Journal Title: Calculus of a single variable /

ISSN:

Volume:
Issue:
Month/Year:
Pages: 628-659

Article Author:

Article Title: Chapter 7: Techniques of Integration (Part A)

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INTRODUCTION

SKI JUMPING DEMANDS a high level of skill and intense concentration. Jumpers slide down a track from a hill 70 to 90 meters high and then leap from a platform, flying through the air for some 90 meters before landing on the ground and gliding to a stop. The competitors require a well-designed ski jump, one that maximizes performance and minimizes danger. The design of a ski jump is a complex task that must address a number of concerns. The goal may be to move the jumpers from one point to another in the fastest time or to ensure that they reach the platform with the greatest velocity. To minimize time or maximize speed may require selecting the appropriate curve for the shape of different sections of the track. The curve may be the graph of a function $y = f(x)$. From physical assumptions, we may find a differential equation whose solution is the desired function f. However a differential equation occurs, it usually involves the derivative f' or the second derivative f'' in an explicit or implicit manner. Solving a differential equation requires recovering functions from their derivatives—that is, finding indefinite integrals.

In this chapter, we consider additional ways to simplify integrals. Foremost among these is *integration by parts*, which we discuss in Section 7.1. This powerful device allows us to obtain indefinite integrals of $\ln x$, $\tan^{-1} x$, and other important transcendental functions. In Sections 7.2–7.5, we develop techniques for simplifying integrals that contain powers of trigonometric functions, radicals, and rational expressions. In Section 7.6, we examine the use of tables of integrals and computer algebra systems. Such tables and systems are always incomplete, and we must frequently use the skills introduced in the preceding sections *before* trying these approaches. Finally, we extend the definition of definite integrals $\int_a^b f(x)\,dx$ in Section 7.7 to handle certain cases where the function f has an infinite discontinuity on the interval $[a, b]$ or where the interval becomes infinitely long.

The techniques we investigate in this chapter extend the range of functions for which we can find antiderivatives explicitly. Sometimes it is impossible to obtain an antiderivative in the form of an expression involving a finite number of sums, products, quotients, or compositions of rational functions, trigonometric functions, or the exponential and logarithmic functions. In such cases, the trapezoidal rule or Simpson's rule can be used to obtain numerical approximations. Then, either a computer or a programmable calculator is invaluable, since it can usually arrive at an accurate approximation in a matter of seconds.

Designing a complicated structure such as a ski jump often involves differential equations whose solutions require evaluating indefinite integrals.

CHAPTER 7

Techniques of Integration

7.1 INTEGRATION BY PARTS

Up to this stage of our work, we have been unable to evaluate integrals such as the following:

$$\int \ln x\, dx, \qquad \int xe^x\, dx, \qquad \int x^2 \sin x\, dx, \qquad \int \tan^{-1} x\, dx$$

The next formula will enable us to evaluate not only these, but also many other types of integrals.

***Integration by Parts Formula* 7.1**

If $u = f(x)$ and $v = g(x)$ and if f' and g' are continuous, then

$$\int u\, dv = uv - \int v\, du.$$

PROOF By the product rule,

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x),$$

or, equivalently,

$$f(x)g'(x) = \frac{d}{dx}(f(x)g(x)) - g(x)f'(x).$$

Integrating both sides of the preceding equation gives us

$$\int f(x)g'(x)\, dx = \int \frac{d}{dx}(f(x)g(x))\, dx - \int g(x)f'(x)\, dx.$$

By Theorem (4.5)(i), the first integral on the right side is $f(x)g(x) + C$. Because another constant of integration is obtained from the second integral, we may omit C in the formula—that is,

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx.$$

Since $dv = g'(x)\, dx$ and $du = f'(x)\, dx$, we may write the preceding formula as in (7.1). ■

When applying Formula (7.1) to an integral, we begin by letting one part of the integrand correspond to dv. The expression we choose for dv must include the differential dx. After selecting dv, we designate the remaining part of the integrand by u and then find du. Since this process involves splitting the integrand into two parts, the use of (7.1) is referred to as **integrating by parts**. A proper choice for dv is crucial. We generally *let dv equal the most complicated part of the integrand that can be readily integrated.* The following examples illustrate this method of integration.

EXAMPLE ■ 1 Evaluate $\int xe^{2x}\, dx$.

SOLUTION The following list contains all possible choices for dv:

$$dx, \quad x\, dx, \quad e^{2x}\, dx, \quad xe^{2x}\, dx$$

The most complicated of these expressions that can be readily integrated is $e^{2x}\, dx$. Thus, we let

$$dv = e^{2x}\, dx.$$

The remaining part of the integrand is u—that is, $u = x$. To find v, we integrate dv, obtaining $v = \frac{1}{2}e^{2x}$. Note that a constant of integration is not added at this stage of the solution. (In Exercise 51, you are asked to prove that if a constant *is* added to v, the same final result is obtained.) If $u = x$, then $du = dx$. For ease of reference, let us display these expressions as follows:

$$dv = e^{2x}\, dx \qquad u = x$$
$$v = \tfrac{1}{2}e^{2x} \qquad du = dx$$

Substituting these expressions in Formula (7.1)—that is, *integrating by parts*—we obtain

$$\int xe^{2x}\, dx = x(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}\, dx.$$

We may find the integral on the right side as in Section 6.4. This gives us

$$\int xe^{2x}\, dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C.$$

It takes considerable practice to become proficient in making a suitable choice for dv. To illustrate, if we had chosen $dv = x\, dx$ in Example 1, then it would have been necessary to let $u = e^{2x}$, giving us

$$dv = x\, dx \qquad u = e^{2x}$$
$$v = \tfrac{1}{2}x^2 \qquad du = 2e^{2x}\, dx.$$

Integrating by parts, we obtain

$$\int xe^{2x}\, dx = \tfrac{1}{2}x^2e^{2x} - \int x^2e^{2x}\, dx.$$

Since the exponent associated with x has increased, the integral on the right is more complicated than the given integral. This indicates that we have made an incorrect choice for dv.

EXAMPLE ■ 2 Evaluate

(a) $\int x \sec^2 x\, dx$ **(b)** $\int_0^{\pi/3} x \sec^2 x\, dx$

SOLUTION

(a) The possible choices for dv are

$$dx, \quad x\,dx, \quad \sec x\,dx, \quad x\sec x\,dx, \quad \sec^2 x\,dx, \quad x\sec^2 x\,dx.$$

The most complicated of these expressions that can be readily integrated is $\sec^2 x\,dx$. Thus, we let

$$dv = \sec^2 x\,dx \qquad u = x$$
$$v = \tan x \qquad du = dx.$$

Integrating by parts gives us

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$$
$$= x\tan x + \ln|\cos x| + C.$$

(b) The indefinite integral obtained in part (a) is an antiderivative of $x\sec^2 x$. Using the fundamental theorem of calculus (and dropping the constant of integration C), we obtain

$$\int_0^{\pi/3} x\sec^2 x\,dx = \left[x\tan x + \ln|\cos x|\right]_0^{\pi/3}$$
$$= \left(\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\left|\cos\frac{\pi}{3}\right|\right) - (0 + \ln 1)$$
$$= \left(\frac{\pi}{3}\sqrt{3} + \ln\frac{1}{2}\right) - (0 + 0)$$
$$= \frac{\pi}{3}\sqrt{3} - \ln 2 \approx 1.12.$$

If, in Example 2, we had chosen $dv = x\,dx$ and $u = \sec^2 x$, then the integration by parts formula (7.1) would have led to a more complicated integral. (Verify this fact.)

In the next example, we use integration by parts to find an antiderivative of the natural logarithmic function.

EXAMPLE 3 Evaluate $\int \ln x\,dx$.

SOLUTION Let

$$dv = dx \qquad u = \ln x$$
$$v = x \qquad du = \frac{1}{x}\,dx$$

and integrate by parts as follows:

$$\int \ln x\,dx = (\ln x)x - \int (x)\frac{1}{x}\,dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C$$

Sometimes it is necessary to use integration by parts more than once in the same problem, as illustrated in the next example.

EXAMPLE 4 Evaluate $\int x^2 e^{2x}\,dx$.

SOLUTION Let

$$dv = e^{2x}\,dx \qquad u = x^2$$
$$v = \tfrac{1}{2}e^{2x} \qquad du = 2x\,dx$$

and integrate by parts as follows:

$$\int x^2 e^{2x}\,dx = x^2\left(\tfrac{1}{2}e^{2x}\right) - \int \left(\tfrac{1}{2}e^{2x}\right)2x\,dx$$
$$= \tfrac{1}{2}x^2 e^{2x} - \int xe^{2x}\,dx$$

To evaluate the integral on the right side of the last equation, we must again integrate by parts. Proceeding exactly as in Example 1 leads to

$$\int x^2 e^{2x}\,dx = \tfrac{1}{2}x^2 e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + C.$$

The following example illustrates another device for evaluating an integral by means of two applications of the integration by parts formula.

