

By long division,

$$\frac{u^3}{u+1} = u^2-u+1-\frac{1}{u+1}.$$

Consequently,

$$\begin{aligned}\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx &= 6\int\left(u^2-u+1-\frac{1}{u+1}\right)du\\ &= 6\left(\tfrac{1}{3}u^3-\tfrac{1}{2}u^2+u-\ln|u+1|\right)+C\\ &= 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(\sqrt[6]{x}+1)+C.\end{aligned}$$

If an integrand is a rational expression in $\sin x$ and $\cos x$, then the substitution

$$u = \tan\frac{x}{2}\quad\text{for}\quad -\pi < x < \pi$$

will transform the integrand into a rational (algebraic) expression in u. To prove this, first note that

$$\cos\frac{x}{2} = \frac{1}{\sec(x/2)} = \frac{1}{\sqrt{1+\tan^2(x/2)}} = \frac{1}{\sqrt{1+u^2}}$$

$$\sin\frac{x}{2} = \tan\frac{x}{2}\cos\frac{x}{2} = u\frac{1}{\sqrt{1+u^2}}.$$

Consequently,

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2u}{1+u^2}$$

$$\cos x = 1 - 2\sin^2\frac{x}{2} = 1 - \frac{2u^2}{1+u^2} = \frac{1-u^2}{1+u^2}.$$

Moreover, since $x/2 = \tan^{-1} u$, we have $x = 2\tan^{-1} u$, and, therefore,

$$dx = \frac{2}{1+u^2}\,du.$$

The following theorem summarizes this discussion.

Theorem 7.6

If an integrand is a rational expression in $\sin x$ and $\cos x$, the following substitutions will produce a rational expression in u:

$$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2}, \qquad dx = \frac{2}{1+u^2}\,du,$$

where $u = \tan\dfrac{x}{2}$ for $-\pi < x < \pi$.

EXAMPLE ■ 5 Evaluate $\displaystyle\int \frac{1}{4\sin x - 3\cos x}\,dx$.

SOLUTION Applying Theorem (7.6) and simplifying the integrand yields

$$\int \frac{1}{4\sin x - 3\cos x}\,dx = \int \frac{1}{4\left(\dfrac{2u}{1+u^2}\right) - 3\left(\dfrac{1-u^2}{1+u^2}\right)} \cdot \frac{2}{1+u^2}\,du$$

$$= \int \frac{2}{8u - 3(1-u^2)}\,du$$

$$= 2\int \frac{1}{3u^2+8u-3}\,du.$$

Using partial fractions, we have

$$\frac{1}{3u^2+8u-3} = \frac{1}{10}\left(\frac{3}{3u-1} - \frac{1}{u+3}\right)$$

and hence

$$\int \frac{1}{4\sin x - 3\cos x}\,dx = \frac{1}{5}\int\left(\frac{3}{3u-1} - \frac{1}{u+3}\right)du$$

$$= \frac{1}{5}(\ln|3u-1| - \ln|u+3|) + C$$

$$= \frac{1}{5}\ln\left|\frac{3u-1}{u+3}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{3\tan(x/2)-1}{\tan(x/2)+3}\right| + C.$$

Theorem (7.6) may be used for any integrand that is a rational expression in $\sin x$ and $\cos x$. However, it is also important to consider simpler substitutions, as illustrated in the next example.

EXAMPLE ■ 6 Evaluate $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx$.

SOLUTION We could use the formulas in Theorem (7.6) to change the integrand into a rational expression in u. The following substitution is simpler:

$$u = \sin x, \qquad du = \cos x\,dx$$

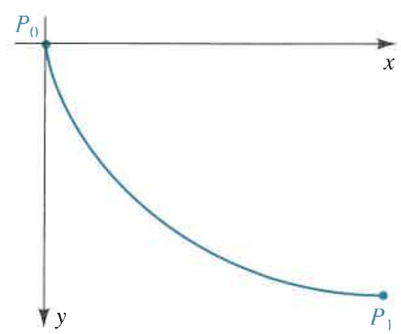
Thus,

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{1}{1+u^2}\,du$$

$$= \arctan u + C$$

$$= \arctan \sin x + C.$$

Figure 7.8



The evaluation of an integral may well involve the application of several techniques in succession. John Bernoulli's solution of the *brachistochrone problem*, for example, requires an algebraic substitution, a trigonometric substitution, and the use of several trigonometric identities. In 1696, Bernoulli published this problem as a challenge to other mathematicians: Find among all smooth curves lying in a vertical plane and connecting a given higher point P_0 to a given lower point P_1, the curve along which a particle will slide in the shortest possible time.

In Figure 7.8, we have set up a coordinate system with the higher point P_0 at the origin and with the positive direction of the y-axis downward. Figure 7.8 also shows a curve joining the points. Such a curve occurs in planning the design of a ski jump, for example. Bernoulli wished to find explicitly an expression for the function $y = f(x)$ whose graph is the curve of fastest descent from P_0 to P_1. Assuming that gravity is the only force acting on the particle, Bernoulli used ideas from optics, mechanics, and calculus to discover that the function f must satisfy the differential equation

$$y\left[1+\left(\frac{dy}{dx}\right)^2\right]=c$$

for some constant c.

We may rewrite this equation as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]=\frac{c}{y}$$

or, equivalently,

$$\left(\frac{dy}{dx}\right)^2=\frac{c}{y}-1=\frac{c-y}{y},$$

so that

$$\frac{dy}{dx}=\sqrt{\frac{c-y}{y}},$$

where we have chosen the positive square root, since in our coordinate system y increases as x increases, making $dy/dx>0$. Since the right side of this differential equation is a function only of the variable y, we separate the variables and obtain

$$\int dx=\int\sqrt{\frac{y}{c-y}}\,dy,$$

so that

$$x=\int\sqrt{\frac{y}{c-y}}\,dy.$$

To carry out the integration of the right side, we first make the algebraic substitution $u=\sqrt{y/(c-y)}$, so that $u^2=y/(c-y)$, which gives $y=cu^2/(1+u^2)$ and, with differentiation, $dy=[2cu/(1+u^2)^2]\,du$. Thus,

$$x=\int\sqrt{\frac{y}{c-y}}\,dy=\int u\cdot\frac{2cu}{(1+u^2)^2}\,du=\int\frac{2cu^2}{(1+u^2)^2}\,du.$$

To evaluate the last integral, we make the trigonometric substitution $u=\tan\phi$, $du=\sec^2\phi\,d\phi$. Thus, the integrand becomes

$$\frac{2cu^2}{(1+u^2)^2}=\frac{2c\tan^2\phi}{(1+\tan^2\phi)^2}=\frac{2c\tan^2\phi}{(\sec^2\phi)^2}$$

so that

$$x=\int\frac{2cu^2}{(1+u^2)^2}\,du=\int\frac{2c\tan^2\phi}{(\sec^2\phi)^2}\sec^2\phi\,d\phi=\int 2c\sin^2\phi\,d\phi.$$

Our final substitution is to use a variation of the double-angle formula for the cosine, $2\sin^2\phi=1-\cos 2\phi$, obtaining

$$x=c\int(1-\cos 2\phi)\,d\phi.$$

Integration yields

$$x=\frac{c}{2}(2\phi-\sin 2\phi)+D$$

for some constant D. Substituting $\phi=0$ into this equation, we see that D is equal to the value of x when $\phi=0$. At $\phi=0$, $u=\tan 0=0$ and hence $y=cu^2/(1+u^2)=0$. But when $y=0$, we also have $x=0$ because the point $P_0(0,0)$ lies on the curve. Thus, $D=0$.

Using $u=\tan\phi$ gives us a formula for y:

$$y=\frac{cu^2}{1+u^2}=\frac{c\tan^2\phi}{1+\tan^2\phi}=\frac{c\tan^2\phi}{\sec^2\phi}=c\sin^2\phi=\frac{c}{2}(1-\cos 2\phi)$$

As a final simplification, we let $a=c/2$ and $\theta=2\phi$. The curve of fastest descent is then described by the set of points $P(x,y)$, where x and y are given by a pair of equations:

$$x=a(\theta-\sin\theta),\qquad y=a(1-\cos\theta)$$

Here we have described the curve by giving separate equations for the x- and y-coordinates in terms of a third variable, θ. This description is called a *parametric representation* of a curve; we will study such representations in more detail in Chapter 9. In this case, we may regard θ as representing time. At $\theta=0$, $x=0$ and $y=0$, and the particle is at the point P_0. As θ increases, the particle moves down the curve toward the point P_1.

EXERCISES 7.5

Exer. 1–18: Evaluate the integral.

1 $\displaystyle\int\frac{1}{(x+1)^2+4}\,dx$

2 $\displaystyle\int\frac{1}{\sqrt{16-(x-3)^2}}\,dx$

3 $\displaystyle\int\frac{1}{x^2-4x+8}\,dx$

4 $\displaystyle\int\frac{1}{x^2-2x+2}\,dx$

5 $\displaystyle\int\frac{1}{\sqrt{4x-x^2}}\,dx$

6 $\displaystyle\int\frac{1}{\sqrt{7+6x-x^2}}\,dx$

7 $\displaystyle\int\frac{2x+3}{\sqrt{9-8x-x^2}}\,dx$

8 $\displaystyle\int\frac{x+5}{9x^2+6x+17}\,dx$

9 $\displaystyle\int\frac{1}{(x^2+4x+5)^2}\,dx$

10 $\displaystyle\int\frac{1}{(x^2-6x+34)^{3/2}}\,dx$

11 $\displaystyle\int\frac{1}{(x^2+6x+13)^{3/2}}\,dx$

12 $\displaystyle\int\sqrt{x(6-x)}\,dx$

13 $\displaystyle\int\frac{1}{2x^2-3x+9}\,dx$

14 $\displaystyle\int\frac{2x}{(x^2+2x+5)^2}\,dx$

15 $\displaystyle\int\frac{e^x}{e^{2x}+3e^x+2}\,dx$

16 $\displaystyle\int\sqrt{x^2+10x}\,dx$

17 $\displaystyle\int_2^3\frac{x^2-4x+6}{x^2-4x+5}\,dx$

18 $\displaystyle\int_0^1\frac{x-1}{x^2+x+1}\,dx$

19 Find the area of the region bounded by the graphs of $y=1/(x^2+4x+29)$, $y=0$, $x=-2$, and $x=3$.

20 The region bounded by the graph of

$$y=1/(x^2+2x+10),$$

the coordinate axes, and the line $x=2$ is revolved about the x-axis. Find the volume of the resulting solid.

