INTRODUCTION

HUMAN CIVILIZATIONS AROSE in the fertile river valleys of China, Egypt, Africa, India, and Mesopotamia. As societies grew highly complex and interdependent, governmental units were created to provide services. To fund these efforts, people paid taxes that were often based then, as now, on the amount and value of their land. Since the annual flooding of the rivers swept away land masses or affected their agricultural value, there occurred early in human history a need to measure accurately land regions, a need we continue to have today.

Land masses and their boundaries are highly irregular. They are subject to significant changes due to such forces as oceans, rivers, and earthquakes. We require accurate ways to estimate precisely lengths, areas, and volumes. As technology will increasingly have an impact on society, there will be an expanding set of situations in which complex quantities must be determined accurately. Calculus provides a powerful means of making such measurements.

This chapter begins with the seemingly unrelated problem of reversing the procedure for finding derivatives: Given a function f, find a function F such that $F' = f$. This problem leads to the definition in Section 4.1 of the closely related ideas of antiderivatives and indefinite integrals. We also consider some elementary differential equations, an important modeling tool for applications. In Section 4.2, we study the technique of change of variables for finding indefinite integrals.

We then turn to a more careful examination of the problem of finding the area of an irregular region, beginning with the area under the graph of a function. In Section 4.3, such an area is defined as a limit of areas of inscribed or circumscribed rectangles. This approach is generalized in Section 4.4, where we give a careful definition of the definite integral of a function as a limit of Riemann sums. We state and show the application of basic properties of the definite integral in Section 4.5.

The principal result in this chapter is the *fundamental theorem of calculus*, proved in Section 4.6. This important theorem enables us to find exact values of definite integrals by using an *antiderivative* or indefinite integral. In addition to providing an important evaluation process, the fundamental theorem shows that there is a relationship between derivatives and integrals—a key result in calculus.

The chapter closes in Section 4.7 with a discussion of methods of *numerical integration*, which are used to approximate definite integrals that we cannot evaluate by the fundamental theorem. These methods are readily programmable for use with calculators and computers and are employed in a wide variety of applied fields.

CHAPTER 4

The definite integral provides a powerful tool for computing lengths of curves, areas of regions, and volumes of solids.

Integrals

4.1 ANTIDERIVATIVES, INDEFINITE INTEGRALS, AND SIMPLE DIFFERENTIAL EQUATIONS

We begin this section with the problem of reversing the process of differentiation and examine two closely related concepts: *antiderivative* and *indefinite integral*. Then we take a first look at using indefinite integrals to solve simple differential equations where we seek to obtain explicit information about a function from given information about its derivative.

ANTIDERIVATIVES

In our previous work, we solved problems of the following type: *Given a function f, find the derivative f'*. We now consider the reverse process: *Given a function f, find a function F such that $F' = f$*. In the next definition, we give F a special name.

Definition 4.1

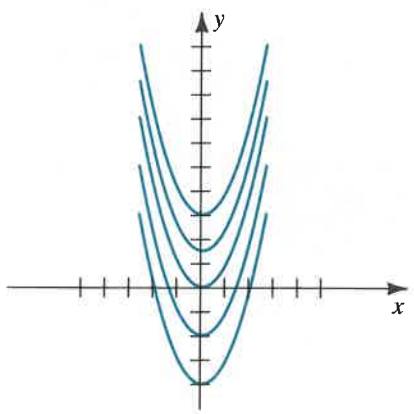
> A function F is an **antiderivative** of the function f on an interval I if $F'(x) = f(x)$ for every x in I.

We shall also call $F(x)$ an antiderivative of $f(x)$. The process of finding F, or $F(x)$, is called **antidifferentiation**.

To illustrate, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, because

$$F'(x) = \frac{d}{dx}(x^2) = 2x = f(x).$$

Figure 4.1



There are many other antiderivatives of $2x$, such as $x^2 + 2$, $x^2 - \frac{5}{3}$, and $x^2 + \sqrt{3}$. In general, if C is *any* constant, then $x^2 + C$ is an antiderivative of $2x$, because

$$\frac{d}{dx}(x^2 + C) = 2x + 0 = 2x.$$

Thus there is a *family of antiderivatives* of $2x$ of the form $F(x) = x^2 + C$, where C is any constant. Graphs of several members of this family are sketched in Figure 4.1.

The next illustration contains other examples of antiderivatives, where C is a constant.

ILLUSTRATION

$f(x)$	Antiderivatives of $f(x)$
x^2	$\frac{1}{3}x^3$, $\frac{1}{3}x^3 + 8$, $\frac{1}{3}x^3 + C$
$8x^3$	$2x^4$, $2x^4 - \sqrt[3]{7}$, $2x^4 + C$
$\cos x$	$\sin x$, $\sin x + \frac{4}{9}$, $\sin x + C$

As in the preceding illustration, if $F(x)$ is an antiderivative of $f(x)$, then so too is $F(x) + C$ for any constant C. It is a consequence of the

mean value theorem (3.12) that *every* antiderivative is of this form. In Corollary (3.14), we proved that two functions with identical derivatives can differ only by some constant. The next theorem restates this result in the language of antiderivatives.

Theorem 4.2

> Let F be an antiderivative of f on an interval I. If G is any antiderivative of f on I, then
>
> $$G(x) = F(x) + C$$
>
> for some constant C and every x in I.

We refer to the constant C in Theorem (4.2) as an **arbitrary constant**. If $F(x)$ is an antiderivative of $f(x)$, then *all* antiderivatives of $f(x)$ can be obtained from $F(x) + C$ by letting C range through the set of real numbers.

INDEFINITE INTEGRALS

We shall use the following notation for a family of antiderivatives of the type given in Theorem (4.2).

Definition 4.3

> The notation
>
> $$\int f(x)\,dx = F(x) + C,$$
>
> where $F'(x) = f(x)$ and C is an arbitrary constant, denotes the family of all antiderivatives of $f(x)$ on an interval I.