EXAMPLE 5 Evaluate $\int e^x \cos x\,dx$.

SOLUTION We could either let $dv = \cos x\,dx$ or let $dv = e^x\,dx$, since each of these expressions is readily integrable. Let us choose

$$dv = \cos x\,dx \qquad u = e^x$$
$$v = \sin x \qquad du = e^x\,dx$$

and integrate by parts as follows:

$$\int e^x \cos x\,dx = e^x \sin x - \int (\sin x)e^x\,dx$$

(1) $$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

We next apply integration by parts to the integral on the right side of equation (1). Since we chose a trigonometric form for dv in the first integration by parts, we shall also choose a trigonometric form for the second. Letting

$$dv = \sin x\,dx \qquad u = e^x$$
$$v = -\cos x \qquad du = e^x\,dx$$

and integrating by parts, we have

$$\int e^x \sin x\,dx = e^x(-\cos x) - \int (-\cos x)e^x\,dx$$

(2) $$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx.$$

If we now use equation (2) to substitute on the right side of equation (1), we obtain

$$\int e^x \cos x\,dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\,dx\right],$$

or

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

Adding $\int e^x \cos x\,dx$ to both sides of the last equation gives us

$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x).$$

Finally, dividing both sides by 2 and adding the constant of integration yields

$$\int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + C.$$

We could have evaluated the given integral by using $dv = e^x\,dx$ for both the first and second applications of the integration by parts formula.

We must choose substitutions carefully when evaluating an integral of the type given in Example 5. To illustrate, suppose that in the evaluation of the integral on the right in equation (1) of the solution we had used

$$dv = e^x\,dx \qquad u = \sin x$$
$$v = e^x \qquad du = \cos x\,dx.$$

Integration by parts then leads to

$$\int e^x \sin x\,dx = (\sin x)e^x - \int e^x \cos x\,dx$$
$$= e^x \sin x - \int e^x \cos x\,dx.$$

If we now substitute in equation (1), we obtain

$$\int e^x \cos x\,dx = e^x \sin x - \left[e^x \sin x - \int e^x \cos x\,dx\right],$$

which reduces to

$$\int e^x \cos x\,dx = \int e^x \cos x\,dx.$$

Although this is a true statement, it is not an evaluation of the given integral.

EXAMPLE ■ 6 Evaluate $\int \sec^3 x\,dx$.

SOLUTION The possible choices for dv are

$$dx, \quad \sec x\,dx, \quad \sec^2 x\,dx, \quad \sec^3 x\,dx.$$

The most complicated of these expressions that can be readily integrated is $\sec^2 x\,dx$. Thus, we let

$$dv = \sec^2 x\,dx \qquad u = \sec x$$
$$v = \tan x \qquad du = \sec x \tan x\,dx$$

and integrate by parts as follows:

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

Instead of applying another integration by parts, let us change the form of the integral on the right by using the identity $1 + \tan^2 x = \sec^2 x$. This gives us

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx,$$

or

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.$$

Adding $\int \sec^3 x\,dx$ to both sides of the last equation gives us

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx.$$

If we now evaluate $\int \sec x\,dx$ and divide both sides of the resulting equation by 2 (and then add the constant of integration), we obtain

$$\int \sec^3 x\,dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C.$$

Integration by parts may sometimes be employed to obtain **reduction formulas** for integrals. We can use such formulas to write an integral involving powers of an expression in terms of integrals that involve lower powers of the expression.

EXAMPLE ■ 7 Find a reduction formula for $\int \sin^n x\,dx$.

SOLUTION Let

$$dv = \sin x\,dx \qquad u = \sin^{n-1} x$$
$$v = -\cos x \qquad du = (n-1)\sin^{n-2} x \cos x\,dx$$

and integrate by parts as follows:

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we may write

$$\int \sin^n x\,dx$$
$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.$$

Consequently,

$$\int \sin^n x\,dx + (n-1)\int \sin^n x\,dx$$
$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx.$$

The left side of the last equation reduces to $n \int \sin^n x\,dx$. Dividing both sides by n, we obtain

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

EXAMPLE 8 Use the reduction formula in Example 7 to evaluate $\int \sin^4 x\,dx$.

SOLUTION Using the formula with $n = 4$ gives us

$$\int \sin^4 x\,dx = -\tfrac{1}{4}\cos x \sin^3 x + \tfrac{3}{4}\int \sin^2 x\,dx.$$

Applying the reduction formula, with $n = 2$, to the integral on the right, we have

$$\int \sin^2 x\,dx = -\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx = -\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x + C.$$

Consequently,

$$\int \sin^4 x\,dx = -\tfrac{1}{4}\cos x \sin^3 x - \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x + D$$

with $D = \tfrac{3}{4}C$.

It should be evident that by repeated applications of the formula in Example 7 we can find $\int \sin^n x\,dx$ for any positive integer n, because these reductions end with either $\int \sin x\,dx$ or $\int dx$, and each of these can be evaluated easily.

EXERCISES 7.1

Exer. 1–38: Evaluate the integral.

1 $\int xe^{-x}\,dx$

2 $\int x \sin x\,dx$

3 $\int x^2 e^{3x}\,dx$

4 $\int x^2 \sin 4x\,dx$

5 $\int x \cos 5x\,dx$

6 $\int xe^{-2x}\,dx$

7 $\int x \sec x \tan x\,dx$

8 $\int x \csc^2 3x\,dx$

9 $\int x^2 \cos x\,dx$

10 $\int x^3 e^{-x}\,dx$

11 $\int \tan^{-1} x\,dx$

12 $\int \sin^{-1} x\,dx$

13 $\int \sqrt{x}\ln x\,dx$

14 $\int x^2 \ln x\,dx$

15 $\int x \csc^2 x\,dx$

16 $\int x \tan^{-1} x\,dx$

17 $\int e^{-x}\sin x\,dx$

18 $\int e^{3x}\cos 2x\,dx$

19 $\int \sin x \ln \cos x\,dx$

20 $\int_0^1 x^3 e^{-x^2}\,dx$

21 $\int \csc^3 x\,dx$

22 $\int \sec^5 x\,dx$

23 $\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$

24 $\int \sin \ln x\,dx$

25 $\int_0^{\pi/2} x \sin 2x\,dx$

26 $\int x \sec^2 5x\,dx$

27 $\int x(2x+3)^{99}\,dx$

28 $\int \frac{x^5}{\sqrt{1-x^3}}\,dx$

29 $\int e^{4x}\sin 5x\,dx$

30 $\int x^3 \cos(x^2)\,dx$

31 $\int (\ln x)^2\,dx$

32 $\int x\,2^x\,dx$

33 $\int x^3 \sinh x\,dx$

34 $\int (x+4)\cosh 4x\,dx$

35 $\int \cos\sqrt{x}\,dx$

36 $\int \tan^{-1} 3x\,dx$

37 $\int \cos^{-1} x\,dx$

38 $\int (x+1)^{10}(x+2)\,dx$

Exer. 39–42: Use integration by parts to derive the reduction formula.

39 $\int x^m e^x\,dx = x^m e^x - m\int x^{m-1}e^x\,dx$

40 $\int x^m \sin x\,dx = -x^m \cos x + m\int x^{m-1}\cos x\,dx$

41 $\int (\ln x)^m\,dx = x(\ln x)^m - m\int (\ln x)^{m-1}\,dx$

42 $\int \sec^m x\,dx = \frac{\sec^{m-2} x \tan x}{m-1} + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx$ for $m \neq 1$.

43 Use Exercise 39 to evaluate $\int x^5 e^x\,dx$.

44 Use Exercise 41 to evaluate $\int (\ln x)^4\,dx$.

45 If $f(x) = \sin\sqrt{x}$, find the area of the region under the graph of f from $x = 0$ to $x = \pi^2$.

46 The region between the graph of $y = x\sqrt{\sin x}$ and the x-axis from $x = 0$ to $x = \pi/2$ is revolved about the x-axis. Find the volume of the resulting solid.

47 The region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$ is revolved about the y-axis. Find the volume of the resulting solid.

48 Suppose that the force $f(x)$ acting at the point with coordinate x on a coordinate line l is given by $f(x) = x^5\sqrt{x^3+1}$. Find the work done in moving an object from $x = 0$ to $x = 1$.

49 Find the centroid of the region bounded by the graphs of the equations $y = e^x$, $y = 0$, $x = 0$, and $x = \ln 3$.

50 The velocity (at time t) of a point moving along a coordinate line is t/e^{2t} ft/sec. If the point is at the origin at $t = 0$, find its position at time t.

51 When applying the integration by parts formula (7.1), show that if, after choosing dv, we use $v + C$ in place of v, the same result is obtained.