Exer. 21–46: Evaluate the integral.

21 $\displaystyle\int x\sqrt[3]{x+9}\,dx$

22 $\displaystyle\int x^2\sqrt{2x+1}\,dx$

23 $\displaystyle\int\frac{x}{\sqrt[5]{3x+2}}\,dx$

24 $\displaystyle\int\frac{5x}{(x+3)^{2/3}}\,dx$

25 $\displaystyle\int_4^9\frac{1}{\sqrt{x}+4}\,dx$

26 $\displaystyle\int_0^{25}\frac{1}{\sqrt{4+\sqrt{x}}}\,dx$

27 $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

28 $\displaystyle\int \frac{1}{\sqrt[4]{x}+\sqrt[3]{x}}\,dx$

29 $\displaystyle\int \frac{1}{(x+1)\sqrt{x-2}}\,dx$

30 $\displaystyle\int_0^4 \frac{2x+3}{\sqrt{1+2x}}\,dx$

31 $\displaystyle\int \frac{x+1}{(x+4)^{1/3}}\,dx$

32 $\displaystyle\int \frac{x^{1/3}+1}{x^{1/3}-1}\,dx$

33 $\displaystyle\int e^{3x}\sqrt{1+e^x}\,dx$

34 $\displaystyle\int \frac{e^{2x}}{\sqrt[3]{1+e^x}}\,dx$

35 $\displaystyle\int \frac{e^{2x}}{e^x+4}\,dx$

36 $\displaystyle\int \frac{\sin 2x}{\sqrt{1+\sin x}}\,dx$

37 $\displaystyle\int \sin\sqrt{x+4}\,dx$

38 $\displaystyle\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

39 $\displaystyle\int_2^3 \frac{x}{(x-1)^6}\,dx$

40 $\displaystyle\int \frac{x^2}{(3x+4)^{10}}\,dx$

41 $\displaystyle\int \frac{\sin x}{\cos x(\cos x-1)}\,dx$

42 $\displaystyle\int \frac{\cos x}{\sin^2 x-\sin x-2}\,dx$

43 $\displaystyle\int \frac{e^x}{e^{2x}-1}\,dx$

44 $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$

45 $\displaystyle\int \frac{\sin 2x}{\sin^2 x-2\sin x-8}\,dx$

46 $\displaystyle\int \frac{\sin x}{5\cos x+\cos^2 x}\,dx$

Exer. 47–52: Use Theorem (7.6) to evaluate the integral.

47 $\displaystyle\int \frac{1}{2+\sin x}\,dx$

48 $\displaystyle\int \frac{1}{3+2\cos x}\,dx$

49 $\displaystyle\int \frac{1}{1+\sin x+\cos x}\,dx$

50 $\displaystyle\int \frac{1}{\tan x+\sin x}\,dx$

51 $\displaystyle\int \frac{\sec x}{4-3\tan x}\,dx$

52 $\displaystyle\int \frac{1}{\sin x-\sqrt{3}\cos x}\,dx$

Exer. 53–54: Use Theorem (7.6) to derive the formula.

53 $\displaystyle\int \sec x\,dx = \ln\left|\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right| + C$

54 $\displaystyle\int \csc x\,dx = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C$

7.6 TABLES OF INTEGRALS AND COMPUTER ALGEBRA SYSTEMS

Mathematicians and scientists who use integrals in their work may refer to a table of integrals or make use of a computer algebra system. We explore these approaches in this section. Many of the formulas contained in tables of integrals can be obtained by the methods that we have studied. A CAS can correctly apply the techniques that we have learned and also use more advanced methods. You should use a table of integrals or a CAS only after gaining some experience with the standard methods of integration.

A CAS may not always be able to perform an integration. However, making a substitution or implementing one of the other techniques may transform the original integral into a new integral that can be found in a table or can be successfully integrated by the CAS. To guard against errors, including data entry errors, when working with a CAS, always check the proposed answers by differentiation. Bear in mind that two answers can look quite different when found using different methods, even though they may be the same (or differ by only a constant).

TABLES OF INTEGRALS

Appendix II contains a brief table of integrals. We shall examine several examples illustrating the use of some of the formulas in this table.

EXAMPLE 1 Evaluate $\int x^3\cos x\,dx$.

SOLUTION We first use reduction Formula 85 in the table of integrals with $n=3$ and $u=x$, obtaining

$$\int x^3\cos x\,dx = x^3\sin x - 3\int x^2\sin x\,dx.$$

Next we apply Formula 84 with $n=2$, and then Formula 83, obtaining

$$\begin{aligned}\int x^2\sin x\,dx &= -x^2\cos x + 2\int x\cos x\,dx\\ &= -x^2\cos x + 2(\cos x + x\sin x) + C.\end{aligned}$$

Substitution in the first expression gives us

$$\int x^3\cos x\,dx = x^3\sin x + 3x^2\cos x - 6\cos x - 6x\sin x + C.$$

EXAMPLE 2 Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{3+5x^2}}\,dx$ for $x>0$.

SOLUTION The integrand suggests that we use that part of the table dealing with the form $\sqrt{a^2+u^2}$. Specifically, Formula 28 states that

$$\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C.$$

(In tables, the differential du is placed in the numerator instead of to the right of the integrand.) To use this formula, we must adjust the given integral so that it matches *exactly* with the formula. If we let

$$a^2 = 3 \quad\text{and}\quad u^2 = 5x^2,$$

then the expression underneath the radical is taken care of; however, we also need

(i) u^2 to the left of the radical

(ii) du in the numerator

We can obtain (i) by writing the integral as

$$5\int \frac{1}{5x^2\sqrt{3+5x^2}}\,dx.$$

For (ii), we note that

$$u = \sqrt{5}x \quad\text{and}\quad du = \sqrt{5}\,dx$$

and write the preceding integral as

$$5\cdot\frac{1}{\sqrt{5}}\int \frac{1}{5x^2\sqrt{3+5x^2}}\sqrt{5}\,dx.$$

The last integral matches exactly with that in Formula 28, and hence

$$\int \frac{1}{x^2\sqrt{3+5x^2}}\,dx = \sqrt{5}\left[-\frac{\sqrt{3+5x^2}}{3(\sqrt{5}x)}\right] + C$$
$$= -\frac{\sqrt{3+5x^2}}{3x} + C.$$

As illustrated in the next example, it may be necessary to make a substitution of some type before a table can be used to help evaluate an integral.

EXAMPLE ■ 3 Evaluate $\displaystyle\int \frac{\sin 2x}{\sqrt{3-5\cos x}}\,dx$.

SOLUTION Let us begin by rewriting the integral:

$$\int \frac{\sin 2x}{\sqrt{3-5\cos x}}\,dx = \int \frac{2\sin x\cos x}{\sqrt{3-5\cos x}}\,dx$$

Since no formulas in the table have this form, we consider making the substitution $u = \cos x$. In this case, $du = -\sin x\,dx$ and the integral may be written

$$2\int \frac{\sin x\cos x}{\sqrt{3-5\cos x}}\,dx = -2\int \frac{\cos x}{\sqrt{3-5\cos x}}(-\sin x)\,dx$$
$$= -2\int \frac{u}{\sqrt{3-5u}}\,du.$$

Referring to the table of integrals, we see that Formula 55 is

$$\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C.$$

Using this result with $a = 3$ and $b = -5$ gives us

$$-2\int \frac{u}{\sqrt{3-5u}}\,du = -2\left(\frac{2}{75}\right)(-5u-6)\sqrt{3-5u} + C.$$

Finally, since $u = \cos x$, we obtain

$$\int \frac{\sin 2x}{\sqrt{3-5\cos x}}\,dx = \frac{4}{75}(5\cos x + 6)\sqrt{3-5\cos x} + C.$$

COMPUTER ALGEBRA SYSTEMS

The next examples illustrate what can be expected from a computer algebra system. In using a CAS to find an antiderivative, the user typically specifies a command to do an indefinite integration, then enters the integrand, and finally gives the variable of integration.

EXAMPLE ■ 4 Evaluate $\displaystyle\int \frac{1}{3+\sqrt{x}}\,dx$ using a CAS.

SOLUTION We use the CAS called *Maple*, which is available for many computer systems. *Maple* displays the symbol · as a prompt to the user that it is ready for a command. The user then types

```
int(1/(3+sqrt(x)),x);
```

where int indicates a request for the indefinite integral of the function, which will appear within the parentheses. The term ,x specifies the variable, and the semicolon at the end of the line denotes the end of the request. The user then presses the ENTER key, and *Maple* responds with an antiderivative, which is displayed on the screen as follows. Note that *Maple* does not include a constant of integration in its answer.

```
• int(1/(3+sqrt(x)),x);

                                                      1/2
                        1/2          - 9 - x + 6 x
 - 3 ln(- 9 + x) + 2 x      + 3 ln(-----------------)
                                          - 9 + x
```

Note that *Maple* and some other computer algebra systems give $\ln u$ as the antiderivative of $1/u$ rather than the more precise result

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Thus, we may write the answer as

$$\int \frac{1}{3+\sqrt{x}}\,dx = -3\ln|-9+x| + 2\sqrt{x} + 3\ln\left|\frac{-9-x+6\sqrt{x}}{-9+x}\right| + C.$$

EXAMPLE ■ 5 Evaluate $\displaystyle\int xe^{-\sqrt{x}}\,dx$ using a CAS.

SOLUTION We use the CAS called *Mathematica*, which is also available for many computer systems. Note that *Mathematica* also does not include a constant of integration. The bold-faced characters on the screen display show what the user enters; *Mathematica* prints the rest:

```
In[6]:=
     Integrate[x*Exp[-1*Sqrt[x]],x]
Output[6]=
                                    3/2
     -12 - 12 Sqrt[x] - 6 x - 2 x
     -------------------------------
               Sqrt[x]
              E
```

Mathematica's prompt to the user in this case has the form *In[6]:=*. *Mathematica* uses Sqrt[x] to denote the function $\sqrt{x}$ and Exp or E for the exponential function. Thus, *Mathematica* writes the function $e^{\sqrt{x}}$ as E$^{\text{Sqrt [x]}}$. We may write the answer as

$$\int xe^{-\sqrt{x}}\,dx = \frac{-12 - 12\sqrt{x} - 6x - 2x^{3/2}}{e^{\sqrt{x}}} + C.$$

EXAMPLE ■ 6 Evaluate $\displaystyle\int \frac{1}{3 + 2\sin x + \cos x}\,dx$ using a CAS.