CAUTION Theorem (4.2) may be false if the interval I is replaced by some other set of real numbers. For example, if A is the set of nonzero real numbers, then the function $F(x) = 1/x$ has derivative $-1/x^2$ for all x in A, as does the function G defined by

$$G(x) = \begin{cases} 1/x & \text{if } x < 0 \\ (1/x) + 1 & \text{if } x > 0 \end{cases}$$

but there is no constant C such that $G(x) = F(x) + C$ for every x in A. Thus, in problems involving antiderivatives for a function f, we always assume, even if not explicitly stated, that the domain of f is an interval.

The symbol $\int$ used in Definition (4.3) is an **integral sign**. We call $\int f(x)\,dx$ the **indefinite integral** of $f(x)$. The expression $f(x)$ is the **integrand**, and C is the **constant of integration**. The process of finding $F(x) + C$, when given $\int f(x)\,dx$, is referred to as **indefinite integration, evaluating the integral,** or **integrating $f(x)$**. The adjective *indefinite* is used because $\int f(x)\,dx$ represents a *family* of antiderivatives, not any *specific* function. Later in the chapter, when we discuss definite integrals, we shall see the reasons for using the integral sign and the differential expression dx that appears to the right of the integrand $f(x)$. For now, we regard dx merely as a symbol that specifies the independent variable x, which we refer to as the **variable of integration**. If we use a different variable of integration, such as t, we write

$$\int f(t)\,dt = F(t) + C,$$

where $F'(t) = f(t)$.

ILLUSTRATION

- $\int x^4\,dx = \frac{1}{5}x^5 + C$ because $\frac{d}{dx}\left(\frac{1}{5}x^5\right) = x^4.$
- $\int t^{-3}\,dt = -\frac{1}{2}t^{-2} + C$ because $\frac{d}{dt}\left(-\frac{1}{2}t^{-2}\right) = t^{-3}.$
- $\int \cos u\,du = \sin u + C$ because $\frac{d}{du}(\sin u) = \cos u.$
- $\int x\cos x\,dx = x\sin x + \cos x + C$ because $\frac{d}{dx}(x\sin x + \cos x) = x\cos x.$

Note that, in general,

$$\int \frac{d}{dx}(f(x))\,dx = f(x) + C$$

because $f'(x) = (d/dx)(f(x))$. This result allows us to use any derivative formula to obtain a corresponding formula for an indefinite integral, as illustrated in the next table. As shown in Formula (1), it is customary to abbreviate $\int 1\,dx$ by $\int dx$.

Brief Table of Indefinite Integrals 4.4

Derivative $\frac{d}{dx}(f(x))$	Indefinite integral $\int \frac{d}{dx}(f(x))\,dx = f(x) + C$
$\frac{d}{dx}(x) = 1$	(1) $\int 1\,dx = \int dx = x + C$
$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = x^r\ (r \neq -1)$	(2) $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C\ (r \neq -1)$
$\frac{d}{dx}(\sin x) = \cos x$	(3) $\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(-\cos x) = \sin x$	(4) $\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	(5) $\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}(-\cot x) = \csc^2 x$	(6) $\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}(\sec x) = \sec x\tan x$	(7) $\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}(-\csc x) = \csc x\cot x$	(8) $\int \csc x\cot x\,dx = -\csc x + C$

Formula (2) is called the *power rule for indefinite integration*. As in the following illustration, it is often necessary to rewrite an integrand before applying the power rule or one of the trigonometric formulas.

ILLUSTRATION

- $\int x^3 \cdot x^5\,dx = \int x^8\,dx = \frac{x^{8+1}}{8+1} + C = \frac{x^9}{9} + C$
- $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2x^2} + C$
- $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \frac{x^{2/3+1}}{\frac{2}{3}+1} + C = \frac{3}{5}x^{5/3} + C$
- $\int \frac{\tan x}{\sec x}\,dx = \int \cos x\,\frac{\sin x}{\cos x}\,dx = \int \sin x\,dx = -\cos x + C$

It is a good idea to check indefinite integrations (such as those in the preceding illustration) by differentiating the final expression to see if either the integrand or an equivalent form of the integrand is obtained.

The next theorem indicates that differentiation and indefinite integration are inverse processes, because each, in a sense, undoes the other. In (i), we assume that f is differentiable, and in (ii), that f has an antiderivative on some interval.

Theorem 4.5

(i) $\int \frac{d}{dx}(f(x))\,dx = f(x) + C$

(ii) $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$

PROOF We have already proved (i). To prove (ii), let F be an antiderivative of f and write

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = \frac{d}{dx}(F(x) + C) = F'(x) + 0 = f(x).$$

EXAMPLE 1 Verify Theorem (4.5) for the special case $f(x) = x^2$.

SOLUTION

(i) If we first differentiate x^2 and then integrate,

$$\int \frac{d}{dx}(x^2)\,dx = \int 2x\,dx = x^2 + C.$$

(ii) If we first integrate x^2 and then differentiate,

$$\frac{d}{dx}\left(\int x^2\,dx\right) = \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2.$$

The next theorem is useful for evaluating many types of indefinite integrals. In the statements, we assume that $f(x)$ and $g(x)$ have antiderivatives on an interval I.

Theorem 4.6

(i) $\int cf(x)\,dx = c\int f(x)\,dx$ for any nonzero constant c

(ii) $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(iii) $\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

PROOF We shall prove (ii). The proofs of (i) and (iii) are similar. If F and G are antiderivatives of f and g, respectively,

$$\frac{d}{dx}(F(x)+G(x)) = F'(x)+G'(x) = f(x)+g(x).$$

Hence, by Definition (4.3),

$$\int [f(x)+g(x)]\,dx = F(x)+G(x)+C,$$

where C is an arbitrary constant. Similarly,

$$\int f(x)\,dx + \int g(x)\,dx = F(x)+C_1+G(x)+C_2$$

for arbitrary constants C_1 and C_2. These give us the same family of antiderivatives, since for any special case, we can choose values of the constants such that $C = C_1 + C_2$. We thus prove (ii). ■

Theorem (4.6)(i) is sometimes stated as follows: *A constant factor in the integrand may be taken outside the integral sign.*

CAUTION It is *not* permissible to take expressions involving variables outside the integral sign in this manner. Note, for example, that

$$\int x\cos x\,dx = x\sin x + \cos x + C,$$

whereas

$$x\int \cos x\,dx = x(\sin x + C) = x\sin x + Cx.$$

These two expressions do not differ by a constant. Hence,

$$\int x\cos x\,dx \neq x\int \cos x\,dx.$$

EXAMPLE ■ 2 Evaluate $\int (5x^3 + 2\cos x)\,dx$.