52 In Section 5.3, the discussion of finding volumes by means of cylindrical shells was incomplete because we did not show that the same result is obtained if the disk method is also applicable. Use integration by parts to prove that if f is differentiable and either $f'(x) > 0$ on $[a, b]$ or $f'(x) < 0$ on $[a, b]$, and if V is the volume of the solid obtained by revolving the region bounded by the graphs of f, $x = a$, and $x = b$ about the x-axis, then the same value of V is obtained using either the disk method or the shell method. (*Hint:* Let g be the inverse function of f, and use integration by parts on $\int_a^b \pi[f(x)]^2\,dx$.)

53 Discuss the following use of Formula (7.1): Given $\int (1/x)\,dx$, let $dv = dx$ and $u = 1/x$ so that $v = x$ and $du = (-1/x^2)\,dx$. Hence

$$\int \frac{1}{x}\,dx = \left(\frac{1}{x}\right)x - \int x\left(-\frac{1}{x^2}\right)dx,$$

or $$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx.$$

Consequently, $0 = 1$.

54 If $u = f(x)$ and $v = g(x)$, prove that the analogue of Formula (7.1) for definite integrals is

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

for values a and b of x.

Mathematicians and Their Times

JOSEPH-LOUIS LAGRANGE

HAILED BY NAPOLEON as "the lofty pyramid of the mathematical sciences," Joseph-Louis Lagrange (1736–1813) was the greatest mathematician of his time. He made fundamental contributions in mechanics, sound, astronomy, and almost every branch of pure mathematics: analysis, calculus of variations, probability, number theory, algebra, differential equations, analytical geometry, and, of course, calculus.

Lagrange was of mixed French and Italian background. His grandfather and father both served in the government of Sardinia. In an era of high infant mortality, only one of Lagrange's ten siblings survived with him beyond early childhood. As a schoolboy, Lagrange was originally attracted to the classics. An essay by Edmund Halley extolling the virtues of calculus captivated Lagrange. He thus turned his attention to mathematics, where he made rapid progress. At the age of sixteen, Lagrange became professor of geometry at the Royal Artillery School in Turin.

In 1764, Lagrange won the Grand Prize of the French Academy of Sciences for his solution to the problem of the "libration" of the moon: Why does the moon always present the same face toward the earth? Soon afterward, Lagrange accepted appointment as court mathematician to Frederick the Great and as director of the physics and mathematics division of the Berlin Academy, serving there for twenty years.

King Louis XVI of France invited Lagrange to return to Paris to continue his work in mathematics as a member of the French Academy, where he remained during the French Revolution. Although he was the beneficiary of royal support for most of his career, he was not sympathetic to the royalists. However, he did not support the revolutionists either because he was indignant at the excesses of terror in the movement, particularly the guillotining of his friend, the chemist Lavosier.

When the École Normale opened in 1795, Lagrange accepted the post of professor of mathematics and turned his considerable talents to teaching. In an effort to lead his students more clearly through calculus, Lagrange wrote two books developing the subject: *Theory of Analytic Functions* (1797) and *Lessons on the Calculus of Functions* (1801). These works had a great influence on the evolution of calculus in the first third of the nineteenth century.

In appearance, Lagrange was of medium height and slightly formed, with pale blue eyes and a colorless complexion. His character, in the words of mathematician W. W. Rouse Ball, was "nervous and timid; he detested controversy, and to avoid it willingly allowed others to take the credit for what he had himself done."

7.2 TRIGONOMETRIC INTEGRALS

In this section, we examine integrals of functions that are products of powers of the trigonometric functions. In particular, we consider integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are integers. In Example 7 of Section 7.1, we obtained a reduction formula for $\int \sin^n x \, dx$. Integrals of this type may also be found without using integration by parts. If n is an odd positive integer, we begin by writing

$$\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx.$$

Since the integer $n - 1$ is even, we may then use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ to obtain a form that is easy to integrate, as illustrated in the next example.

EXAMPLE ■ 1 Evaluate $\int \sin^5 x \, dx$.

SOLUTION As in the preceding discussion, we write

$$\begin{aligned}\int \sin^5 x \, dx &= \int \sin^4 x \sin x \, dx \\ &= \int (\sin^2 x)^2 \sin x \, dx \\ &= \int (1 - \cos^2 x)^2 \sin x \, dx \\ &= \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx.\end{aligned}$$

If we substitute

$$u = \cos x, \qquad du = -\sin x\, dx,$$

we obtain

$$\begin{aligned}\int \sin^5 x\, dx &= -\int (1 - 2\cos^2 x + \cos^4 x)(-\sin x)\, dx \\ &= -\int (1 - 2u^2 + u^4)\, du \\ &= -u + \tfrac{2}{3}u^3 - \tfrac{1}{5}u^5 + C \\ &= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.\end{aligned}$$

Similarly, for odd powers of $\cos x$, we write

$$\int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$$

and use the fact that $\cos^2 x = 1 - \sin^2 x$ to obtain an integrable form.

If the integrand is $\sin^n x$ or $\cos^n x$ and n is *even*, then the half-angle formula

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{or} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

may be used to simplify the integrand.

EXAMPLE 2 Evaluate $\int \cos^2 x\, dx$.

SOLUTION Using a half-angle formula, we have

$$\begin{aligned}\int \cos^2 x\, dx &= \tfrac{1}{2}\int (1 + \cos 2x)\, dx \\ &= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C.\end{aligned}$$

EXAMPLE 3 Evaluate $\int \sin^4 x\, dx$.

SOLUTION

$$\begin{aligned}\int \sin^4 x\, dx &= \int (\sin^2 x)^2\, dx \\ &= \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx \\ &= \tfrac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx\end{aligned}$$

We apply a half-angle formula again and write

$$\cos^2 2x = \tfrac{1}{2}(1 + \cos 4x) = \tfrac{1}{2} + \tfrac{1}{2}\cos 4x.$$

Substituting in the last integral and simplifying gives us

$$\begin{aligned}\int \sin^4 x\, dx &= \tfrac{1}{4}\int \left(\tfrac{3}{2} - 2\cos 2x + \tfrac{1}{2}\cos 4x\right) dx \\ &= \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.\end{aligned}$$

Integrals involving only products of $\sin x$ and $\cos x$ may be evaluated using the following guidelines.

Guidelines for Evaluating $\int \sin^m x \cos^n x\, dx$ **7.2**

1 **If m is an odd integer:** Write the integral as

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cos^n x \sin x\, dx$$

and express $\sin^{m-1} x$ in terms of $\cos x$ by using the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. Make the substitution

$$u = \cos x, \quad du = -\sin x\, dx$$

and evaluate the resulting integral.

2 **If n is an odd integer:** Write the integral as

$$\int \sin^m x \cos^n x\, dx = \int \sin^m x \cos^{n-1} x \cos x\, dx$$

and express $\cos^{n-1} x$ in terms of $\sin x$ by using the trigonometric identity $\cos^2 x = 1 - \sin^2 x$. Make the substitution

$$u = \sin x, \qquad du = \cos x\, dx$$

and evaluate the resulting integral.

3 **If m and n are even:** Use half-angle formulas for $\sin^2 x$ and $\cos^2 x$ to reduce the exponents by one-half.

EXAMPLE 4 Evaluate $\int \cos^3 x \sin^4 x\, dx$.

SOLUTION By guideline (2) of (7.2),

$$\begin{aligned}\int \cos^3 x \sin^4 x\, dx &= \int \cos^2 x \sin^4 x \cos x\, dx \\ &= \int (1 - \sin^2 x)\sin^4 x \cos x\, dx.\end{aligned}$$

If we let $u = \sin x$, then $du = \cos x\, dx$, and the integral may be written

$$\begin{aligned}\int \cos^3 x \sin^4 x\, dx &= \int (1 - u^2)u^4\, du = \int (u^4 - u^6)\, du \\ &= \tfrac{1}{5}u^5 - \tfrac{1}{7}u^7 + C \\ &= \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C.\end{aligned}$$

The following guidelines are analogous to those in (7.2) for integrands of the form $\tan^m x \sec^n x$.

Guidelines for Evaluating $\int \tan^m x \sec^n x\,dx$ 7.3

1 **If m is an odd integer:** Write the integral as

$$\int \tan^m x \sec^n x\,dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x\,dx$$

and express $\tan^{m-1} x$ in terms of $\sec x$ by using the trigonometric identity $\tan^2 = \sec^2 x - 1$. Make the substitution

$$u = \sec x, \qquad du = \sec x \tan x\,dx$$

and evaluate the resulting integral.