SOLUTION We now use a CAS for MS-DOS machines called *Derive*. *Derive* also does not include a constant of integration.

$$1:\quad \frac{1}{3+2\ \text{SIN}(x)+\text{COS}(x)}$$

$$2:\quad \int \frac{1}{3+2\ \text{SIN}(x)+\text{COS}(x)}\,dx$$

$$3:\quad \text{ATAN}\left[\frac{\text{COS}(x)+\text{SIN}(x)+1}{\text{COS}(x)+1}\right] - \text{ATAN}\left[\frac{\text{SIN}(x)}{\text{COS}(x)+1}\right] + \frac{x}{2}$$

With *Derive*, the user selects commands from a set of on-screen menus by either typing in the first letter of the command or using a "mouse" or "trackball." The screen then shows the result of executing the command. In this example, the user selects an Author command before entering the function on line 1 and then selects Calculus to request a new menu from which Integrate is chosen. There are also commands for selecting x as the variable of integration and for requesting a simplification of the resulting antiderivative.

Derive uses ATAN to represent the inverse tangent function that we have written as $\tan^{-1}$. We may write the answer as

$$\int \frac{1}{3 + 2\sin x + \cos x}\,dx$$
$$= \tan^{-1}\left(\frac{\cos x + \sin x + 1}{\cos x + 1}\right) - \tan^{-1}\left(\frac{\sin x}{\cos x + 1}\right) + \frac{x}{2} + C.$$

Each CAS uses its own set of rules for simplification. Thus, the same integrand may give different-looking antiderivatives for different computer algebra systems. It may take some effort to see that these results are either

the same or differ only by a constant. For example, the three computer algebra systems discussed so far give the following results for $\int \sec x\,dx$:

Maple: `ln(sec(x) + tan(x))`

Mathematica: $-\text{Log}[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]] + \text{Log}[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]]$

Derive: $\text{LN}\left[\frac{\text{SIN}(x)+1}{\text{COS}(x)}\right]$

Note that each of these three computer algebra systems uses a different notation for the natural logarithm: ln in *Maple*, Log in *Mathematica*, and LN in *Derive*. Although these may appear to be three different functions, they are all equivalent. For example,

$$\frac{\sin x + 1}{\cos x} = \frac{\sin x}{\cos x} + \frac{1}{\cos x} = \tan x + \sec x.$$

As an exercise in trigonometric identities and elementary properties of logarithms, you may want to show that

$$\text{LN}\left[\frac{\text{SIN}(x)+1}{\text{COS}(x)}\right]$$
$$= -\text{Log}[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]] + \text{Log}[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]].$$

As in Example 2, we observe that computer algebra systems often display solutions to integration problems in the form $\ln u$ when the more precise answer should be $\ln |u|$. The results for $\int \sec x\,dx$ should be written as

$$\ln\left|\frac{1 + \sin x}{\cos x}\right| + C, \qquad \ln|\sec x + \tan x| + C,$$

and

$$-\ln\left|\cos\frac{x}{2} - \sin\frac{x}{2}\right| + \ln\left|\cos\frac{x}{2} + \sin\frac{x}{2}\right| + C.$$

We have discussed various methods for evaluating indefinite integrals; however, the types of integrals we have considered constitute only a small percentage of those that occur in applications. The following are examples of indefinite integrals for which antiderivatives of the integrands cannot be expressed in terms of a finite number of algebraic or transcendental functions:

$$\int \sqrt[3]{x^2 + 4x - 1}\,dx, \qquad \int \sqrt{3\cos^2 x + 1}\,dx, \qquad \int e^{-x^2}\,dx$$

In Chapter 8, we shall consider methods involving *infinite* sums that are sometimes useful in evaluating such integrals.

EXERCISES 7.6

Exer. 1–30: Use the table of integrals in Appendix II to evaluate the integral.

1 $\int \frac{\sqrt{4+9x^2}}{x}\,dx$

2 $\int \frac{1}{x\sqrt{2+3x^2}}\,dx$

3 $\int (16-x^2)^{3/2}\,dx$

4 $\int x^2\sqrt{4x^2-16}\,dx$

5 $\int x\sqrt{2-3x}\,dx$

6 $\int x^2\sqrt{5+2x}\,dx$

7 $\int \sin^6 3x\,dx$

8 $\int x\cos^5(x^2)\,dx$

9 $\int \csc^4 x\,dx$

10 $\int \sin 5x\cos 3x\,dx$

11 $\int x\sin^{-1} x\,dx$

12 $\int x^2\tan^{-1} x\,dx$

13 $\int e^{-3x}\sin 2x\,dx$

14 $\int x^5\ln x\,dx$

15 $\int \frac{\sqrt{5x-9x^2}}{x}\,dx$

16 $\int \frac{1}{x\sqrt{3x-2x^2}}\,dx$

17 $\int \frac{x}{5x^4-3}\,dx$

18 $\int \cos x\sqrt{\sin^2 x-\frac{1}{4}}\,dx$

19 $\int e^{2x}\cos^{-1} e^x\,dx$

20 $\int \sin^2 x\cos^3 x\,dx$

21 $\int x^3\sqrt{2+x}\,dx$

22 $\int \frac{7x^3}{\sqrt{2-x}}\,dx$

23 $\int \frac{\sin 2x}{4+9\sin x}\,dx$

24 $\int \frac{\tan x}{\sqrt{4+3\sec x}}\,dx$

25 $\int \frac{\sqrt{9+2x}}{x}\,dx$

26 $\int \sqrt{8x^3-3x^2}\,dx$

27 $\int \frac{1}{x(4+\sqrt[3]{x})}\,dx$

28 $\int \frac{1}{2x^{3/2}+5x^2}\,dx$

29 $\int \sqrt{16-\sec^2 x}\tan x\,dx$

30 $\int \frac{\cot x}{\sqrt{4-\csc^2 x}}\,dx$

c **Exer. 31–40: Use a CAS to evaluate the integral, if possible.**

31 $\int \frac{dx}{3+2\cos x+3\sin x}$

32 $\int \frac{dx}{2+2\sin x+\cos x}$

33 $\int x^3e^{4x}\sin(2x)\,dx$

34 $\int x^2e^{-x}\sin(5x)\,dx$

35 $\int \frac{\sqrt{x}}{3+x+\sqrt{x}}\,dx$

36 $\int \frac{\sin x}{\sin x+\cos x}\,dx$

37 $\int \csc x\,dx$

38 $\int \sqrt{\tan x}\,dx$

39 $\int \frac{dx}{1+\sqrt{x}}$

40 $\int \frac{dx}{\sqrt{1+\sqrt[3]{x}}}$

7.7 IMPROPER INTEGRALS

In our work with definite integrals of the form $\int_a^b f(x)\,dx$, we have considered almost exclusively *proper integrals*—that is, situations in which the function f is continuous on a closed interval $[a, b]$ of finite length. In this section, we extend the definite integral to cases where the interval may be of infinite length or where the function f has isolated discontinuities on the interval.

INTEGERS WITH INFINITE LIMITS OF INTEGRATION

Suppose that a function f is continuous and nonnegative on an infinite interval $[a, \infty)$ and $\lim_{x\to\infty} f(x) = 0$. If $t > a$, then the area $A(t)$ under the graph of f from a to t, as illustrated in Figure 7.9, is

$$A(t) = \int_a^t f(x)\,dx.$$

If $\lim_{t\to\infty} A(t)$ exists, then the limit may be interpreted as the area of the region that lies under the graph of f, over the x-axis, and to the right of $x = a$, as illustrated in Figure 7.10. The symbol $\int_a^\infty f(x)\,dx$ is used to denote this number. If $\lim_{t\to\infty} A(t) = \infty$, we cannot assign an area to this (unbounded) region.

Figure 7.9 $\int_a^t f(x)\,dx$

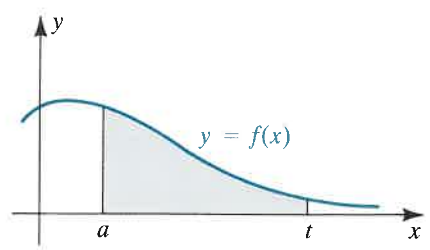
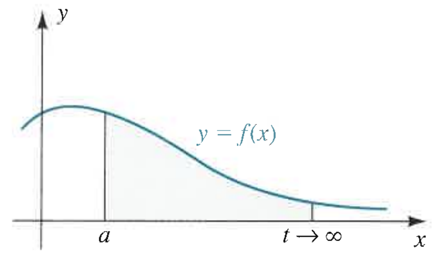


Figure 7.10 $\int_a^\infty f(x)\,dx$



Part (i) of the next definition generalizes the preceding remarks to the case where $f(x)$ may be negative for some x in $[a, \infty)$.

Definition 7.7

(i) If f is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx,$$

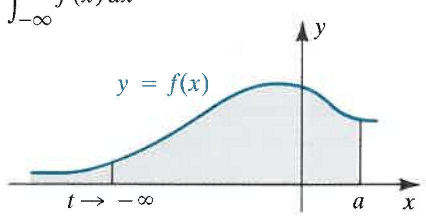
provided the limit exists.

(ii) If f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx,$$

provided the limit exists.

Figure 7.11 $\int_{-\infty}^a f(x)\,dx$



If $f(x) \ge 0$ for every x, then the limit in Definition (7.7)(ii) may be regarded as the area under the graph of f, over the x-axis, and to the *left* of $x = a$ (see Figure 7.11).

The expressions in Definition (7.7) are **improper integrals**. They differ from definite integrals in that one of the limits of integration is not a real number. An improper integral is said to **converge** if the limit exists, and the limit is the **value** of the improper integral. If the limit does not exist, the improper integral **diverges**.