SOLUTION We first use (ii) and (i) of Theorem (4.6) and then formulas from (4.4):

$$\begin{aligned}\int (5x^3+2\cos x)\,dx &= \int 5x^3\,dx + \int 2\cos x\,dx\\ &= 5\int x^3\,dx + 2\int \cos x\,dx\\ &= 5\left(\frac{x^4}{4}+C_1\right) + 2(\sin x + C_2)\\ &= \tfrac{5}{4}x^4 + 5C_1 + 2\sin x + 2C_2\\ &= \tfrac{5}{4}x^4 + 2\sin x + C,\end{aligned}$$

where $C = 5C_1 + 2C_2$.

In Example 2, we added the two constants $5C_1$ and $2C_2$ to obtain one arbitrary constant C. We can always manipulate arbitrary constants in this way, so it is not necessary to introduce a constant for each indefinite integration as we did in Example 2. Instead, if an integrand is a sum, *we integrate each term of the sum without introducing constants and then add one arbitrary constant C after the last integration.* We also often bypass the step $\int cf(x)\,dx = c\int f(x)\,dx$, as in the next example.

EXAMPLE ■ 3 Evaluate $\int \left(8t^3 - 6\sqrt{t} + \frac{1}{t^3}\right) dt$.

SOLUTION First we find an antiderivative for each of the three terms in the integrand and then add an arbitrary constant C. We rewrite $\sqrt{t}$ as $t^{1/2}$ and $1/t^3$ as t^{-3} and then use the power rule for integration:

$$\begin{aligned}\int \left(8t^3 - 6\sqrt{t} + \frac{1}{t^3}\right) dt &= \int (8t^3 - 6t^{1/2} + t^{-3})\,dt\\ &= 8\cdot\frac{t^4}{4} - 6\cdot\frac{t^{3/2}}{\frac{3}{2}} + \frac{t^{-2}}{-2} + C\\ &= 2t^4 - 4t^{3/2} - \frac{1}{2t^2} + C\end{aligned}$$

EXAMPLE ■ 4 Evaluate $\int \frac{(x^2-1)^2}{x^2}\,dx$.

SOLUTION First we change the form of the integrand, because the degree of the numerator is greater than or equal to the degree of the denominator. We then find an antiderivative for each term, adding an arbitrary constant C after the last integration:

$$\begin{aligned}\int \frac{(x^2-1)^2}{x^2}\,dx &= \int \frac{x^4-2x^2+1}{x^2}\,dx\\ &= \int (x^2-2+x^{-2})\,dx\\ &= \frac{x^3}{3}-2x+\frac{x^{-1}}{-1}+C\\ &= \frac{1}{3}x^3-2x-\frac{1}{x}+C\end{aligned}$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{1}{\cos u \cot u}\,du$.

SOLUTION We use trigonometric identities to change the integrand and then apply Formula (7) from Table (4.4):

$$\begin{aligned}\int \frac{1}{\cos u \cot u}\,du &= \int \sec u \tan u\,du\\ &= \sec u + C\end{aligned}$$

NOTE While most work on computers involves numerical calculations, computer algebra systems (CAS) can perform operations on symbolic formulas. Using CAS software, you may be able to enter the expression for a function and then request the indefinite integral. If an antiderivative can be found using the techniques discussed in this text, there is an excellent chance that the CAS will be successful in finding one. If you have access to a CAS, you should investigate the rules for entering functions symbolically and requesting derivatives and antiderivatives.

SIMPLE DIFFERENTIAL EQUATIONS

An applied problem may be stated in terms of a **differential equation**—that is, an equation that involves derivatives or differentials of an unknown function. A function f is a **solution** of a differential equation if it satisfies the equation—that is, if substitution of f for the unknown function produces a true statement. To **solve** a differential equation means to find all solutions. Sometimes, in addition to the differential equation, we may know certain values of f or f', called **initial conditions**.

Indefinite integrals are useful for solving certain differential equations, because if we are given a derivative $f'(x)$, we can integrate and use Theorem (4.5)(i) to obtain an equation involving the unknown function f:

$$\int f'(x)\,dx = f(x) + C$$

If we are also given an initial condition for f, it may be possible to find $f(x)$ explicitly, as in the next example.

EXAMPLE 6 Solve the differential equation

$$f'(x) = 6x^2 + x - 5$$

subject to the initial condition $f(0) = 2$.

SOLUTION We proceed as follows:

$$\begin{aligned}f'(x) &= 6x^2 + x - 5\\ \int f'(x)\,dx &= \int (6x^2+x-5)\,dx\\ f(x) &= 2x^3 + \tfrac{1}{2}x^2 - 5x + C\end{aligned}$$

for some number C. (It is unnecessary to add a constant of integration to *each* side of the equation.) Letting $x = 0$ and using the given initial condition $f(0) = 2$ gives us

$$f(0) = 0 + 0 - 0 + C, \quad \text{or} \quad 2 = C.$$

Hence the solution f of the differential equation with the initial condition $f(0) = 2$ is

$$f(x) = 2x^3 + \tfrac{1}{2}x^2 - 5x + 2.$$

If we are given a *second* derivative $f''(x)$, then we must employ two successive indefinite integrals to find $f(x)$. First we use Theorem (4.5)(i) as follows:

$$\int f''(x)\,dx = \int \frac{d}{dx}(f'(x))\,dx = f'(x) + C$$

After finding $f'(x)$, we proceed as in Example 6.

EXAMPLE 7 Solve the differential equation

$$f''(x) = 5\cos x + 2\sin x$$

subject to the initial conditions $f(0) = 3$ and $f'(0) = 4$.