2 **If n is an even integer:** Write the integral as

$$\int \tan^m x \sec^n x\,dx = \int \tan^m x \sec^{n-2} x \sec^2 x\,dx$$

and express $\sec^{n-2} x$ in terms of $\tan x$ by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$. Make the substitution

$$u = \tan x, \qquad du = \sec^2 x\,dx$$

and evaluate the resulting integral.

3 **If m is even and n is odd:** There is no standard method of evaluation. Possibly use integration by parts.

EXAMPLE 5 Evaluate $\int \tan^3 x \sec^5 x\,dx$.

SOLUTION By guideline (1) of (7.3),

$$\int \tan^3 x \sec^5 x\,dx = \int \tan^2 x \sec^4 x(\sec x \tan x)\,dx$$
$$= \int (\sec^2 x - 1)\sec^4 x(\sec x \tan x)\,dx.$$

Substituting $u = \sec x$ and $du = \sec x \tan x\,dx$, we obtain

$$\int \tan^3 x \sec^5 x\,dx = \int (u^2 - 1)u^4\,du$$
$$= \int (u^6 - u^4)\,du$$
$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + C.$$

EXAMPLE 6 Evaluate $\int \tan^2 x \sec^4 x\,dx$.

SOLUTION By guideline (2) of (7.3),

$$\int \tan^2 x \sec^4 x\,dx = \int \tan^2 x \sec^2 x \sec^2 x\,dx$$
$$= \int \tan^2 x(\tan^2 x + 1)\sec^2 x\,dx.$$

If we let $u = \tan x$, then $du = \sec^2 x\,dx$, and

$$\int \tan^2 x \sec^4 x\,dx = \int u^2(u^2 + 1)\,du$$
$$= \int (u^4 + u^2)\,du$$
$$= \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C$$
$$= \tfrac{1}{5}\tan^2 x + \tfrac{1}{3}\tan^3 x + C.$$

Integrals of the form $\int \cot^m x \csc^n x\,dx$ may be evaluated in similar fashion.

Finally, if an integrand has one of the forms $\cos mx \cos nx$, $\sin mx \sin nx$, or $\sin mx \cos nx$, we use a product-to-sum formula to help evaluate the integral, as illustrated in the next example.

EXAMPLE 7 Evaluate $\int \cos 5x \cos 3x\,dx$.

SOLUTION Using the product-to-sum formula for $\cos u \cos v$, we obtain

$$\int \cos 5x \cos 3x\,dx = \int \tfrac{1}{2}(\cos 8x + \cos 2x)\,dx$$
$$= \tfrac{1}{16}\sin 8x + \tfrac{1}{4}\sin 2x + C.$$

EXERCISES 7.2

Exer. 1–30: Evaluate the integral.

1 $\int \cos^3 x\,dx$

2 $\int \sin^2 2x\,dx$

3 $\int \sin^2 x \cos^2 x\,dx$

4 $\int \cos^7 x\,dx$

5 $\int \sin^3 x \cos^2 x\,dx$

6 $\int \sin^5 x \cos^3 x\,dx$

7 $\int \sin^6 x\,dx$

8 $\int \sin^4 x \cos^2 x\,dx$

9 $\int \tan^3 x \sec^4 x \, dx$ **10** $\int \sec^6 x \, dx$

11 $\int \tan^3 x \sec^3 x \, dx$ **12** $\int \tan^5 x \sec x \, dx$

13 $\int \tan^6 x \, dx$ **14** $\int \cot^4 x \, dx$

15 $\int \sqrt{\sin x} \, \cos^3 x \, dx$ **16** $\int \frac{\cos^3 x}{\sqrt{\sin x}} \, dx$

17 $\int (\tan x + \cot x)^2 \, dx$ **18** $\int \cot^3 x \csc^3 x \, dx$

19 $\int_0^{\pi/4} \sin^3 x \, dx$ **20** $\int_0^1 \tan^2(\tfrac{1}{4}\pi x) \, dx$

21 $\int \sin 5x \sin 3x \, dx$ **22** $\int_0^{\pi/4} \cos x \cos 5x \, dx$

23 $\int_0^{\pi/2} \sin 3x \cos 2x \, dx$ **24** $\int \sin 4x \cos 3x \, dx$

25 $\int \csc^4 x \cot^4 x \, dx$ **26** $\int (1 + \sqrt{\cos x})^2 \sin x \, dx$

27 $\int \frac{\cos x}{2 - \sin x} \, dx$ **28** $\int \frac{\tan^2 x - 1}{\sec^2 x} \, dx$

29 $\int \frac{\sec^2 x}{(1 + \tan x)^2} \, dx$ **30** $\int \frac{\sec x}{\cot^5 x} \, dx$

31 The region bounded by the x-axis and the graph of $y = \cos^2 x$ from $x = 0$ to $x = 2\pi$ is revolved about the x-axis. Find the volume of the resulting solid.

32 The region between the graphs of $y = \tan^2 x$ and $y = 0$ from $x = 0$ to $x = \pi/4$ is revolved about the x-axis. Find the volume of the resulting solid.

33 The velocity (at time t) of a point moving on a coordinate line is $\cos^2 \pi t$ ft/sec. How far does the point travel in 5 sec?

34 The acceleration (at time t) of a point moving on a coordinate line is $\sin^2 t \cos t$ ft/sec². At $t = 0$, the point is at the origin and its velocity is 10 ft/sec. Find its position at time t.

35 **(a)** Prove that if m and n are positive integers,

$$\int \sin mx \sin nx \, dx = \begin{cases} \dfrac{\sin(m-n)x}{2(m-n)} - \dfrac{\sin(m+n)x}{2(m+n)} + C & \text{if } m \neq n \\[2ex] \dfrac{x}{2} - \dfrac{\sin 2mx}{4m} + C & \text{if } m = n \end{cases}$$

(b) Obtain formulas similar to that in part (a) for

$$\int \sin mx \cos nx \, dx$$

and $$\int \cos mx \cos nx \, dx.$$

36 **(a)** Use part (a) of Exercise 35 to prove that

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

(b) Find

(i) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$

(ii) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$

7.3 TRIGONOMETRIC SUBSTITUTIONS

In Example 4 on p. 42, we saw how to change an expression of the form $\sqrt{a^2 - x^2}$, with $a > 0$, into a trigonometric expression without radicals, by using the *trigonometric substitution* $x = a \sin\theta$. We can use a similar procedure for $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$. This technique is useful for eliminating radicals from certain types of integrands. The substitutions are listed in the following table.

***Trigonometric Substitutions* 7.4**

Expression in integrand	Trigonometric substitution
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$

When making a trigonometric substitution, we shall assume that θ is in the range of the corresponding inverse trigonometric function. Thus, for the substitution $x = a\sin\theta$, we have $-\pi/2 \le \theta \le \pi/2$. In this case, $\cos\theta \ge 0$ and

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2\theta} \\ &= \sqrt{a^2(1 - \sin^2\theta)} \\ &= \sqrt{a^2 \cos^2\theta} \\ &= a\cos\theta. \end{aligned}$$

If $\sqrt{a^2 - x^2}$ occurs in a denominator, we add the restriction $|x| \neq a$, or, equivalently, $-\pi/2 < \theta < \pi/2$.

EXAMPLE ■ 1 Evaluate $\int \frac{1}{x^2\sqrt{16 - x^2}} \, dx$.

SOLUTION The integrand contains $\sqrt{16 - x^2}$, which is of the form $\sqrt{a^2 - x^2}$ with $a = 4$. Hence, by (7.4), we let

$$x = 4\sin\theta \quad \text{for} \quad -\pi/2 < \theta < \pi/2.$$

It follows that

$$\begin{aligned} \sqrt{16 - x^2} &= \sqrt{16 - 16\sin^2\theta} \\ &= 4\sqrt{1 - \sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\cos\theta. \end{aligned}$$

Since $x = 4\sin\theta$, we have $dx = 4\cos\theta \, d\theta$. Substituting in the given integral yields

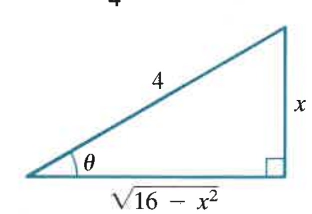
$$\begin{aligned} \int \frac{1}{x^2\sqrt{16 - x^2}} \, dx &= \int \frac{1}{(16\sin^2\theta)\, 4\cos\theta} 4\cos\theta \, d\theta \\ &= \frac{1}{16} \int \frac{1}{\sin^2\theta} \, d\theta \\ &= \tfrac{1}{16} \int \csc^2\theta \, d\theta \\ &= -\tfrac{1}{16} \cot\theta + C. \end{aligned}$$

We must now return to the original variable of integration, x. Since $\theta = \arcsin(x/4)$, we could write $-\frac{1}{16}\cot\theta$ as $-\frac{1}{16}\cot\arcsin(x/4)$, but this is a cumbersome expression. Since the integrand contains $\sqrt{16 - x^2}$, it is preferable that the evaluated form also contain this radical. There is a simple geometric method for ensuring that it does. If $0 < \theta < \pi/2$ and $\sin\theta = x/4$, we may interpret θ as an acute angle of a right triangle having opposite side and hypotenuse of lengths x and 4, respectively (see Figure 7.1). By the Pythagorean theorem, the length of the adjacent side is $\sqrt{16 - x^2}$. Referring to the triangle, we find

$$\cot\theta = \frac{\sqrt{16 - x^2}}{x}.$$

Figure 7.1

$\sin\theta = \dfrac{x}{4}$



It can be shown that the last formula is also true if $-\pi/2 < \theta < 0$. Thus, Figure 7.1 may be used if θ is either positive or negative.