Definition (7.7) is useful in many applications. In Example 4, we shall use an improper integral to calculate the work required to project an object from the surface of the earth to a point outside of the earth's gravitational field. Another important application occurs in the investigation of infinite series.

EXAMPLE ■ 1 Determine whether the integral converges or diverges, and if it converges, find its value.

(a) $\displaystyle\int_2^{\infty} \frac{1}{(x-1)^2}\,dx$ **(b)** $\displaystyle\int_2^{\infty} \frac{1}{x-1}\,dx$

SOLUTION

(a) By Definition (7.7)(i),

$$\int_2^{\infty} \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty}\int_2^{t} \frac{1}{(x-1)^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{x-1}\right]_2^t$$
$$= \lim_{t\to\infty}\left(\frac{-1}{t-1}+\frac{1}{2-1}\right) = 0+1 = 1.$$

Thus, the integral converges and has the value 1.

(b) By Definition (7.7)(i),

$$\int_2^{\infty} \frac{1}{x-1}\,dx = \lim_{t\to\infty}\int_2^{t} \frac{1}{x-1}\,dx$$
$$= \lim_{t\to\infty}\left[\ln(x-1)\right]_2^t$$
$$= \lim_{t\to\infty}\left[\ln(t-1)-\ln(2-1)\right]$$
$$= \lim_{t\to\infty}\ln(t-1) = \infty.$$

Since the limit does not exist, the improper integral diverges.

Figure 7.12

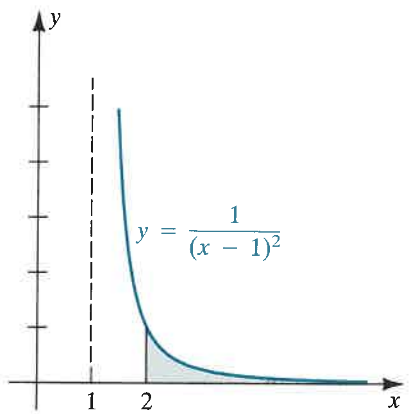
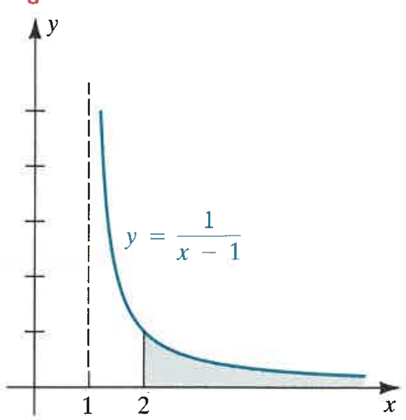


Figure 7.13



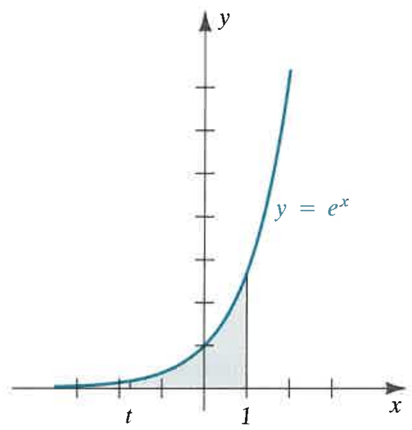
The graphs of the two functions given by the integrands in Example 1, together with the (unbounded) regions that lie under the graphs for $x \geq 2$, are sketched in Figures 7.12 and 7.13. Note that although the graphs have the same general shape for $x \geq 2$, we may assign an area to the region under the graph shown in Figure 7.12, but not to that shown in Figure 7.13.

The graph in Figure 7.13 has an interesting property. If we revolve the region under the graph of $y = 1/(x-1)$ about the x-axis, we obtain an unbounded solid of revolution. We may regard the improper integral

$$\int_2^{\infty} \pi\frac{1}{(x-1)^2}\,dx$$

as the volume of this solid. By Example 1(a), the value of this improper integral is π. This gives us the curious fact that although we cannot assign an area to the region in Figure 7.13, the volume of the solid of revolution generated by the region is finite.

Figure 7.14



EXAMPLE ■ 2 Assign an area to the region that lies under the graph of $y = e^x$, over the x-axis, and to the left of $x = 1$.

SOLUTION The region bounded by the graphs of $y = e^x$, $y = 0$, $x = 1$, and $x = t$, for $t < 1$, is sketched in Figure 7.14. The area of the *unbounded* region to the left of $x = 1$ is

$$\int_{-\infty}^{1} e^x\,dx = \lim_{t\to-\infty}\int_t^1 e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^1$$
$$= \lim_{t\to-\infty}(e-e^t) = e-0 = e.$$

An improper integral may have *two* infinite limits of integration, as in the following definition.

Definition 7.8

> Let f be continuous for every x. If a is any real number, then
> $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx,$$
> provided both of the improper integrals on the right converge.

If either of the integrals on the right in Definition (7.8) diverges, then $\int_{-\infty}^{\infty} f(x)\,dx$ is said to **diverge**. It can be shown that (7.8) does not depend on the choice of the real number a. It can also be shown that $\int_{-\infty}^{\infty} f(x)\,dx$ is not necessarily the same as $\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$ (consider $f(x) = x$).

EXAMPLE ■ 3

(a) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

(b) Sketch the graph of $f(x) = 1/(1+x^2)$ and interpret the integral in part (a) as an area.

SOLUTION

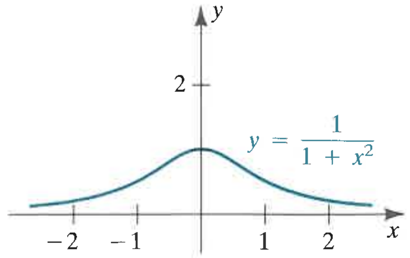
(a) Using Definition (7.8), with $a = 0$, yields

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx.$$

Next, applying Definition (7.7)(i), we have

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^{t} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\left[\arctan x\right]_0^t$$
$$= \lim_{t\to\infty}(\arctan t - \arctan 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Figure 7.15



Similarly, we may show, by using Definition (7.7)(ii), that

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Consequently, the given improper integral converges and has the value $(\pi/2) + (\pi/2) = \pi$.

(b) The graph of $y = 1/(1+x^2)$ is sketched in Figure 7.15. As in our previous discussion, the unbounded region that lies under the graph and above the x-axis may be assigned an area of π square units.

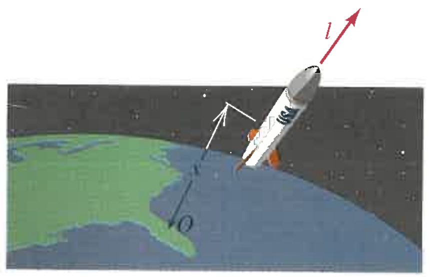
Figure 7.16

We now consider a physical application of an improper integral. If a and b are the coordinates of two points A and B on a coordinate line l (see Figure 7.16) and if $f(x)$ is the force acting at the point P with coordinate x, then, by Definition (5.21), the work done as P moves from A to B is given by

$$W = \int_a^b f(x)\,dx.$$

In similar fashion, the improper integral $\int_a^\infty f(x)\,dx$ may be used to define the work done as P moves indefinitely to the right (in applications, we use the terminology *P moves to infinity*). For example, if $f(x)$ is the force of attraction between a particle fixed at point A and a (movable) particle at P and if $c > a$, then $\int_c^\infty f(x)\,dx$ represents the work required to move P from the point with coordinate c to infinity.

Figure 7.17



EXAMPLE 4 Let l be a coordinate line with origin O at the center of the earth, as shown in Figure 7.17. The gravitational force exerted at a point on l that is a distance x from O is given by $f(x) = k/x^2$, for some constant k. Using 4000 mi as the radius of the earth, find the work required to project an object weighing 100 lb along l, from the surface to a point outside of the earth's gravitational field.

SOLUTION Theoretically, there is *always* a gravitational force $f(x)$ acting on the object; however, we may think of projecting the object from the surface to infinity. From the preceding discussion, we wish to find

$$W = \int_{4000}^{\infty} f(x)\,dx.$$

By definition, $f(x) = k/x^2$ is the weight of an object that is a distance x from O, and hence

$$100 = f(4000) = \frac{k}{(4000)^2},$$

or, equivalently,

$$k = 100(4000)^2 = 10^2 \cdot 16 \cdot 10^6 = 16 \cdot 10^8.$$

Thus,

$$f(x) = (16 \cdot 10^8)\frac{1}{x^2}$$

and the required work is

$$W = \int_{4000}^{\infty} (16 \cdot 10^8)\frac{1}{x^2}\,dx = 16 \cdot 10^8 \lim_{t\to\infty} \int_{4000}^{t} \frac{1}{x^2}\,dx$$

$$= 16 \cdot 10^8 \lim_{t\to\infty} \left[-\frac{1}{x}\right]_{4000}^{t} = 16 \cdot 10^8 \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{4000}\right)$$

$$= \frac{16 \cdot 10^8}{4000} = 4 \cdot 10^5 \text{ mi-lb.}$$

In terms of foot-pounds,

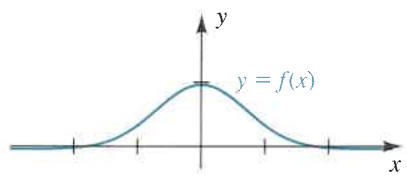
$$W = 5280 \cdot 4 \cdot 10^5 \approx (2.1)10^9 \text{ ft-lb},$$

or approximately 2 billion ft-lb.

In applications, we frequently encounter integrals for which there exist no antiderivatives that can be expressed in simple terms involving standard functions we have studied. The indefinite integral $\int e^{-x^2}\,dx$ is an example. If one of these integrals occurs as a definite integral, then we must use numerical integration techniques for evaluation. The next example illustrates how we may use such techniques for an improper integral.

EXAMPLE 5 The improper integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ occurs frequently in the study of probability and statistics. Estimate the value of this integral using Simpson's rule.