SOLUTION We proceed as follows:

$$\begin{aligned}f''(x) &= 5\cos x + 2\sin x\\ \int f''(x)\,dx &= \int (5\cos x + 2\sin x)\,dx\\ f'(x) &= 5\sin x - 2\cos x + C\end{aligned}$$

Letting $x = 0$ and using the initial condition $f'(0) = 4$ gives us

$$\begin{aligned}f'(0) &= 5\sin 0 - 2\cos 0 + C\\ 4 &= 0 - 2\cdot 1 + C, \quad \text{or} \quad C = 6.\end{aligned}$$

Thus,

$$f'(x) = 5\sin x - 2\cos x + 6.$$

We integrate a second time:

$$\int f'(x)\,dx = \int (5\sin x - 2\cos x + 6)\,dx$$
$$f(x) = -5\cos x - 2\sin x + 6x + D$$

Letting $x = 0$ and using the initial condition $f(0) = 3$, we find that

$$f(0) = -5\cos 0 - 2\sin 0 + 6\cdot 0 + D$$
$$3 = -5 - 0 + 0 + D, \quad \text{or} \quad D = 8.$$

Therefore, the solution of the differential equation with the given initial condition is

$$f(x) = -5\cos x - 2\sin x + 6x + 8.$$

Suppose that a point P is moving on a coordinate line with an acceleration $a(t)$ at time t, and the corresponding velocity is $v(t)$. By Definition (3.23), $a(t) = v'(t)$ and hence

$$\int a(t)\,dt = \int v'(t)\,dt = v(t) + C$$

for some constant C.

Similarly, if we know $v(t)$, then since $v(t) = s'(t)$, where s is the position function of P, we can find a formula that involves $s(t)$ by indefinite integration:

$$\int v(t)\,dt = \int s'(t)\,dt = s(t) + D$$

for some constant D. In the next example, we shall use this technique to find the position function for an object that is moving with a given acceleration function $a(t)$.

EXAMPLE 8 A particle moving along a coordinate line at time $t = 0$ is at a position 3 cm from the origin and traveling at a velocity of 7 cm/sec. If the acceleration of the particle is given by

$$a(t) = 2 - 2(t+1)^{-3},$$

find the velocity and the position of the particle as functions of t.

SOLUTION Since the velocity $v(t) = \int v'(t)\,dt = \int a(t)\,dt$, we have

$$v(t) = \int [2 - 2(t+1)^{-3}]\,dt$$
$$= 2t + (t+1)^{-2} + C$$

for some number C. Substituting 0 for t and using the fact that $v(0) = 7$ gives us $7 = 0 + 1 + C$, or $C = 6$. Consequently,

$$v(t) = 2t + (t+1)^{-2} + 6.$$

Since $s'(t) = v(t)$, we obtain

$$s'(t) = 2t + (t+1)^{-2} + 6$$
$$\int s'(t)\,dt = \int [2t + (t+1)^{-2} + 6]\,dt$$
$$s(t) = t^2 - (t+1)^{-1} + 6t + D$$

for some number D. Using the fact that $s(0) = 3$ gives $3 = 0 - 1 + 0 + D$, or $D = 4$. Thus, the position of the particle from the origin at time t is given by

$$s(t) = t^2 - (t+1)^{-1} + 6t + 4 \text{ cm},$$

and the particle travels at a velocity of

$$v(t) = 2t + (t+1)^{-2} + 6 \text{ cm/sec}.$$

In economics applications, if a marginal function is known (see page 330), then we can use indefinite integration to find the function, as illustrated in the next example.

EXAMPLE 9 A manufacturer finds that the marginal cost (in dollars) associated with the production of x units of a photocopier component is given by $30 - 0.02x$. If the cost of producing one unit is \$35, find the cost function and the cost of producing 100 units.

SOLUTION If C is the cost function, then the marginal cost is the rate of change of C with respect to x—that is,

$$C'(x) = 30 - 0.02x.$$

Hence

$$\int C'(x)\,dx = \int (30 - 0.02x)\,dx$$

and

$$C(x) = 30x - 0.01x^2 + K$$

for some K. Letting $x = 1$ and using $C(1) = 35$, we obtain

$$35 = 30 - 0.01 + K, \quad \text{or} \quad K = 5.01.$$

Consequently,

$$C(x) = 30x - 0.01x^2 + 5.01.$$

In particular, the cost of producing 100 units is

$$C(100) = 3000 - 100 + 5.01$$
$$= \$2905.01.$$

EXERCISES 4.1

Exer. 1–40: Evaluate.