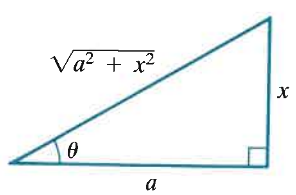
Substituting $\sqrt{16-x^2}/x$ for $\cot\theta$ in our integral evaluation gives us

$$\begin{aligned}\int \frac{1}{x^2\sqrt{16-x^2}}\,dx &= -\frac{1}{16}\cdot\frac{\sqrt{16-x^2}}{x}+C\\ &= -\frac{\sqrt{16-x^2}}{16x}+C.\end{aligned}$$

If an integrand contains $\sqrt{a^2+x^2}$ for $a>0$, then, by (7.4), we use the substitution $x = a\tan\theta$ to eliminate the radical. When using this substitution, we assume that θ is in the range of the inverse tangent function—that is, $-\pi/2<\theta<\pi/2$. In this case, $\sec\theta>0$ and

Figure 7.2

$\tan\theta = \dfrac{x}{a}$



$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+a^2\tan^2\theta}\\ &= \sqrt{a^2(1+\tan^2\theta)}\\ &= \sqrt{a^2\sec^2\theta}\\ &= a\sec\theta.\end{aligned}$$

After substituting and evaluating the resulting trigonometric integral, it is necessary to return to the variable x. We can do so by using the formula $\tan\theta = x/a$ and referring to the right triangle shown in Figure 7.2.

EXAMPLE ▪ 2 Evaluate $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$.

SOLUTION The denominator of the integrand has the form $\sqrt{a^2+x^2}$ with $a=2$. Hence, by (7.4), we make the substitution

$$x = 2\tan\theta,\qquad dx = 2\sec^2\theta\,d\theta.$$

Consequently,

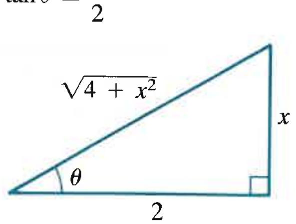
$$\begin{aligned}\sqrt{4+x^2} &= \sqrt{4+4\tan^2\theta}\\ &= 2\sqrt{1+\tan^2\theta} = 2\sqrt{\sec^2\theta} = 2\sec\theta\end{aligned}$$

and

$$\begin{aligned}\int\frac{1}{\sqrt{4+x^2}}\,dx &= \int\frac{1}{2\sec\theta}2\sec^2\theta\,d\theta\\ &= \int\sec\theta\,d\theta\\ &= \ln|\sec\theta+\tan\theta|+C.\end{aligned}$$

Figure 7.3

$\tan\theta = \dfrac{x}{2}$



Using $\tan\theta = x/2$, we sketch the triangle in Figure 7.3, from which we obtain

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Hence,

$$\int\frac{1}{\sqrt{4+x^2}}\,dx = \ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right|+C.$$

The expression on the right may be written

$$\ln\left|\frac{\sqrt{4+x^2}+x}{2}\right|+C = \ln\left|\sqrt{4+x^2}+x\right|-\ln 2+C.$$

Since $\sqrt{4+x^2}+x>0$ for every x, the absolute value sign is unnecessary. If we also let $D = -\ln 2 + C$, then

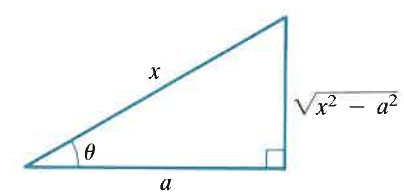
$$\int\frac{1}{\sqrt{4+x^2}}\,dx = \ln\left(\sqrt{4+x^2}+x\right)+D.$$

If an integrand contains $\sqrt{x^2-a^2}$, then using (7.4) we substitute $x = a\sec\theta$, where θ is chosen in the range of the inverse secant function—that is, either $0\le\theta<\pi/2$ or $\pi\le\theta<3\pi/2$. In this case, $\tan\theta\ge 0$ and

$$\begin{aligned}\sqrt{x^2-a^2} &= \sqrt{a^2\sec^2\theta-a^2}\\ &= \sqrt{a^2(\sec^2\theta-1)}\\ &= \sqrt{a^2\tan^2\theta}\\ &= a\tan\theta.\end{aligned}$$

Figure 7.4

$\sec\theta = \dfrac{x}{a}$



Since

$$\sec\theta = \frac{x}{a},$$

we may refer to the triangle in Figure 7.4 when changing from the variable θ to the variable x.

EXAMPLE ▪ 3 Evaluate $\displaystyle\int\frac{\sqrt{x^2-9}}{x}\,dx$.

SOLUTION The integrand contains $\sqrt{x^2-9}$, which is of the form $\sqrt{x^2-a^2}$ with $a=3$. Referring to (7.4), we substitute as follows:

$$x = 3\sec\theta,\qquad dx = 3\sec\theta\tan\theta\,d\theta$$

Consequently,

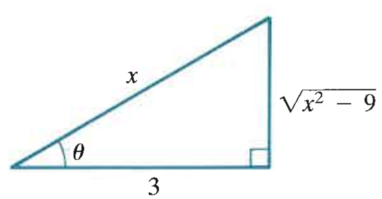
$$\begin{aligned}\sqrt{x^2-9} &= \sqrt{9\sec^2\theta-9}\\ &= 3\sqrt{\sec^2\theta-1} = 3\sqrt{\tan^2\theta} = 3\tan\theta\end{aligned}$$

and

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta$$
$$= 3\int \tan^2\theta\,d\theta$$
$$= 3\int (\sec^2\theta - 1)\,d\theta = 3\int \sec^2\theta\,d\theta - 3\int d\theta$$
$$= 3\tan\theta - 3\theta + C.$$

Since $\sec\theta = x/3$, we may refer to the right triangle in Figure 7.5. Using $\tan\theta = \sqrt{x^2-9}/3$ and $\theta = \sec^{-1}(x/3)$, we obtain

Figure 7.5
$\sec\theta = \dfrac{x}{3}$



$$\int \frac{\sqrt{x^2-9}}{x}\,dx = 3\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + C$$
$$= \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + C.$$

As shown in the next example, we can use trigonometric substitutions to evaluate certain integrals that involve $(a^2-x^2)^r$, $(a^2+x^2)^r$, or $(x^2-a^2)^r$, in cases other than $r = \frac{1}{2}$.

EXAMPLE ■ 4 Evaluate $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx.$

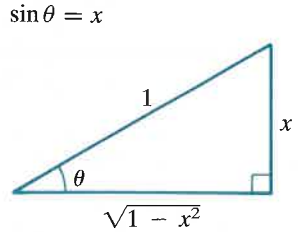
SOLUTION The integrand contains the expression $1-x^2$, which is of the form a^2-x^2 with $a = 1$. Using (7.4), we substitute

$$x = \sin\theta, \qquad dx = \cos\theta\,d\theta.$$

Thus, $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$, and

$$\int \frac{(1-x^2)^{3/2}}{x^6}\,dx = \int \frac{(\cos^2\theta)^{3/2}}{\sin^6\theta}\cos\theta\,d\theta$$
$$= \int \frac{\cos^4\theta}{\sin^6\theta}\,d\theta = \int \frac{\cos^4\theta}{\sin^4\theta}\cdot\frac{1}{\sin^2\theta}\,d\theta$$
$$= \int \cot^4\theta\csc^2\theta\,d\theta$$
$$= -\tfrac{1}{5}\cot^5\theta + C.$$

Figure 7.6
$\sin\theta = x$



To return to the variable x, we note that $\sin\theta = x = x/1$ and refer to the right triangle in Figure 7.6, obtaining $\cot\theta = \sqrt{1-x^2}/x$. Hence,

$$\int \frac{(1-x^2)^{3/2}}{x^6}\,dx = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C = -\frac{(1-x^2)^{5/2}}{5x^5} + C.$$

Trigonometric substitutions may also be used with trigonometric identities in the evaluation of definite integrals. In the next example, we use the substitution $x = a\sin\theta$ and the identity $2\cos^2\theta = 1 + \cos 2\theta$ to find the area bounded by an ellipse.