Figure 7.18 $f(x) = e^{-x^2}$



SOLUTION The function $f(x) = e^{-x^2}$ has the property that $f(-x) = f(x)$. Hence, the graph of $y = f(x)$ is symmetric about the y-axis (see Figure 7.18). Thus,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx.$$

Since

$$\int_0^\infty e^{-x^2}\,dx = \lim_{t\to\infty} \int_0^t e^{-x^2}\,dx,$$

we estimate the improper integral by numerical integration of $\int_0^t e^{-x^2}\,dx$ for successively larger values of t. Using Simpson's rule with $\Delta x = 0.01$, we obtain the values listed in the following table:

t	$\int_0^t e^{-x^2}\,dx$
1	0.746824132818
2	0.882081390760
3	0.886207348259
4	0.886226911790
5	0.886226925451
6	0.886226925453
7	0.886226925453

The numerical values in the table appear to converge rather rapidly for relatively small values of t. This result is consistent with Figure 7.18, which shows the graph of $f(x) = e^{-x^2}$ quickly approaching 0 as x moves away from 0.

It appears then that $\int_0^\infty e^{-x^2}\,dx \approx 0.886226925453$. Thus, our final estimate is $\int_{-\infty}^\infty e^{-x^2}\,dx \approx 2(0.886226925453) = 1.772453850906$. In Chapter 9, we shall determine that the exact value of the improper integral is $\sqrt{\pi} \approx 1.772453850906$.

In Example 5, we estimated $\int_0^\infty e^{-x^2}\,dx$ by a numerical approximation of $\int_0^7 e^{-x^2}\,dx$. With this approach, we ignored the contribution to the improper integral due to $\int_7^\infty e^{-x^2}\,dx$. By comparing our integral to one whose antiderivative we *can* find, we can estimate the error made by using this approach.

EXAMPLE ▪ 6 Obtain an upper bound for $\int_7^\infty e^{-x^2}\,dx$ by comparing this improper integral to $\int_7^\infty xe^{-x^2}\,dx$.

SOLUTION For $x > 1$, we have $0 < e^{-x^2} < xe^{-x^2}$. Hence,

$$\int_t^\infty e^{-x^2}\,dx < \int_t^\infty xe^{-x^2}\,dx = \lim_{s\to\infty}\int_t^s xe^{-x^2}\,dx$$

$$= \lim_{s\to\infty}\left(-\tfrac{1}{2}e^{-x^2}\right)\Big|_t^s = \tfrac{1}{2}e^{-t^2}.$$

Thus, the error made by ignoring $\int_7^\infty e^{-x^2}\,dx$ is less than $\frac{1}{2}e^{-49}$, or about 2.621E−22.

In economics, improper integrals often occur when considering the entire *future* amount of a quantity whose rate of flow is known as a function of time. For example, if the revenue flow from sales of a particular item is estimated to be $R(t)$ dollars per time unit at time t, with $t = 0$ corresponding to the present, then the entire future revenue from sales is given by $\int_0^\infty R(t)\,dt$. Since t is the variable of integration, we can modify Definition (7.7) as follows: $\int_0^\infty R(t)\,dt = \lim_{N\to\infty}\int_0^N R(t)\,dt$. In the next example, we consider another application from economics.

EXAMPLE ▪ 7 In assessing the potential revenue or profit from a mineral or energy source, economists must estimate the total amount of the resource that can be recovered from the site. Mining engineers determine that t years from now, a newly opened natural gas well will produce gas at a rate of

$$W(t) = 750e^{-0.1t} - 450e^{-0.3t}$$

thousand cubic feet per year. Estimate the total amount of gas that this well could produce.

SOLUTION We wish to estimate the entire future production of the well if it continues to pump indefinitely. This amount is given by

$$\int_0^\infty W(t)\,dt$$

$$= \int_0^\infty (750e^{-0.1t} - 450e^{-0.3t})\,dt$$

$$= \lim_{N\to\infty}\int_0^N (750e^{-0.1t} - 450e^{-0.3t})\,dt$$

$$= \lim_{N\to\infty}\left[\frac{750}{-0.1}e^{-0.1t} - \frac{450}{-0.3}e^{-0.3t}\right]_{t=0}^{t=N}$$

$$= \lim_{N\to\infty}\left[\frac{750}{-0.1}e^{-0.1N} - \frac{450}{-0.3}e^{-0.3N} - \left(\frac{750}{-0.1}e^0 - \frac{450}{-0.3}e^0\right)\right]$$

$$= \lim_{N\to\infty}\left[\frac{750}{(-0.1)e^{0.1N}} - \frac{450}{(-0.3)e^{0.3N}} - (-7500 + 1500)\right]$$

$$= 0 - 0 - (-6000) = 6000.$$

Thus, we estimate that this well will produce $(6000)(1000) = 6$ million cubic feet of natural gas.

INTEGRALS WITH DISCONTINUOUS INTEGRANDS

If a function f is continuous on a closed interval $[a, b]$, then, by Theorem (4.20), the definite integral $\int_a^b f(x)\,dx$ exists. If f has an infinite discontinuity at some number in the interval, it may still be possible to assign

Figure 7.19 $\int_a^t f(x)\,dx$

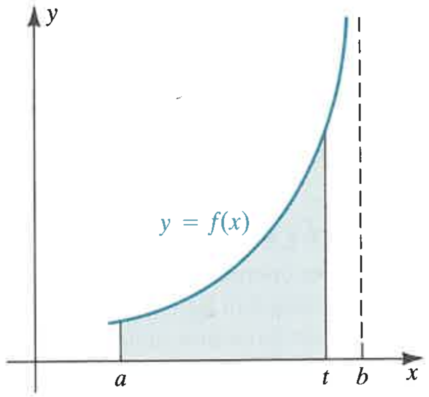
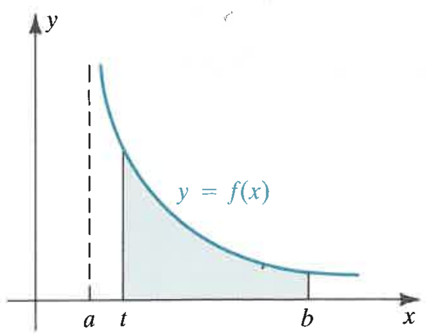


Figure 7.20 $\int_t^b f(x)\,dx$



a value to the integral. Suppose, for example, that f is continuous and nonnegative on the half-open interval $[a, b)$ and $\lim_{x\to b^-} f(x) = \infty$. If $a < t < b$, then the area $A(t)$ under the graph of f from a to t (see Figure 7.19) is

$$A(t) = \int_a^t f(x)\,dx.$$

If $\lim_{t\to b^-} A(t)$ exists, then the limit may be interpreted as the area of the unbounded region that lies under the graph of f, over the x-axis, and between $x = a$ and $x = b$. We shall denote this number by $\int_a^b f(x)\,dx$.

For the situation illustrated in Figure 7.20, $\lim_{x\to a^+} f(x) = \infty$, and we define $\int_a^b f(x)\,dx$ as the limit of $\int_t^b f(x)\,dx$ as $t \to a^+$.

These remarks are the motivation for the following definition.

Definition 7.9

(i) If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx,$$

provided the limit exists.

(ii) If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx,$$

provided the limit exists.

As in the preceding section, the integrals defined in (7.9) are referred to as *improper integrals* and they *converge* if the limits exist. The limits

are called the *values* of the improper integrals. If the limits do not exist, the improper integrals *diverge*.

Another type of improper integral is defined as follows.

Definition 7.10

If f has a discontinuity at a number c in the open interval (a, b) but is continuous elsewhere on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

provided *both* of the improper integrals on the right converge. If both converge, then the value of the improper integral $\int_a^b f(x)\,dx$ is the sum of the two values.

Figure 7.21

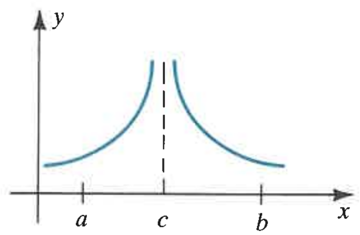
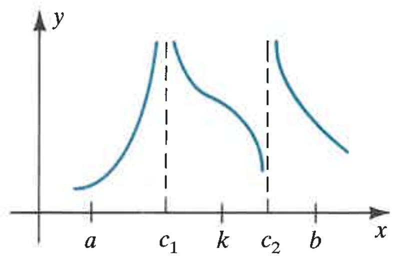


Figure 7.22



The graph of a function satisfying the conditions of Definition (7.10) is sketched in Figure 7.21.

A definition similar to (7.10) is used if f has any finite number of discontinuities in (a, b). For example, suppose f has discontinuities at c_1 and c_2, with $c_1 < c_2$, but is continuous elsewhere on $[a, b]$. One possibility is illustrated in Figure 7.22. In this case, we choose a number k between c_1 and c_2 and express $\int_a^b f(x)\,dx$ as a sum of four improper integrals over the intervals $[a, c_1]$, $[c_1, k]$, $[k, c_2]$, and $[c_2, b]$, respectively. By definition, $\int_a^b f(x)\,dx$ converges if and only if each of the four improper integrals in the sum converges. We can show that this definition is independent of the number k.

Finally, if f is continuous on (a, b) but has infinite discontinuities at a and b, then we again define $\int_a^b f(x)\,dx$ by means of (7.10).

EXAMPLE ■ 8 Evaluate $\displaystyle\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$.

SOLUTION Since the integrand has an infinite discontinuity at the number $x = 3$, we apply Definition (7.9)(i) as follows:

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}\,dx\\ &= \lim_{t\to 3^-} \Big[-2\sqrt{3-x}\Big]_0^t\\ &= \lim_{t\to 3^-} (-2\sqrt{3-t} + 2\sqrt{3})\\ &= 0 + 2\sqrt{3} = 2\sqrt{3}\end{aligned}$$

EXAMPLE 9 Determine whether the improper integral $\int_0^1 \frac{1}{x}\,dx$ converges or diverges.

SOLUTION The integrand is undefined at $x = 0$. Applying (7.9)(ii) gives us

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}[\ln x]_t^1 = \lim_{t\to 0^+}(0 - \ln t) = \infty.$$

Since the limit does not exist, the improper integral diverges.

EXAMPLE 10 Determine whether the improper integral

$$\int_0^4 \frac{1}{(x-3)^2}\,dx$$

converges or diverges.