1 $\int (4x+3)\,dx$

2 $\int (4x^2 - 8x + 1)\,dx$

3 $\int (9t^2 - 4t + 3)\,dt$

4 $\int (2t^3 - t^2 + 3t - 7)\,dt$

5 $\int \left(\frac{1}{z^3} - \frac{3}{z^2}\right) dz$

6 $\int \left(\frac{4}{z^7} - \frac{7}{z^4} + z\right) dz$

7 $\int \left(3\sqrt{u} + \frac{1}{\sqrt{u}}\right) du$

8 $\int \left(\sqrt{u^3} - \tfrac{1}{2}u^{-2} + 5\right) du$

9 $\int (2v^{5/4} + 6v^{1/4} + 3v^{-4})\,dv$

10 $\int (3v^5 - v^{5/3})\,dv$

11 $\int (3x-1)^2\,dx$

12 $\int \left(x - \frac{1}{x}\right)^2 dx$

13 $\int x(2x+3)\,dx$

14 $\int (2x-5)(3x+1)\,dx$

15 $\int \frac{8x-5}{\sqrt[3]{x}}\,dx$

16 $\int \frac{2x^2 - x + 3}{\sqrt{x}}\,dx$

17 $\int \frac{x^3-1}{x-1}\,dx$

18 $\int \frac{x^3 + 3x^2 - 9x - 2}{x-2}\,dx$

19 $\int \frac{(t^2+3)^2}{t^6}\,dt$

20 $\int \frac{(\sqrt{t}+2)^2}{t^3}\,dt$

21 $\int \tfrac{3}{4}\cos u\,du$

22 $\int -\tfrac{3}{5}\sin u\,du$

23 $\int \frac{7}{\csc x}\,dx$

24 $\int \frac{1}{4\sec x}\,dx$

25 $\int (\sqrt{t} + \cos t)\,dt$

26 $\int \left(\sqrt[3]{t^2} - \sin t\right) dt$

27 $\int \frac{\sec t}{\cos t}\,dt$

28 $\int \frac{1}{\sin^2 t}\,dt$

29 $\int (\csc v \cot v \sec v)\,dv$

30 $\int (4 + 4\tan^2 v)\,dv$

31 $\int \frac{\sec w \sin w}{\cos w}\,dw$

32 $\int \frac{\csc w \cos w}{\sin w}\,dw$

33 $\int \frac{(1+\cot^2 z)\cot z}{\csc z}\,dz$

34 $\int \frac{\tan z}{\cos z}\,dz$

35 $\int \frac{d}{dx}\left(\sqrt{x^2+4}\right) dx$

36 $\int \frac{d}{dx}\left(\sqrt[3]{x^3-8}\right) dx$

37 $\int \frac{d}{dx}(\sin \sqrt[3]{x})\,dx$

38 $\int \frac{d}{dx}(\sqrt{\tan x})\,dx$

39 $\frac{d}{dx}\int (x^3\sqrt{x-4})\,dx$

40 $\frac{d}{dx}\int \left(x^4\sqrt[3]{x^2+9}\right) dx$

41 Show that $\int x^2\,dx \neq x\int x\,dx$.

42 Show that $\int (1+x)\,dx \neq 1 + \int x\,dx$.

Exer. 43–48: Evaluate the integral if a and b are constants.

43 $\int a^2\,dx$

44 $\int ab\,dx$

45 $\int (at+b)\,dt$

46 $\int \left(\frac{a}{b^2}t\right) dt$

47 $\int (a+b)\,du$

48 $\int (b-a^2)\,du$

Exer. 49–56: Solve the differential equation subject to the given conditions.

49 $f'(x) = 12x^2 - 6x + 1$; $f(1) = 5$

50 $f'(x) = 9x^2 + x - 8$; $f(-1) = 1$

51 $\frac{dy}{dx} = 4x^{1/2}$; $y = 21$ if $x = 4$

52 $\frac{dy}{dx} = 5x^{-1/3}$; $y = 70$ if $x = 27$

53 $f''(x) = 4x - 1$; $f'(2) = -2$; $f(1) = 3$

54 $f''(x) = 6x - 4$; $f'(2) = 5$; $f(2) = 4$

55 $\frac{d^2y}{dx^2} = 3\sin x - 4\cos x$; $y = 7$ and $y' = 2$ if $x = 0$

56 $\frac{d^2y}{dx^2} = 2\cos x - 5\sin x$; $y = 2 + 6\pi$ and $y' = 3$ if $x = \pi$

Exer. 57–58: If a point is moving on a coordinate line with the given acceleration $a(t)$ and initial conditions, find $s(t)$.

57 $a(t) = 2 - 6t$; $v(0) = -5$; $s(0) = 4$

58 $a(t) = 3t^2$; $v(0) = 20$; $s(0) = 5$

59 A projectile is fired vertically upward from ground level with a velocity of 1600 ft/sec. Disregarding air resistance, find

(a) its distance $s(t)$ above the ground at time t

(b) its maximum height

60 An object is dropped from a height of 1000 ft. Disregarding air resistance, find

(a) the distance it falls in t seconds

(b) its velocity at the end of 3 sec

(c) when it strikes the ground

61 A stone is thrown directly downward from a height of 96 ft with an initial velocity of 16 ft/sec. Find

(a) its distance above the ground after t seconds

(b) when it strikes the ground

(c) the velocity at which it strikes the ground

62 A gravitational constant for objects near the surface of the moon is 5.3 ft/sec^2.

(a) If an astronaut on the moon throws a stone directly upward with an initial velocity of 60 ft/sec, find the maximum altitude of the stone.

(b) If, after returning to earth, the astronaut throws the same stone directly upward with the same initial velocity, find the maximum altitude of the stone.

63 If a projectile is fired vertically upward from a height of s_0 feet above the ground with a velocity of v_0 ft/sec, prove that if air resistance is disregarded, its distance $s(t)$ above the ground after t seconds is given by $s(t) = -\tfrac{1}{2}gt^2 + v_0t + s_0$, where g is a gravitational constant.

64 A ball rolls down an inclined plane with an acceleration of 2 ft/sec^2.

(a) If the ball is given no initial velocity, how far will it roll in t seconds?

(b) What initial velocity must be given for the ball to roll 100 ft in 5 sec?

65 If an automobile starts from rest, what constant acceleration will enable it to travel 500 ft in 10 sec?

66 If a car is traveling at a speed of 60 mi/hr, what constant (negative) acceleration will enable it to stop in 9 sec?

67 A small country has natural gas reserves of 100 billion ft^3. If $A(t)$ denotes the total amount of natural gas consumed after t years, then dA/dt is the *rate of consumption*. If the rate of consumption is predicted to be $5 + 0.01t$ billion ft^3/yr, in approximately how many years will the country's natural gas reserves be depleted?

68 Refer to Exercise 67. Based on U.S. Department of Energy statistics, the rate of consumption of gasoline in the United States (in billions of gallons per year) is approximated by $dA/dt = 2.74 - 0.11t - 0.01t^2$, with $t = 0$ corresponding to the year 1980. Estimate the number of gallons of gasoline consumed in the United States between 1980 and 1984.

69 A sportswear manufacturer determines that the marginal cost in dollars of producing x warmup suits is given by $20 - 0.015x$. If the cost of producing one suit is \$25, find the cost function and the cost of producing 50 suits.

70 If the marginal cost function of a product is given by $2/x^{1/3}$ and if the cost of producing 8 units is \$20, find the cost function and the cost of producing 64 units.

c **Exer. 71–76: Use the commands of a computer algebra system (CAS) to find the derivative and the indefinite integral of the following functions.**

71 $f(x) = 2x^5 + x^4 + 9x^3 - 5x^2 + 4x + 10$

72 $f(x) = 3x^4 + 6x^2 + 8\sqrt{x}$

73 $g(x) = x^2e^{3x}\cos 4x$

74 $g(x) = x^4e^x\sin 2x$

75 $s(t) = \frac{t-5}{2t^3 + 3t^2 - 5t - 6}$

76 $s(t) = \frac{t^2+1}{2t^3 + 3t^2 - 5t - 6}$

77 (a) Show that each of the functions $\sin^2 x$, $-\cos^2 x$, and $-\tfrac{1}{2}\cos 2x$ are each antiderivatives of $2\sin x\cos x$.