EXAMPLE ■ 5 Find the area of the region bounded by an ellipse whose major and minor axes have lengths $2a$ and $2b$, respectively.

SOLUTION By Theorem 34 (page 70), we see that an equation for the ellipse is $(x^2/a^2) + (y^2/b^2) = 1$. Solving for y gives us

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}.$$

The graph of the ellipse has the general shape shown in Figure 64 (page 69) and hence, by symmetry, it is sufficient to find the area of the region in the first quadrant and multiply the result by 4. By Theorem (4.19),

$$A = 4\int_0^a y\,dx = 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$$

If we make the trigonometric substitution $x = a\sin\theta$, then

$$\sqrt{a^2-x^2} = a\cos\theta \quad\text{and}\quad dx = a\cos\theta\,d\theta.$$

Since the values of θ that correspond to $x = 0$ and $x = a$ are $\theta = 0$ and $\theta = \pi/2$, respectively, we obtain

$$A = 4\frac{b}{a}\int_0^{\pi/2} a^2\cos^2\theta\,d\theta = 4ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$
$$= 2ab\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$
$$= 2ab\left[\frac{\pi}{2}\right] = \pi ab.$$

Thus, the area of an ellipse with axes of lengths $2a$ and $2b$ is πab. As a special case, if $b = a$, the ellipse is a circle and $A = \pi a^2$.

Although we now have additional integration techniques available, it is a good idea to keep earlier methods in mind. For example, the integral $\int (x/\sqrt{9+x^2})\,dx$ could be evaluated by means of the trigonometric substitution $x = 3\tan\theta$. However, it is simpler to use the algebraic substitution $u = 9 + x^2$ and $du = 2x\,dx$, for in this event the integral takes on the form $\frac{1}{2}\int u^{-1/2}\,du$, which is readily integrated by means of the power rule. The following exercises include integrals that can be evaluated using simpler techniques than trigonometric substitutions.

EXERCISES 7.3

Exer. 1–22: Evaluate the integral.

1 $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx$

2 $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

3 $\displaystyle\int \frac{1}{x\sqrt{9+x^2}}\,dx$

4 $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$

5 $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$

6 $\displaystyle\int \frac{1}{x^3\sqrt{x^2-25}}\,dx$

7 $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

8 $\displaystyle\int \frac{x}{x^2+9}\,dx$

9 $\displaystyle\int \frac{1}{(x^2-1)^{3/2}}\,dx$

10 $\displaystyle\int \frac{1}{\sqrt{4x^2-25}}\,dx$

11 $\displaystyle\int \frac{1}{(36+x^2)^2}\,dx$

12 $\displaystyle\int \frac{1}{(16-x^2)^{5/2}}\,dx$

13 $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

14 $\displaystyle\int \frac{1}{49+x^2}\,dx$

15 $\displaystyle\int \frac{x}{(16-x^2)^2}\,dx$

16 $\displaystyle\int x\sqrt{x^2-9}\,dx$

17 $\displaystyle\int \frac{x^3}{\sqrt{9x^2+49}}\,dx$

18 $\displaystyle\int \frac{1}{x\sqrt{25x^2+16}}\,dx$

19 $\displaystyle\int \frac{1}{x^4\sqrt{x^2-3}}\,dx$

20 $\displaystyle\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx$

21 $\displaystyle\int \frac{(4+x^2)^2}{x^3}\,dx$

22 $\displaystyle\int \frac{3x-5}{\sqrt{1-x^2}}\,dx$

23 The region bounded by the graphs of $y=0$, $x=5$, and $y=x(x^2+25)^{-1/2}$ is revolved about the y-axis. Find the volume of the resulting solid.

24 Find the area of the region bounded by the graph of $y=x^3(10-x^2)^{-1/2}$, the x-axis, and the line $x=1$.

25 The shape of the earth's surface can be approximated by revolving the ellipse $(x^2/a^2)+(y^2/b^2)=1$, with $a=6378$ km and $b=6356$ km, about the x-axis. Approximate the surface area of the earth to the nearest 10^6 km^2. (*Hint:* Use (5.19) with $f(x)=\sqrt{b^2-(b^2/a^2)x^2}$, and make the substitution $u=(b/a)x$.)

26 Let R be the region bounded by the right branch of the hyperbola $x^2-y^2=8$ and the vertical line through the focus. Find the area of the curved surface of the solid obtained by revolving R about the x-axis.

Exer. 27–28: Solve the differential equation subject to the given initial condition.

27 $x\,dy=\sqrt{x^2-16}\,dx;\quad y=0$ if $x=4$

28 $\sqrt{1-x^2}\,dy=x^3\,dx;\quad y=0$ if $x=0$

Exer. 29–34: Use a trigonometric substitution to derive the formula. (See Formulas 21, 27, 31, 36, 41, and 44 in Appendix II.)

29 $\displaystyle\int \sqrt{a^2+u^2}\,du$
$$=\frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right|+C$$

30 $\displaystyle\int \frac{1}{u\sqrt{a^2+u^2}}\,du=-\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right|+C$

31 $\displaystyle\int u^2\sqrt{a^2-u^2}\,du$
$$=\frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C$$

32 $\displaystyle\int \frac{1}{u^2\sqrt{a^2-u^2}}\,du=-\frac{1}{a^2u}\sqrt{a^2-u^2}+C$

33 $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du=\sqrt{u^2-a^2}-a\sec^{-1}\frac{u}{a}+C$

34 $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du$
$$=\frac{u}{2}\sqrt{u^2-a^2}+\frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right|+C$$

7.4 INTEGRALS OF RATIONAL FUNCTIONS

Recall that if q is a rational function, then $q(x)=f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials. In this section, we examine the rules for evaluating $\int q(x)\,dx$.

Let us consider the specific case $q(x)=2/(x^2-1)$. It is easy to verify that

$$\frac{1}{x-1}+\frac{-1}{x+1}=\frac{2}{x^2-1}.$$

The expression on the left side of the equation is called the *partial fraction decomposition* of $2/(x^2-1)$. To find $\int q(x)\,dx$, we integrate each of the fractions that make up the decomposition, obtaining

$$\int \frac{2}{x^2-1}\,dx=\int \frac{1}{x-1}\,dx+\int \frac{-1}{x+1}\,dx$$
$$=\ln|x-1|-\ln|x+1|+C$$
$$=\ln\left|\frac{x-1}{x+1}\right|+C.$$

It is theoretically possible to write *any* rational expression $f(x)/g(x)$ as a sum of rational expressions whose denominators involve powers of polynomials of degree not greater than two. Specifically, if $f(x)$ and $g(x)$ are polynomials *and the degree of $f(x)$ is less than the degree of $g(x)$*, then it can be proved that

$$\frac{f(x)}{g(x)}=F_1+F_2+\cdots+F_r$$

such that each term F_k of the sum has one of the forms

$$\frac{A}{(ax+b)^n}\quad\text{or}\quad\frac{Ax+B}{(ax^2+bx+c)^n}$$

for real numbers A and B and a nonnegative integer n, where ax^2+bx+c is **irreducible** in the sense that this quadratic polynomial has no real zeros (that is, $b^2-4ac<0$). In this case, ax^2+bx+c cannot be expressed as a product of two first-degree polynomials with real coefficients.

The sum $F_1+F_2+\cdots+F_r$ is the **partial fraction decomposition** of $f(x)/g(x)$, and each F_k is a **partial fraction**. We shall not prove this algebraic result but shall, instead, state guidelines for obtaining the decomposition.

The guidelines for finding the partial fraction decomposition of $f(x)/g(x)$ should be used only if $f(x)$ has lower degree than $g(x)$. If this is not the case, then we may use long division to arrive at the proper form. For example, given

$$\frac{x^3-6x^2+5x-3}{x^2-1},$$

we obtain, by long division,

$$\frac{x^3-6x^2+5x-3}{x^2-1}=x-6+\frac{6x-9}{x^2-1}.$$

We then find the partial fraction decomposition for $(6x-9)/(x^2-1)$.

Guidelines for Partial Fraction Decompositions of $f(x)/g(x)$ 7.5

1 If the degree of $f(x)$ is not lower than the degree of $g(x)$, use long division to obtain the proper form.

2 Express $g(x)$ as a product of linear factors $ax + b$ or irreducible quadratic factors $ax^2 + bx + c$, and collect repeated factors so that $g(x)$ is a product of *different* factors of the form $(ax + b)^n$ or $(ax^2 + bx + c)^n$ for a nonnegative integer n.

3 Apply the following rules.

Rule a For each factor $(ax + b)^n$ with $n \geq 1$, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n},$$

where each numerator A_k is a real number.

Rule b For each factor $(ax^2 + bx + c)^n$ with $n \geq 1$ and with $ax^2 + bx + c$ irreducible, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2+bx+c} + \frac{A_2x + B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2+bx+c)^n},$$

where each A_k and B_k is a real number.