SOLUTION The integrand is undefined at $x = 3$. Since this number is in the interval $(0, 4)$, we use Definition (7.10) with $c = 3$:

$$\int_0^4 \frac{1}{(x-3)^2}\,dx = \int_0^3 \frac{1}{(x-3)^2}\,dx + \int_3^4 \frac{1}{(x-3)^2}\,dx$$

For the integral on the left to converge, *both* integrals on the right must converge. Equivalently, the integral on the left diverges if either of the integrals on the right diverges. Applying Definition (7.9)(i) to the first integral on the right gives us

$$\int_0^3 \frac{1}{(x-3)^2}\,dx = \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}\,dx = \lim_{t\to 3^-}\left[\frac{-1}{x-3}\right]_0^t$$

$$= \lim_{t\to 3^-}\left(\frac{-1}{t-3} - \frac{1}{3}\right) = \infty.$$

Thus, the given improper integral diverges.

It is important to note that the fundamental theorem of calculus cannot be applied to the integral in Example 10, since the function given by the integrand is not continuous on $[0, 4]$. If we had (incorrectly) applied the fundamental theorem, we would have obtained

$$\left[\frac{-1}{x-3}\right]_0^4 = -1 - \frac{1}{3} = -\frac{4}{3}.$$

This result is obviously incorrect, since the integrand is never negative.

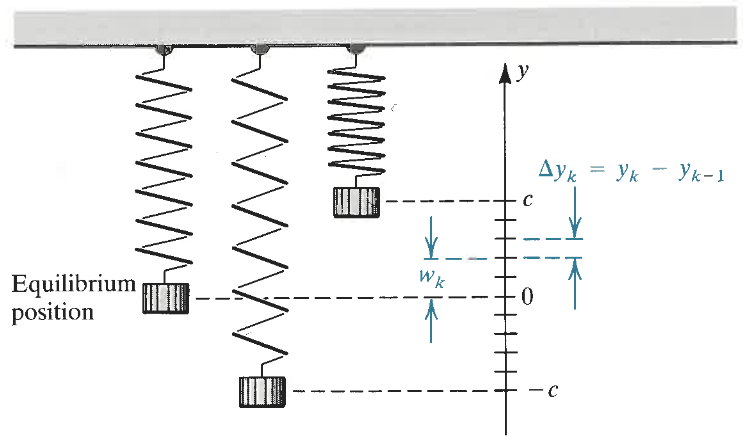
An improper integral may have both a discontinuity in the integrand and an infinite limit of integration. Integrals of this type may be investigated by expressing them as sums of improper integrals, each of which has *one* of the forms previously defined. As an illustration, since the integrand of $\int_0^\infty (1/\sqrt{x})\,dx$ is discontinuous at $x = 0$, we choose any number greater than 0—say, 1—and write

$$\int_0^\infty \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{\sqrt{x}}\,dx.$$

We can show that the first integral on the right side of the equation converges and the second diverges. Hence (by definition) the given integral diverges.

Improper integrals of the types considered in this section occur in physical applications. Figure 7.23 is a schematic drawing of a spring with an attached weight that is oscillating between points with coordinates $-c$ and c on a coordinate line y (the y-axis has been positioned at the right for clarity). The **period** T is the time required for one complete oscillation—that is, *twice* the time required for the weight to cover the interval $[-c, c]$. The next example illustrates how an improper integral results when we derive a formula for T.

Figure 7.23



EXAMPLE 11 Let $v(y)$ denote the velocity of the weight in Figure 7.23 when it is at the point with coordinate y in $[-c, c]$. Show that the period T is given by

$$T = 2\int_{-c}^{c} \frac{1}{v(y)}\,dy.$$

SOLUTION Let us partition $[-c, c]$ in the usual way, and let $\Delta y_k = y_k - y_{k-1}$ denote the distance that the weight travels during the time interval Δt_k. If w_k is any number in the subinterval $[y_{k-1}, y_k]$, then $v(w_k)$ is the velocity of the weight when it is at the point with coordinate w_k. If the norm of the partition is small and if we assume that v is a continuous function, then the distance Δy_k may be approximated by the product $v(w_k)\Delta t_k$; that is,

$$\Delta y_k \approx v(w_k)\Delta t_k.$$

Hence, the time required for the weight to cover the distance Δy_k may be approximated by

$$\Delta t_k \approx \frac{1}{v(w_k)}\Delta y_k$$

and, therefore,

$$T = 2\sum_k \Delta t_k \approx 2\sum_k \frac{1}{v(w_k)}\Delta y_k.$$

By considering the limit of the sums on the right and using the definition of definite integral, we conclude that

$$T = 2\int_{-c}^{c} \frac{1}{v(y)}\,dy.$$

Note that $v(c) = 0$ and $v(-c) = 0$, so the integral is improper.

EXERCISES 7.7

Exer. 1–20: Determine whether the integral converges or diverges, and if it converges, find its value.

1 $\displaystyle\int_1^\infty \frac{1}{x^{4/3}}\,dx$ 2 $\displaystyle\int_{-\infty}^0 \frac{1}{(x-1)^3}\,dx$

3 $\displaystyle\int_1^\infty \frac{1}{x^{3/4}}\,dx$ 4 $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,dx$

5 $\displaystyle\int_{-\infty}^2 \frac{1}{5-2x}\,dx$ 6 $\displaystyle\int_{-\infty}^\infty \frac{x}{x^4+9}\,dx$

7 $\displaystyle\int_0^\infty e^{-2x}\,dx$ 8 $\displaystyle\int_{-\infty}^0 e^x\,dx$

9 $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$ 10 $\displaystyle\int_0^\infty \frac{1}{\sqrt[3]{x+1}}\,dx$

11 $\displaystyle\int_{-\infty}^0 \frac{1}{(x-8)^{2/3}}\,dx$ 12 $\displaystyle\int_1^\infty \frac{x}{(1+x^2)^2}\,dx$

13 $\displaystyle\int_0^\infty \frac{\cos x}{1+\sin^2 x}\,dx$ 14 $\displaystyle\int_{-\infty}^2 \frac{1}{x^2+4}\,dx$

15 $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$ 16 $\displaystyle\int_{-\infty}^\infty \cos^2 x\,dx$

17 $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$ 18 $\displaystyle\int_3^\infty \frac{1}{x^2-1}\,dx$

19 $\displaystyle\int_{-\infty}^{\pi/2} \sin 2x\,dx$ 20 $\displaystyle\int_0^\infty xe^{-x}\,dx$

Exer. 21–24: If f and g are continuous functions and $0 \le f(x) \le g(x)$ for every x in $[a, \infty)$, then the following *comparison tests for improper integrals* are true:

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

(ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Determine whether the first integral converges by comparing it with the second integral.

21 $\displaystyle\int_1^\infty \frac{1}{1+x^4}\,dx;\quad \int_1^\infty \frac{1}{x^4}\,dx$

22 $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x^2-1}}\,dx;\quad \int_2^\infty \frac{1}{\sqrt[3]{x^2}}\,dx$

23 $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx;\quad \int_2^\infty \frac{1}{x}\,dx$

24 $\displaystyle\int_1^\infty e^{-x^2}\,dx;\quad \int_1^\infty e^{-x}\,dx$

Exer. 25–26: Assign, if possible, a value to (a) the area of the region R and (b) the volume of the solid obtained by revolving R about the x-axis.

25 $R = \{(x, y): x \ge 1, 0 \le y \le 1/x\}$

26 $R = \{(x, y): x \ge 1, 0 \le y \le 1/\sqrt{x}\}$

27 The unbounded region to the right of the y-axis and between the graphs of $y = e^{-x^2}$ and $y = 0$ is revolved about the y-axis. Show that a volume can be assigned to the resulting unbounded solid, and find the volume.

28 The graph of $y = e^{-x}$ for $x \ge 0$ is revolved about the x-axis. Show that an area can be assigned to the resulting unbounded surface, and find the area.

29 The solid of revolution known as *Gabriel's horn* is generated by rotating the region under the graph of $y = 1/x$ for $x \ge 1$ about the x-axis (see figure).

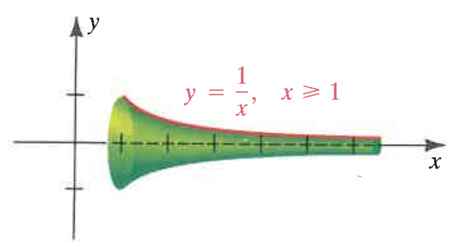
(a) Show that Gabriel's horn has a finite volume of π cubic units.

(b) Is a finite volume obtained if the graph is rotated about the y-axis?

(c) Show that the surface area of Gabriel's horn is given by $\int_1^\infty 2\pi(1/x)\sqrt{1+(1/x^4)}\,dx$. Use a comparison test (see Exercises 21–24) with $f(x) = 2\pi/x$ to establish that this integral diverges.

(d) Comment on the following: "If Gabriel's Horn has finite volume but infinite surface area, then we can fill it with a finite amount of paint but we would never be able to paint its surface. On the other hand, if we fill it with paint, then the entire inside surface area is also covered with paint. Thus, we *can* paint the inside surface area. But the outside and inside surface areas are equal, so we can paint it with only finitely much paint!"

Exercise 29



30 A spacecraft carries a fuel supply of mass m. As a conservation measure, the captain decides to burn fuel at a rate of $R(t) = mke^{-kt}$ g/sec, for some positive constant k.

(a) What does the improper integral $\int_0^\infty R(t)\,dt$ represent?

(b) When will the spacecraft run out of fuel?

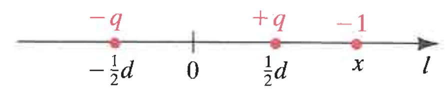
31 The force (in joules) with which two electrons repel one another is inversely proportional to the square of the distance (in meters) between them. If, in Figure 7.16, one electron is fixed at A, find the work done if another electron is repelled along l from a point B, which is 1 meter from A, to infinity.

32 An electric dipole consists of opposite charges separated by a small distance d. Suppose that charges of $+q$ and $-q$ units are located on a coordinate line l at $\frac{1}{2}d$ and $-\frac{1}{2}d$, respectively (see figure). By Coulomb's law, the net force acting on a unit charge of -1 unit at $x > \frac{1}{2}d$ is given by

$$f(x) = \frac{-kq}{(x-\frac{1}{2}d)^2} + \frac{kq}{(x+\frac{1}{2}d)^2}$$

for some positive constant k. If $a > \frac{1}{2}d$, find the work done in moving the unit charge along l from a to infinity.