(b) Reconcile the results of part (a) with the conclusion of Theorem (4.2).

Mathematicians and Their Times

GOTTFRIED WILHELM LEIBNIZ

THE THIRTY YEARS' WAR began as a religious struggle between German Protestants and Roman Catholics in 1618 and spread to a general European struggle for territory and political power. At the war's end, Germany was in ruins. Thousands of people had been killed and entire cities and towns had disappeared.

In this unhappy period of European history were also sown the seeds of modern rationalist thought. Such profound thinkers as Galileo, Newton, Descartes, Pascal, Bacon, Spinoza, Locke, and Leibniz, "the great teachers of the seventeenth century," as one historian called them, "disciplined the minds of men for impartial inquiry and . . . produced a passionate love of truth which has revolutionized all departments of knowledge."* Gottfried Wilhelm Leibniz was the most versatile genius of all, making notable contributions to logic, philosophy, law, history, geology, theology, physics, and mathematics while carrying out an active diplomatic career.

Born in Leipzig, Germany, on July 1, 1646, Leibniz showed an early interest in his studies. He mastered Latin and Greek as an essentially self-taught youth. At age 15, he entered the University of Leipzig, where he earned a philosophy degree at age 17. By age 20, he had completed a brilliant doctoral thesis. Leibniz entered the diplomatic service, first for the Elector of Mainz and later, for 40 years, for the Elector of Hanover. He died on November 14, 1716.

Leibniz formulated many plans to avoid a recurrence of the bloodshed prompted by earlier religious and political rivalries, seeking, unsuccessfully, to reconcile Catholicism and Protestantism. Leibniz went to Paris to try to persuade Louis XIV, the king of France, to turn his attention from attacks against the German states to seizing Egypt. Although Louis XIV chose not to attack Egypt, Leibniz spent four fruitful years

*W. E. Lecky, *History of the Rise and Influence of the Spirit of Rationalism in Europe*. London: Longmans, 1866.

in Paris, absorbing the latest scientific advances and training himself in mathematics. During this period, he conceived the principal features of calculus, developing a general method for the calculation of derivatives and integrals and discovering the fundamental theorem of calculus.

A bitter controversy arose concerning the "discovery" of calculus. Newton apparently made his own discoveries in 1666, but did not publish them until 1692. Leibniz independently reached the same results in 1676, publishing them in 1686. Some British mathematicians unfairly charged Leibniz with plagiarizing Newton's ideas; the resulting furor drove a wedge between the British and Continental intellectual communities that hampered the development of calculus.

4.2 CHANGE OF VARIABLES IN INDEFINITE INTEGRALS

The formulas for indefinite integrals in Table (4.4) are limited in scope, because we cannot use them directly to evaluate integrals such as

$$\int \sqrt{5x+7}\,dx \quad \text{or} \quad \int \cos 4x\,dx.$$

In this section, we shall develop a simple but powerful method for changing the variable of integration so that these integrals (and many others) can be evaluated by using the formulas in Table (4.4).

To justify this method, we shall apply Formula (i) of Theorem (4.5) to a *composite* function. We intend to consider several functions f, g, and F, so it will simplify our work if we state the formula in terms of a function h as follows:

$$\int \frac{d}{dx}(h(x))\,dx = h(x) + C$$

Suppose that F is an antiderivative of a function f and that g is a differentiable function such that $g(x)$ is in the domain of F for every x in some interval. If we let h denote the composite function $F \circ g$, then $h(x) = F(g(x))$ and hence

$$\int \frac{d}{dx}(F(g(x)))\,dx = F(g(x)) + C.$$

Applying the chain rule (2.26) to the integrand $(d/dx)(F(g(x)))$ and using the fact that $F' = f$, we obtain

$$\frac{d}{dx}(F(g(x))) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Substitution in the preceding indefinite integral gives us

$$(*) \qquad \int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

We can use differential notation to help remember this formula. We formally identify the expression dx with the increment Δx—that is, $dx = \Delta x$. We then introduce the variable $u = g(x)$ and note that Definition (2.34) gives us the statement:

$$\text{If} \quad u = g(x), \quad \text{then} \quad du = g'(x)\,dx$$

If we *formally substitute* u and du into $(*)$, we obtain

$$\int f(u)\,du = F(u) + C.$$

This equation has the same *form* as the integral in Definition (4.3); however, u represents a *function*, not an independent variable x, as before. The equation indicates that $g'(x)\,dx$ in $(*)$ may be regarded as the product of $g'(x)$ and dx. Since the variable x has been replaced by a new variable u, finding indefinite integrals in this way is referred to as a **change of variable**, or as the **method of substitution**. We may summarize our discussion as follows, where we assume that f and g have the properties described previously.

***Method of Substitution* 4.7**

If F is an antiderivative of f, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$ and $du = g'(x)\,dx$, then

$$\int f(u)\,du = F(u) + C.$$

After we have made the substitution $u = g(x)$ as indicated in (4.7), it may be necessary to insert a constant factor k into the integrand in order to arrive at the proper form $\int f(u)\,du$. We must then also multiply by $1/k$ to maintain equality, as illustrated in the next examples.

EXAMPLE ■ 1 Evaluate $\int \sqrt{5x+7}\,dx$.

SOLUTION We let $u = 5x + 7$ and calculate du:

$$u = 5x + 7, \qquad du = 5\,dx$$

Since du contains the factor 5, the integral is not in the proper form $\int f(u)\,du$ required by (4.7). However, we can *introduce* the factor 5 into the integrand, provided we also multiply by $\frac{1}{5}$. Doing so and using Theorem (4.6)(i) gives us

$$\begin{aligned}\int \sqrt{5x+7}\,dx &= \int \sqrt{5x+7}\left(\tfrac{1}{5}\right)5\,dx\\ &= \tfrac{1}{5}\int \sqrt{5x+7}\,5\,dx.\end{aligned}$$

We now substitute and use the power rule for integration:

$$\begin{aligned}\int \sqrt{5x+7}\,dx &= \tfrac{1}{5}\int \sqrt{u}\,du\\ &= \tfrac{1}{5}\int u^{1/2}\,du\\ &= \tfrac{1}{5}\frac{u^{3/2}}{\frac{3}{2}} + C\\ &= \tfrac{2}{15}u^{3/2} + C\\ &= \tfrac{2}{15}(5x+7)^{3/2} + C\end{aligned}$$

In the future, after inserting a factor k into an integrand, as in Example 1, we shall simply multiply the integral by $1/k$, skipping the intermediate steps of first writing $(1/k)k$ and then bringing $1/k$ *outside*—that is, to the left of—the integral sign.