EXAMPLE 1 Evaluate $\displaystyle\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x}\,dx.$

SOLUTION We may factor the denominator of the integrand as follows:

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) = x(x + 3)(x - 1)$$

Each factor has the form stated in rule (a) of (7.5), with $n = 1$. Thus, to the factor x there corresponds a partial fraction of the form A/x. Similarly, to the factors $x + 3$ and $x - 1$ there correspond partial fractions $B/(x + 3)$ and $C/(x - 1)$, respectively. Therefore, the partial fraction decomposition has the form

$$\frac{4x^2 + 13x - 9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}.$$

Multiplying by the lowest common denominator gives us

$$(*) \qquad 4x^2 + 13x - 9 = A(x + 3)(x - 1) + Bx(x - 1) + Cx(x + 3).$$

In a case such as this, in which the factors are all linear and nonrepeated, the values for A, B, and C can be found by substituting values for x that make the various factors zero. If we let $x = 0$ in $(*)$, then

$$-9 = -3A, \quad \text{or} \quad A = 3.$$

Letting $x = 1$ in $(*)$ gives us

$$8 = 4C, \quad \text{or} \quad C = 2.$$

Finally, if $x = -3$ in $(*)$, then

$$-12 = 12B, \quad \text{or} \quad B = -1.$$

The partial fraction decomposition is, therefore,

$$\frac{4x^2 + 13x - 9}{x(x+3)(x-1)} = \frac{3}{x} + \frac{-1}{x+3} + \frac{2}{x-1}.$$

Integrating and letting K denote the sum of the constants of integration, we have

$$\begin{aligned}\int \frac{4x^2 + 13x - 9}{x(x+3)(x-1)}\,dx &= \int \frac{3}{x}\,dx + \int \frac{-1}{x+3}\,dx + \int \frac{2}{x-1}\,dx \\ &= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + K \\ &= \ln|x^3| - \ln|x+3| + \ln|x-1|^2 + K \\ &= \ln\left|\frac{x^3(x-1)^2}{x+3}\right| + K.\end{aligned}$$

Another technique for finding A, B, and C is to expand the right-hand side of $(*)$ and collect like powers of x as follows:

$$4x^2 + 13x - 9 = (A + B + C)x^2 + (2A - B + 3C)x - 3A$$

We now use the fact that if two polynomials are equal, then coefficients of like powers of x are the same. It is convenient to arrange our work in the following way, which we call **comparing coefficients of x**:

$$\begin{aligned}&\textit{coefficients of } x^2\text{:} && A + B + C = 4 \\ &\textit{coefficients of } x\text{:} && 2A - B + 3C = 13 \\ &\textit{constant terms}\text{:} && -3A = -9\end{aligned}$$

We may show that the solution of this system of equations is $A = 3$, $B = -1$, and $C = 2$.

EXAMPLE 2 Evaluate $\displaystyle\int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3}\,dx.$

SOLUTION By rule (a) of (7.5), there is a partial fraction of the form $A/(x + 1)$ corresponding to the factor $x + 1$ in the denominator of the integrand. For the factor $(x - 2)^3$, we apply rule (a), with $n = 3$, obtaining a sum of three partial fractions $B/(x - 2)$, $C/(x - 2)^2$, and $D/(x - 2)^3$. Consequently, the partial fraction decomposition has the form

$$\frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}.$$

Multiplying both sides by $(x + 1)(x - 2)^3$ gives us

$$\begin{aligned}(*) \qquad 3x^3 - 18x^2 + 29x - 4 = {} & A(x - 2)^3 + B(x + 1)(x - 2)^2 \\ & + C(x + 1)(x - 2) + D(x + 1).\end{aligned}$$

Two of the unknown constants may be determined easily. If we let $x = 2$ in (∗), we obtain

$$6 = 3D, \quad \text{or} \quad D = 2.$$

Similarly, letting $x = -1$ in (∗) yields

$$-54 = -27A, \quad \text{or} \quad A = 2.$$

The remaining constants may be found by comparing coefficients. Examining the right-hand side of (∗), we see that the coefficient of x^3 is $A + B$. This must equal the coefficient of x^3 on the left. Thus, by comparison,

$$\textit{coefficients of } x^3\text{:} \quad 3 = A + B.$$

Since $A = 2$, it follows that $B = 1$.

Finally, we compare the constant terms in (∗) by letting $x = 0$. This gives us the following:

$$\textit{constant terms:} \quad -4 = -8A + 4B - 2C + D$$

Substituting the values we have found for A, B, and D into the preceding equation yields

$$-4 = -16 + 4 - 2C + 2,$$

which has the solution $C = -3$. The partial fraction decomposition is, therefore,

$$\frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} = \frac{2}{x+1} + \frac{1}{x-2} + \frac{-3}{(x-2)^2} + \frac{2}{(x-2)^3}.$$

To find the given integral, we integrate each of the partial fractions on the right side of the last equation, obtaining

$$2\ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + K$$

with K the sum of the four constants of integration. This may be written in the form

$$\ln\left[(x+1)^2|x-2|\right] + \frac{3}{x-2} - \frac{1}{(x-2)^2} + K.$$

EXAMPLE ■ 3 Evaluate $\displaystyle\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4}\,dx.$

SOLUTION The denominator may be factored by grouping as follows:

$$2x^3 - x^2 + 8x - 4 = x^2(2x-1) + 4(2x-1) = (x^2+4)(2x-1)$$

Applying rule (b) of (7.5) to the irreducible quadratic factor $x^2 + 4$, we see that one of the partial fractions has the form $(Ax + B)/(x^2 + 4)$. By rule (a), there is also a partial fraction $C/(2x - 1)$ corresponding to the factor $2x - 1$. Consequently,

$$\frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} = \frac{Ax + B}{x^2 + 4} + \frac{C}{2x - 1}.$$

As in previous examples, this result leads to

$$(*) \qquad x^2 - x - 21 = (Ax + B)(2x - 1) + C(x^2 + 4).$$

We can find one constant easily. Substituting $x = \frac{1}{2}$ in (∗) gives us

$$-\tfrac{85}{4} = \tfrac{17}{4}C, \quad \text{or} \quad C = -5.$$

The remaining constants may be found by comparing coefficients of x in (∗):

$$\begin{aligned} \textit{coefficients of } x^2\text{:} &\quad 1 = 2A + C \\ \textit{coefficients of } x\text{:} &\quad -1 = -A + 2B \\ \textit{constant terms:} &\quad -21 = -B + 4C \end{aligned}$$

Since $C = -5$, it follows from $1 = 2A + C$ that $A = 3$. Similarly, using the coefficients of x with $A = 3$ gives us $-1 = -3 + 2B$, or $B = 1$. Thus, the partial fraction decomposition of the integrand is

$$\begin{aligned} \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} &= \frac{3x + 1}{x^2 + 4} + \frac{-5}{2x - 1} \\ &= \frac{3x}{x^2 + 4} + \frac{1}{x^2 + 4} - \frac{5}{2x - 1}. \end{aligned}$$

The given integral may now be found by integrating the right side of the last equation. This gives us

$$\frac{3}{2}\ln(x^2 + 4) + \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{5}{2}\ln|2x - 1| + K.$$

EXAMPLE ■ 4 Evaluate $\displaystyle\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2}\,dx.$

SOLUTION Applying rule (b) of (7.5), with $n = 2$, yields

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

Multiplying by the lowest common denominator $(x^2 + 1)^2$ gives us

$$\begin{aligned} 5x^3 - 3x^2 + 7x - 3 &= (Ax + B)(x^2 + 1) + Cx + D \\ 5x^3 - 3x^2 + 7x - 3 &= Ax^3 + Bx^2 + (A + C)x + (B + D). \end{aligned}$$

We next compare coefficients as follows:

$$\begin{aligned} \textit{coefficients of } x^3\text{:} &\quad 5 = A \\ \textit{coefficients of } x^2\text{:} &\quad -3 = B \\ \textit{coefficients of } x\text{:} &\quad 7 = A + C \\ \textit{constant terms:} &\quad -3 = B + D \end{aligned}$$

We now have $A = 5$, $B = -3$, $C = 7 - A = 2$, and $D = -3 - B = 0$; therefore,

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{5x - 3}{x^2+1} + \frac{2x}{(x^2+1)^2}$$
$$= \frac{5x}{x^2+1} - \frac{3}{x^2+1} + \frac{2x}{(x^2+1)^2}.$$

Integrating yields

$$\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2}\,dx = \frac{5}{2}\ln(x^2+1) - 3\tan^{-1} x - \frac{1}{x^2+1} + K.$$

Many of the steps in the partial fraction decomposition of $f(x)/g(x)$ are straightforward algebraic manipulations that may be tedious to perform if the degree of g is large. Computer algebra systems provide a useful tool for automating some of these steps in such cases. The next example illustrates a few of the capabilities of these systems.