Exercise 32



33 The reliability $R(t)$ of a product is the probability that it will not require repair for at least t years. To design a warranty guarantee, a manufacturer must know the average time of service before first repair of a product. This is given by the improper integral $\int_0^\infty (-t)R'(t)\,dt$.

(a) For many high-quality products, $R(t)$ has the form e^{-kt} for some positive constant k. Find an expression in terms of k for the average time of service before repair.

(b) Is it possible to manufacture a product for which $R(t) = 1/(t+1)$?

34 A sum of money is deposited into an account that pays interest at 8% per year, compounded continuously. Starting T years from now, money will be withdrawn at the *capital flow rate* of $f(t)$ dollars per year, continuing indefinitely. For future income to be generated at this rate, the minimum amount A that must be deposited, or the *present value of the capital flow*, is given by the improper integral $A = \int_T^\infty f(t)e^{-0.08t}\,dt$. Find A if the income desired 20 years from now is

(a) 12,000 dollars per year

(b) $12{,}000e^{0.04t}$ dollars per year

35 **(a)** Use integration by parts to establish the formula

$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{2a^{3/2}}\int_0^\infty e^{-u^2}\,du.$$

It can be shown that the value of this integral is $\sqrt{\pi}/2$.

(b) The relative number of gas molecules in a container that travel at a speed of v cm/sec can be found by using the *Maxwell–Boltzmann speed distribution* F:

$$F(v) = cv^2 e^{-mv^2/(2kT)},$$

where T is the temperature (in °K), m is the mass of a molecule, and c and k are positive constants. The constant c must be selected so that $\int_0^\infty F(v)\,dv = 1$. Use part (a) to express c in terms of k, T, and m.

36 The *Fourier transform* is useful for solving certain differential equations. The *Fourier cosine transform* of a function f is defined by

$$F_c[f(x)] = \int_0^\infty f(x)\cos sx\,dx$$

for every real number s for which the improper integral converges. Find $F_c[e^{-ax}]$ for $a > 0$.

Exer. 37–42: In the theory of differential equations, if f is a function, then the *Laplace transform* L of $f(x)$ is defined by

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)\,dx$$

for every real number s for which the improper integral converges. Find $L[f(x)]$ if $f(x)$ is the given expression.

37 1 **38** x **39** $\cos x$

40 $\sin x$ **41** e^{ax} **42** $\sin ax$

43 The *gamma function* Γ is defined by $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$ for every positive real number n.

(a) Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.

(b) Prove that $\Gamma(n+1) = n\Gamma(n)$.

(c) Use mathematical induction to prove that if n is any positive integer, then $\Gamma(n+1) = n!$. (This shows that factorials are special values of the gamma function.)

44 Refer to Exercise 43. Functions given by $f(x) = cx^k e^{-ax}$ with $x > 0$ are called *gamma distributions* and play an important role in probability theory. The constant c must be selected so that $\int_0^\infty f(x)\,dx = 1$. Express c in terms of the positive constants k and a and the gamma function Γ.

c **Exer. 45–46: Approximate the improper integral by making the substitution $u = 1/x$ and then using Simpson's rule with $n = 2$.**

45 $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^4+x}}\,dx$ **46** $\displaystyle\int_{-\infty}^{-10} \frac{\sqrt{|x|}}{x^3+1}\,dx$

Exer. 47–70: Determine whether the integral converges or diverges, and if it converges, find its value.

47 $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$ **48** $\displaystyle\int_0^9 \frac{1}{\sqrt{x}}\,dx$

49 $\displaystyle\int_{-3}^1 \frac{1}{x^2}\,dx$ **50** $\displaystyle\int_{-2}^{-1} \frac{1}{(x+2)^{5/4}}\,dx$

51 $\displaystyle\int_0^{\pi/2} \sec^2 x\,dx$ **52** $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

53 $\displaystyle\int_0^4 \frac{1}{(4-x)^{3/2}}\,dx$ **54** $\displaystyle\int_0^{-1} \frac{1}{\sqrt[3]{x+1}}\,dx$

55 $\displaystyle\int_0^4 \frac{1}{(4-x)^{2/3}}\,dx$ **56** $\displaystyle\int_1^2 \frac{x}{x^2-1}\,dx$

57 $\displaystyle\int_{-2}^2 \frac{-1}{(x+1)^3}\,dx$ **58** $\displaystyle\int_{-1}^1 x^{-4/3}\,dx$

59 $\displaystyle\int_{-2}^0 \frac{1}{\sqrt{4-x^2}}\,dx$ **60** $\displaystyle\int_{-2}^0 \frac{x}{\sqrt{4-x^2}}\,dx$

61 $\displaystyle\int_{-1}^2 \frac{1}{x}\,dx$ **62** $\displaystyle\int_0^4 \frac{1}{x^2-x-2}\,dx$

63 $\displaystyle\int_0^1 x\ln x\,dx$ **64** $\displaystyle\int_0^{\pi/2} \tan^2 x\,dx$

65 $\displaystyle\int_0^{\pi/2} \tan x\,dx$ **66** $\displaystyle\int_0^{\pi/2} \frac{1}{1-\cos x}\,dx$

67 $\displaystyle\int_2^4 \frac{x-2}{x^2-5x+4}\,dx$ **68** $\displaystyle\int_{1/e}^e \frac{1}{x(\ln x)^2}\,dx$

69 $\displaystyle\int_{-1}^2 \frac{1}{x^2}\cos\frac{1}{x}\,dx$ **70** $\displaystyle\int_0^\pi \sec x\,dx$

Exer. 71–74: Suppose that f and g are continuous and $0 \le f(x) \le g(x)$ for every x in $(a, b]$. If f and g are discontinuous at $x = a$, then the following *comparison tests* can be proved:

(i) If $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.

(ii) If $\int_a^b f(x)\,dx$ diverges, then $\int_a^b g(x)\,dx$ diverges.

Analogous tests may be stated for continuity on $[a, b)$ with a discontinuity at $x = b$. Determine whether the first integral converges or diverges by comparing it with the second integral.

71 $\displaystyle\int_0^\pi \frac{\sin x}{\sqrt{x}}\,dx$; $\displaystyle\int_0^\pi \frac{1}{\sqrt{x}}\,dx$

72 $\displaystyle\int_0^{\pi/4} \frac{\sec x}{x^3}\,dx$; $\displaystyle\int_0^{\pi/4} \frac{1}{x^3}\,dx$

73 $\displaystyle\int_0^2 \frac{\cosh x}{(x-2)^2}\,dx$; $\displaystyle\int_0^2 \frac{1}{(x-2)^2}\,dx$

74 $\displaystyle\int_0^1 \frac{e^{-x}}{x^{2/3}}\,dx$; $\displaystyle\int_0^1 \frac{1}{x^{2/3}}\,dx$

Exer. 75–76: Find all values of n for which the integral converges.

75 $\displaystyle\int_0^1 x^n\,dx$ **76** $\displaystyle\int_0^1 x^n \ln x\,dx$

Exer. 77–78: Assign, if possible, a value to (a) the area of the region R and (b) the volume of the solid obtained by revolving R about the x-axis.

77 $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1/\sqrt{x}\}$

78 $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1/\sqrt[3]{x}\}$

c **79** Approximate $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$ by making the substitution $u = \sqrt{x}$ and then using the trapezoidal rule with $n = 4$.

c **80** Approximate $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ by removing the discontinuity at $x = 0$ and then using Simpson's rule, with $n = 2$.

81 Refer to Example 11. If the weight in Figure 7.23 has mass m and if the spring obeys Hooke's law (with spring constant $k > 0$), then, in the absence of frictional forces, the velocity v of the weight is a solution of the differential equation

$$mv\frac{dv}{dy} + ky = 0.$$

(a) Use separation of variables (see Section 6.6) to show that $v^2 = (k/m)(c^2 - y^2)$. (*Hint:* Recall from Example 11 that $v(c) = v(-c) = 0$.)

(b) Find the period T of the oscillation.

82 A simple pendulum consists of a bob of mass m attached to a string of length L (see figure). If we assume that the string is weightless and that no other frictional forces are present, then the angular velocity $v = d\theta/dt$ is a solution of the differential equation

$$v\frac{dv}{d\theta} + \frac{g}{L}\sin\theta = 0,$$

where g is a gravitational constant.

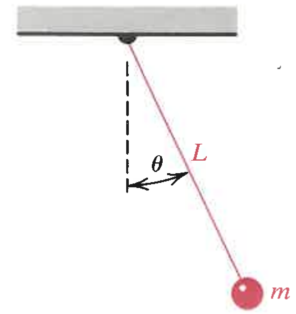
(a) If $v = 0$ at $\theta = \pm\theta_0$, use separation of variables to show that

$$v^2 = \frac{2g}{L}(\cos\theta - \cos\theta_0).$$

(b) The period T of the pendulum is twice the amount of time needed for θ to change from $-\theta_0$ to θ_0. Show that T is given by the improper integral

$$T = 2\sqrt{\frac{2L}{g}}\int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,d\theta.$$

Exercise 82



83 When a dose of y_0 milligrams of a drug is injected directly into the bloodstream, the average length of time T that a molecule remains in the bloodstream is given by the formula $T = (1/y_0)\int_0^{y_0} t\,dy$ for the time t at which exactly y milligrams is still present.

(a) If $y = y_0 e^{-kt}$ for some positive constant k, explain why the integral for T is improper.

(b) If τ is the half-life of the drug in the bloodstream, show that $T = \tau/\ln 2$.

84 In fishery science, the collection of fish that results from one annual reproduction is referred to as a *cohort*. The number N of fish still alive after t years is usually given by an exponential function. For North Sea haddock with initial size of a cohort N_0, $N = N_0 e^{-0.2t}$. The average life expectancy T (in years) of a fish in a cohort is given by $T = (1/N_0)\int_0^{N_0} t\,dN$ for the time t when precisely N fish are still alive.

(a) Find the value of T for North Sea haddock.

(b) Is it possible to have a species such that $N = N_0/(1 + kN_0 t)$ for some positive constant k? If so, compute T for such a species.

CHAPTER 7 REVIEW EXERCISES

Exer. 1–100: Evaluate the integral.