EXAMPLE ■ 2 Evaluate $\int \cos 4x\,dx$.

SOLUTION We make the substitution

$$u = 4x, \qquad du = 4\,dx.$$

Since du contains the factor 4, we adjust the integrand by multiplying by 4 and compensate by multiplying the integral by $\frac{1}{4}$ before substituting:

$$\begin{aligned}\int \cos 4x\,dx &= \tfrac{1}{4}\int (\cos 4x)4\,dx\\ &= \tfrac{1}{4}\int \cos u\,du\\ &= \tfrac{1}{4}\sin u + C\\ &= \tfrac{1}{4}\sin 4x + C\end{aligned}$$

It is not always easy to decide what substitution $u = g(x)$ is needed to transform an indefinite integral into a form that can be readily evaluated. It may be necessary to try several different possibilities before finding a suitable substitution. In most cases, *no* substitution will simplify the integrand properly. The following guidelines may be helpful.

Guidelines for Changing Variables in Indefinite Integrals 4.8

1. Decide on a reasonable substitution $u = g(x)$.
2. Calculate $du = g'(x)\,dx$.
3. Using guidelines (1) and (2), try to transform the integral into a form that involves only the variable u. If necessary, introduce a *constant* factor k into the integrand and compensate by multiplying the integral by $1/k$. If any part of the resulting integrand contains the variable x, use a different substitution in guideline (1).
4. Evaluate the integral obtained in guideline (3), obtaining an antiderivative involving u.
5. Replace u in the antiderivative obtained in guideline (4) by $g(x)$. The final result should contain only the variable x.

The next examples illustrate the use of the guidelines.

EXAMPLE 3 Evaluate $\int (2x^3+1)^7 x^2\,dx$.

SOLUTION If an integrand involves an expression raised to a power, such as $(2x^3+1)^7$, we substitute u for the expression. Thus we let

$$u = 2x^3 + 1, \qquad du = 6x^2\,dx.$$

Comparing $du = 6x^2\,dx$ with $x^2\,dx$ in the integral suggests that we introduce the factor 6 into the integrand. Doing so and compensating by multiplying the integral by $\frac{1}{6}$, we obtain the following:

$$\begin{aligned}\int (2x^3+1)^7x^2\,dx &= \tfrac{1}{6}\int (2x^3+1)^7 6x^2\,dx\\ &= \tfrac{1}{6}\int u^7\,du\\ &= \frac{1}{6}\left(\frac{u^8}{8}\right) + C\\ &= \tfrac{1}{48}(2x^3+1)^8 + C\end{aligned}$$

A substitution in an indefinite integral can sometimes be made in several different ways. To illustrate, another method for evaluating the integral in Example 3 is to consider

$$u = 2x^3 + 1, \qquad du = 6x^2\,dx, \qquad \tfrac{1}{6}\,du = x^2\,dx.$$

We then substitute $\frac{1}{6}\,du$ for $x^2\,dx$,

$$\int (2x^3+1)^7x^2\,dx = \int u^7\tfrac{1}{6}\,du = \tfrac{1}{6}\int u^7\,du,$$

and integrate as before.

EXAMPLE 4 Evaluate $\int x\sqrt[3]{7-6x^2}\,dx$.

SOLUTION Note that the integrand contains the term $x\,dx$. If the factor x were missing or if x were raised to a higher power, the problem would be more complicated. For integrands that involve a radical, we often substitute for the expression under the radical sign. Thus we let

$$u = 7 - 6x^2, \qquad du = -12x\,dx.$$

Next, we introduce the factor -12 into the integrand, compensate by multiplying the integral by $-\frac{1}{12}$, and proceed as follows:

$$\begin{aligned}\int x\sqrt[3]{7-6x^2}\,dx &= -\tfrac{1}{12}\int \sqrt[3]{7-6x^2}(-12)x\,dx\\ &= -\tfrac{1}{12}\int \sqrt[3]{u}\,du = -\tfrac{1}{12}\int u^{1/3}\,du\\ &= -\frac{1}{12}\left(\frac{u^{4/3}}{4/3}\right) + C = -\frac{1}{16}u^{4/3} + C\\ &= -\tfrac{1}{16}(7-6x^2)^{4/3} + C\end{aligned}$$

We could also have written

$$u = 7 - 6x^2, \qquad du = -12x\,dx, \qquad -\tfrac{1}{12}\,du = x\,dx$$

and substituted directly for $x\,dx$. Thus,

$$\int \sqrt[3]{7-6x^2}\,x\,dx = \int \sqrt[3]{u}\left(-\tfrac{1}{12}\right)du = -\tfrac{1}{12}\int \sqrt[3]{u}\,du.$$

The remainder of the solution would proceed exactly as before.

EXAMPLE 5 Evaluate $\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx$.

SOLUTION Let

$$u = x^3 - 3x + 1, \qquad du = (3x^2-3)\,dx = 3(x^2-1)\,dx$$

and proceed as follows:

$$\begin{aligned}\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx &= \frac{1}{3}\int \frac{3(x^2-1)}{(x^3-3x+1)^6}\,dx\\ &= \frac{1}{3}\int \frac{1}{u^6}\,du = \frac{1}{3}\int u^{-6}\,du\\ &= \frac{1}{3}\left(\frac{u^{-5}}{-5}\right) + C = -\frac{1}{15}\left(\frac{1}{u^5}\right) + C\\ &= -\frac{1}{15}\frac{1}{(x^3-3x+1)^5} + C\end{aligned}$$

If the variable of integration is different from x, we can make use of Guidelines (4.8), with an appropriate change of notation. In the next example, t is the original variable of integration; our substitution takes the form $u = g(t)$ for an appropriately chosen function g.