EXAMPLE ■ 5 Use a computer algebra system to evaluate

$$\int \frac{264x^3 - 553x^2 - 310x - 543}{24x^4 - 142x^3 - 59x^2 + 267x + 90}\,dx.$$

SOLUTION We first use a CAS to help factor the denominator. The exact commands and the rules for using them depend on the particular CAS being used. In this example, we illustrate some commands available in *Theorist*®. We *enter* the rational function and then *select* the denominator. The command to *factor* yields

$$\frac{264x^3 - 553x^2 - 310x - 543}{24x^4 - 142x^3 - 59x^2 + 267x + 90}$$
$$= \frac{264x^3 - 553x^2 - 310x - 543}{24(x-6)(x+\frac{5}{4})(x-\frac{3}{2})(x+\frac{1}{3})}.$$

We see that the denominator is the product of distinct linear factors. Next, we *select* the entire expression on the right side of the equation and give the command to *expand*. The CAS responds with the decomposition:

$$\frac{264x^3 - 553x^2 - 310x - 543}{24(x-6)(x+\frac{5}{4})(x-\frac{3}{2})(x+\frac{1}{3})}$$
$$= 7\frac{1}{x-6} + \frac{7}{2}\frac{1}{x+\frac{5}{4}} + \frac{5}{2}\frac{1}{x-\frac{3}{2}} - 2\frac{1}{x+\frac{1}{3}}$$

We may now integrate the original fraction by integrating each of the four terms on the right:

$$\int \frac{264x^3 - 553x^2 - 310x - 543}{24x^4 - 142x^3 - 59x^2 + 267x + 90}\,dx$$
$$= 7\int \frac{dx}{x-6} + \frac{7}{2}\int \frac{dx}{x+\frac{5}{4}} + \frac{5}{2}\int \frac{dx}{x-\frac{3}{2}} - 2\int \frac{dx}{x+\frac{1}{3}}$$
$$= 7\ln|x-6| + \frac{7}{2}\ln\left|x+\frac{5}{4}\right| + \frac{5}{2}\ln\left|x-\frac{3}{2}\right| - 2\ln\left|x+\frac{1}{3}\right| + K$$

The next example is an application in which a partial fraction decomposition is used to solve a differential equation.

EXAMPLE ■ 6 If x represents the number of people in a population of constant size N who have certain information, then a model of social diffusion for the rate by which x changes is

$$\frac{dx}{dt} = kx(N - x)$$

for some positive constant k (see Section 3.8).

(a) Find the number of people $x(t)$ who have the information at time t as an explicit function of t.

(b) Find $\lim_{t\to\infty} x(t)$ and interpret the result.

SOLUTION

(a) Beginning with the differential equation

$$\frac{dx}{dt} = kx(N - x),$$

we separate the variables and integrate to obtain

$$\int dt = \int \frac{1}{kx(N-x)}\,dx.$$

To integrate the expression on the right side of this equation, we make the partial fraction decomposition

$$\frac{1}{kx(N-x)} = \frac{1/N}{kx} + \frac{1/kN}{N-x},$$

so that

$$\int N\,dt = \int \frac{N}{kx(N-x)}\,dx$$
$$= \int \left(\frac{1}{kx} + \frac{1/k}{N-x}\right)dx$$
$$= \int \frac{1}{kx}\,dx + \int \frac{1/k}{N-x}\,dx.$$

Integrating, we have

$$Nt = \frac{1}{k}\ln|x| - \frac{1}{k}\ln|N-x| + D,$$

or

$$kNt = \ln|x| - \ln|N-x| + kD$$

for some constant D. Since x and $N - x$ represent numbers of people, they are positive, so $\ln|x| = \ln x$ and $\ln|N-x| = \ln(N-x)$. Thus,

$$kNt = \ln x - \ln(N-x) + kD.$$

Using properties of the logarithm,

$$\ln A + \ln B = \ln(AB) \quad \text{and} \quad \ln A - \ln B = \ln(A/B),$$

we have

$$kNt = \ln\frac{x}{N-x} + \ln e^{kD} = \ln\frac{Cx}{N-x},$$

where the constant C represents e^{kD}. Since $y = \ln x$ is equivalent to $x = e^y$, we can write

$$kNt = \ln\frac{Cx}{N-x} \quad \text{as} \quad \frac{Cx}{N-x} = e^{kNt}.$$

We now solve this equation for x:

$$\begin{aligned} Cx &= e^{kNt}(N-x) \\ Cx + e^{kNt}x &= Ne^{kNt} \\ x(C + e^{kNt}) &= Ne^{kNt} \\ x &= \frac{Ne^{kNt}}{C + e^{kNt}} \end{aligned}$$

Thus, the solution of the differential equation gives $x(t)$, the number of people x who have the information at time t, as

$$x(t) = \frac{Ne^{kNt}}{C + e^{kNt}}.$$

If the number of people who have the information at time $t = 0$ is x_0, then we can determine the value of C since

$$x_0 = \frac{Ne^0}{C + e^0} = \frac{N}{C+1}.$$

Thus,

$$C = \frac{N}{x_0} - 1 = \frac{N - x_0}{x_0},$$

and we can write $x(t)$ as

$$x(t) = \frac{Nx_0e^{kNt}}{N - x_0 + x_0e^{kNt}}.$$

(b) To determine

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{Ne^{kNt}}{C + e^{kNt}},$$

we first divide the numerator and the denominator of the fraction by e^{kNt} to obtain

$$x = \frac{N}{Ce^{-kNt} + 1},$$

so that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{N}{Ce^{-kNt} + 1}.$$

Since k and N are positive,

$$\lim_{t\to\infty} e^{-kNt} = 0$$

and hence

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{N}{Ce^{-kNt} + 1} = \frac{N}{C \cdot 0 + 1} = N.$$

We conclude that the model predicts that eventually everyone in the population will have the information.

NOTE The differential equation $dx/dt = kx(N-x)$ in Example 6 is called the *logistic model*. It occurs in many applications in which the growth of a population is under consideration.

EXERCISES 7.4

Exer. 1–28: Evaluate the integral.

1 $\displaystyle\int \frac{5x-12}{x(x-4)}\,dx$

2 $\displaystyle\int \frac{x+34}{(x-6)(x+2)}\,dx$

3 $\displaystyle\int \frac{37-11x}{(x+1)(x-2)(x-3)}\,dx$

4 $\displaystyle\int \frac{4x^2+54x+134}{(x-1)(x+5)(x+3)}\,dx$

5 $\displaystyle\int \frac{6x-11}{(x-1)^2}\,dx$

6 $\displaystyle\int \frac{-19x^2+50x-25}{x^2(3x-5)}\,dx$

7 $\displaystyle\int \frac{x+16}{x^2+2x-8}\,dx$

8 $\displaystyle\int \frac{11x+2}{2x^2-5x-3}\,dx$

9 $\displaystyle\int \frac{5x^2-10x-8}{x^3-4x}\,dx$

10 $\displaystyle\int \frac{4x^2-5x-15}{x^3-4x^2-5x}\,dx$

11 $\displaystyle\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx$

12 $\displaystyle\int \frac{2x^2-12x+4}{x^3-4x^2}\,dx$

13 $\displaystyle\int \frac{9x^4+17x^3+3x^2-8x+3}{x^5+3x^4}\,dx$

14 $\displaystyle\int \frac{5x^2+30x+43}{(x+3)^3}\,dx$

15 $\displaystyle\int \frac{x^3+6x^2+3x+16}{x^3+4x}\,dx$

16 $\displaystyle\int \frac{2x^2+7x}{x^2+6x+9}\,dx$

17 $\displaystyle\int \frac{5x^2+11x+17}{x^3+5x^2+4x+20}\,dx$

18 $\displaystyle\int \frac{4x^3-3x^2+6x-27}{x^4+9x^2}\,dx$

19 $\displaystyle\int \frac{x^2+3x+1}{x^4+5x^2+4}\,dx$

20 $\displaystyle\int \frac{4x}{(x^2+1)^3}\,dx$

21 $\displaystyle\int \frac{2x^3+10x}{(x^2+1)^2}\,dx$

22 $\displaystyle\int \frac{x^4+2x^2+4x+1}{(x^2+1)^3}\,dx$

23 $\displaystyle\int \frac{x^3+3x-2}{x^2-x}\,dx$

24 $\displaystyle\int \frac{x^4+2x^2+3}{x^3-4x}\,dx$

25 $\displaystyle\int \frac{x^6-x^3+1}{x^4+9x^2}\,dx$

26 $\displaystyle\int \frac{x^5}{(x^2+4)^2}\,dx$