1 $\displaystyle\int x\sin^{-1} x\,dx$ **2** $\displaystyle\int \sec^3(3x)\,dx$

3 $\displaystyle\int_0^1 \ln(1+x)\,dx$ **4** $\displaystyle\int_0^1 e^{\sqrt{x}}\,dx$

5 $\displaystyle\int \cos^3 2x\sin^2 2x\,dx$ **6** $\displaystyle\int \cos^4 x\,dx$

7 $\displaystyle\int \tan x\sec^5 x\,dx$ **8** $\displaystyle\int \tan x\sec^6 x\,dx$

9 $\int \frac{1}{(x^2+25)^{3/2}}\,dx$ 10 $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$

11 $\int \frac{\sqrt{4-x^2}}{x}\,dx$ 12 $\int \frac{x}{(x^2+1)^2}\,dx$

13 $\int \frac{x^3+1}{x(x-1)^3}\,dx$ 14 $\int \frac{1}{x+x^3}\,dx$

15 $\int \frac{x^3-20x^2-63x-198}{x^4-81}\,dx$

16 $\int \frac{x-1}{(x+2)^5}\,dx$

17 $\int \frac{x}{\sqrt{4+4x-x^2}}\,dx$ 18 $\int \frac{x}{x^2+6x+13}\,dx$

19 $\int \frac{\sqrt[3]{x+8}}{x}\,dx$ 20 $\int \frac{\sin x}{2\cos x+3}\,dx$

21 $\int e^{2x}\sin 3x\,dx$ 22 $\int \cos(\ln x)\,dx$

23 $\int \sin^3 x\cos^3 x\,dx$ 24 $\int \cot^2 3x\,dx$

25 $\int \frac{x}{\sqrt{4-x^2}}\,dx$ 26 $\int \frac{1}{x\sqrt{9x^2+4}}\,dx$

27 $\int \frac{x^5-x^3+1}{x^3+2x^2}\,dx$ 28 $\int \frac{x^3}{x^3-3x^2+9x-27}\,dx$

29 $\int \frac{1}{x^{3/2}+x^{1/2}}\,dx$ 30 $\int \frac{2x+1}{(x+5)^{100}}\,dx$

31 $\int e^x\sec e^x\,dx$ 32 $\int x\tan x^2\,dx$

33 $\int x^2\sin 5x\,dx$ 34 $\int \sin 2x\cos x\,dx$

35 $\int \sin^3 x\cos^{1/2} x\,dx$ 36 $\int \sin 3x\cot 3x\,dx$

37 $\int e^x\sqrt{1+e^x}\,dx$ 38 $\int x(4x^2+25)^{-1/2}\,dx$

39 $\int \frac{x^2}{\sqrt{4x^2+25}}\,dx$ 40 $\int \frac{3x+2}{x^2+8x+25}\,dx$

41 $\int \sec^2 x\tan^2 x\,dx$ 42 $\int \sin^2 x\cos^5 x\,dx$

43 $\int x\cot x\csc x\,dx$ 44 $\int (1+\csc 2x)^2\,dx$

45 $\int x^2(8-x^3)^{1/3}\,dx$ 46 $\int x(\ln x)^2\,dx$

47 $\int \sqrt{x}\sin\sqrt{x}\,dx$ 48 $\int x\sqrt{5-3x}\,dx$

49 $\int \frac{e^{3x}}{1+e^x}\,dx$ 50 $\int \frac{e^{2x}}{4+e^{4x}}\,dx$

51 $\int \frac{x^2-4x+3}{\sqrt{x}}\,dx$ 52 $\int \frac{\cos^3 x}{\sqrt{1+\sin x}}\,dx$

53 $\int \frac{x^3}{\sqrt{16-x^2}}\,dx$ 54 $\int \frac{x}{25-9x^2}\,dx$

55 $\int \frac{1-2x}{x^2+12x+35}\,dx$ 56 $\int \frac{7}{x^2-6x+18}\,dx$

57 $\int \tan^{-1} 5x\,dx$ 58 $\int \sin^4 3x\,dx$

59 $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$ 60 $\int \frac{x}{\csc 5x^2}\,dx$

61 $\int \frac{1}{\sqrt{7+5x^2}}\,dx$ 62 $\int \frac{2x+3}{x^2+4}\,dx$

63 $\int \cot^6 x\,dx$ 64 $\int \cot^5 x\csc x\,dx$

65 $\int x^3\sqrt{x^2-25}\,dx$ 66 $\int (\sin x)10^{\cos x}\,dx$

67 $\int (x^2-\operatorname{sech}^2 4x)\,dx$ 68 $\int x\cosh x\,dx$

69 $\int x^2e^{-4x}\,dx$ 70 $\int x^5\sqrt{x^3+1}\,dx$

71 $\int \frac{3}{\sqrt{11-10x-x^2}}\,dx$ 72 $\int \frac{12x^3+7x}{x^4}\,dx$

73 $\int \tan 7x\cos 7x\,dx$ 74 $\int e^{1+\ln 5x}\,dx$

75 $\int \frac{4x^2-12x^2-10}{(x-2)(x^2-4x+3)}\,dx$

76 $\int \frac{1}{x^4\sqrt{16-x^2}}\,dx$

77 $\int (x^3+1)\cos x\,dx$ 78 $\int (x-3)^2(x+1)\,dx$

79 $\int \frac{\sqrt{9-4x^2}}{x^2}\,dx$

80 $\int \frac{4x^3-15x^2-6x+81}{x^4-18x^2+81}\,dx$

81 $\int (5-\cot 3x)^2\,dx$ 82 $\int x(x^2+5)^{3/4}\,dx$

83 $\int \frac{1}{x(\sqrt{x}+\sqrt[4]{x})}\,dx$ 84 $\int \frac{x}{\cos^2 4x}\,dx$

85 $\int \frac{\sin x}{\sqrt{1+\cos x}}\,dx$ 86 $\int \frac{4x^2-6x+4}{(x^2+4)(x-2)}\,dx$

87 $\int \frac{x^2}{(25+x^2)^2}\,dx$ 88 $\int \sin^4 x\cos^3 x\,dx$

89 $\int \tan^3 x\sec x\,dx$ 90 $\int \frac{x}{\sqrt{4+9x^2}}\,dx$

91 $\int \frac{2x^3+4x^2+10x+13}{x^4+9x^2+20}\,dx$

92 $\int \frac{\sin x}{(1+\cos x)^3}\,dx$

93 $\int \frac{(x^2-2)^2}{x}\,dx$ 94 $\int \cot^2 x\csc x\,dx$

95 $\int x^{3/2}\ln x\,dx$ 96 $\int \frac{x}{\sqrt[3]{x}-1}\,dx$

97 $\int \frac{x^2}{\sqrt[3]{2x+3}}\,dx$ 98 $\int \frac{1-\sin x}{\cot x}\,dx$

99 $\int x^3e^{(x^2)}\,dx$ 100 $\int (x+2)^2(x+1)^{10}\,dx$

Exer. 101–112: Determine whether the integral converges or diverges, and if it converges, find its value.

101 $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$ 102 $\int_4^\infty \frac{1}{x\sqrt{x}}\,dx$

103 $\int_{-\infty}^0 \frac{1}{x+2}\,dx$ 104 $\int_0^\infty \sin x\,dx$

105 $\int_{-8}^1 \frac{1}{\sqrt[3]{x}}\,dx$ 106 $\int_{-4}^0 \frac{1}{x+4}\,dx$

107 $\int_0^2 \frac{x}{(x^2-1)^2}\,dx$ 108 $\int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx$

109 $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$ 110 $\int_{-\infty}^0 xe^x\,dx$

111 $\int_0^1 \frac{\ln x}{x}\,dx$ 112 $\int_0^{\pi/2} \csc x\,dx$

c **Exer. 113–114: Approximate the improper integral by making the substitution $u = 1/x$ and then using Simpson's rule, with $n = 2$.**

113 $\int_1^\infty e^{-x^2}\,dx$ 114 $\int_1^\infty e^{-x}\sin\sqrt{x}\,dx$

Exer. 115–118: Assign, if possible, a value to (a) the area of the region R and (b) the volume of the solid obtained by revolving R about the x-axis.

115 $R = \{(x, y): x \ge 4, 0 \le y \le x^{-3/2}\}$

116 $R = \{(x, y): x \ge 8, 0 \le y \le x^{-2/3}\}$

117 $R = \{(x, y): -4 \le x \le 4, 0 \le y \le 1/(x+4)\}$

118 $R = \{(x, y): 1 \le x \le 2, 0 \le y \le 1/(x-1)\}$

EXTENDED PROBLEMS AND GROUP PROJECTS

1 (a) As an alternative to partial fractions, show that an integral of the form

$$\int \frac{1}{ax^2+bx}\,dx$$

may be evaluated by writing it as

$$\int \frac{1/x^2}{a+(b/x)}\,dx$$

and using the substitution $u = a + (b/x)$.

(b) Generalize part (a) to integrals of the form

$$\int \frac{1}{ax^n+bx}\,dx.$$

2 (a) Use integration by parts on $\int f(x)\,dx$ with $u = f(x)$ and $dv = dx$ to find

(i) $\int \ln x\,dx$ (ii) $\int \tan^{-1} x\,dx$

(iii) $\int \sin^{-1} x\,dx$ (iv) $\int \cos^{-1} x\,dx$

(v) $\int \sqrt{x}\,dx$

(b) Use integration by parts on $\int f^{-1}(x)\,dx$ with $u = f^{-1}(x)$ and $dv = dx$ to show that

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F(f^{-1}(x)),$$

where F is any antiderivative of f.

(c) Verify that the formula for $\int f^{-1}(x)\,dx$ given in part (b) is valid for the functions appearing in part (a).

(d) In what sense is the statement "If we can integrate f, then we can integrate f^{-1}" true?

3 If $f(x)$ and $g(x)$ are polynomials with f having a smaller degree than g, then we claimed that the rational function $f(x)/g(x)$ can be decomposed as a finite sum, where each term has the form

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n},$$

where A and B are real numbers, n is a nonnegative integer, and (ax^2+bx+c) is an irreducible quadratic. Prove this claim. Are the terms in the partial fraction decomposition unique? (For a set of exercises outlining an approach to this problem, see Nathan Jacobson, *Basic Algebra I*, New York: Freeman, 1985, p. 150.)