EXAMPLE 6 Evaluate $\int \cos^3 5t \sin 5t\, dt$.

SOLUTION The form of the integrand suggests that we use the power rule (2) in (4.4) with $\int u^3\, du = \frac{1}{4}u^4 + C$. Thus we let

$$u = g(t) = \cos 5t, \qquad du = -5 \sin 5t\, dt.$$

The form of du indicates that we should introduce the factor -5 into the integrand, multiply the integral by $-\frac{1}{5}$, and then integrate as follows:

$$\begin{aligned}\int \cos^3 5t \sin 5t\, dt &= -\tfrac{1}{5}\int \cos^3 5t(-5\sin 5t)\, dt \\ &= -\tfrac{1}{5}\int u^3\, du \\ &= -\frac{1}{5}\left(\frac{u^4}{4}\right) + C \\ &= -\tfrac{1}{20}\cos^4 5t + C\end{aligned}$$

The method of substitution or change of variable is also quite useful in solving differential equations. The next example illustrates an application of differential equations in which we make use of a change of variable.

EXAMPLE 7 Studies have shown that the rate at which students learn new vocabulary words in a foreign language decreases as the size of the known vocabulary increases. If $W(t)$ is the number of words known after t days and the rate of change of W is modeled by the differential equation

$$W'(t) = \frac{8200}{W(t)} \quad \text{for} \quad 0 \le t \le 365,$$

(a) find W as an explicit function of t, if the student knows 400 words at time $t = 0$

(b) find the number of words known after 1 day and 2 days

SOLUTION

(a) From the differential equation, we have

$$W(t)W'(t) = 8200 \quad \text{for} \quad 0 \le t \le 365.$$

Since the expressions on each side of the equation are identical on an interval [0, 365], the indefinite integrals of the functions they represent will be identical. Thus, we have

$$\int W(t)W'(t)\, dt = \int 8200\, dt.$$

We make the substitution

$$u = W(t), \qquad du = W'(t)\, dt$$

on the left-hand side of this equation, obtaining

$$\int u\, du = \int 8200\, dt$$

so that

$$\frac{u^2}{2} = 8200t + C.$$

Changing back to our original variables, we have

$$\frac{[W(t)]^2}{2} = 8200t + C.$$

Evaluating at $t = 0$ (with $W(0) = 400$) yields

$$\frac{400^2}{2} = 8200(0) + C = 0 + C = C,$$

so $C = 80{,}000$. Hence,

$$\frac{[W(t)]^2}{2} = 8200t + 80{,}000.$$

This result gives us

$$[W(t)]^2 = 16{,}400t + 160{,}000$$

and

$$W(t) = \sqrt{16{,}400t + 160{,}000} = 20\sqrt{41t + 400}.$$

Thus the number of words $W(t)$ known after t days is $20\sqrt{41t + 400}$.

(b) After 1 day of additional study, the number of words the student knows is given by

$$W(1) = 20\sqrt{41 + 400} = 20\sqrt{441} = (20)(21) = 420.$$

After 2 days, the number of words is

$$W(2) = 20\sqrt{482} \approx 439.$$

In this model of learning, the student masters an additional 20 words on the first day, but only 19 more words on the second day.

EXERCISES 4.2

Exer. 1–8: Evaluate the integral using the given substitution, and express the answer in terms of x.

1 $\int x(2x^2 + 3)^{10}\, dx; \quad u = 2x^2 + 3$

2 $\int \frac{x}{(x^2 + 5)^3}\, dx; \quad u = x^2 + 5$

3 $\int x^2 \sqrt[3]{3x^3 + 7}\, dx; \quad u = 3x^3 + 7$

4 $\int \frac{5x}{\sqrt{x^2 - 3}}\, dx; \quad u = x^2 - 3$

5 $\int \frac{(1 + \sqrt{x})^3}{\sqrt{x}}\, dx; \quad u = 1 + \sqrt{x}$

6 $\int \frac{1}{(5x-4)^{10}}\,dx;\quad u = 5x - 4$

7 $\int \sqrt{x}\cos\sqrt{x^3}\,dx;\quad u = x^{3/2}$

8 $\int \tan x \sec^2 x\,dx;\quad u = \tan x$

Exer. 9 – 48: Evaluate the integral.

9 $\int \sqrt{3x-2}\,dx$ 10 $\int \sqrt[4]{2x+5}\,dx$

11 $\int \sqrt[3]{8t+5}\,dt$ 12 $\int \frac{1}{\sqrt{4-5t}}\,dt$

13 $\int (3z+1)^4\,dz$ 14 $\int (2z^2-3)^5 z\,dz$

15 $\int v^2\sqrt{v^3-1}\,dv$ 16 $\int v\sqrt{9-v^2}\,dv$

17 $\int \frac{x}{\sqrt[3]{1-2x^2}}\,dx$ 18 $\int (3-x^4)^3 x^3\,dx$

19 $\int (s^2+1)^2\,ds$ 20 $\int (3-s^3)^2 s\,ds$

21 $\int \frac{(\sqrt{x}+3)^4}{\sqrt{x}}\,dx$ 22 $\int \left(1+\frac{1}{x}\right)^{-3}\left(\frac{1}{x^2}\right)dx$

23 $\int \frac{t-2}{(t^2-4t+3)^3}\,dt$ 24 $\int \frac{t^2+t}{(4-3t^2-2t^3)^4}\,dt$

25 $\int 3\sin 4x\,dx$ 26 $\int 4\cos\tfrac{1}{2}x\,dx$

27 $\int \cos(4x-3)\,dx$ 28 $\int \sin(1+6x)\,dx$

29 $\int v\sin(v^2)\,dv$ 30 $\int \frac{\cos\sqrt[3]{v}}{\sqrt[3]{v^2}}\,dv$

31 $\int \cos 3x\sqrt[3]{\sin 3x}\,dx$ 32 $\int \frac{\sin 2x}{\sqrt{1-\cos 2x}}\,dx$

33 $\int (\sin x + \cos x)^2\,dx$ (*Hint:* $\sin 2\theta = 2\sin\theta\cos\theta$.)

34 $\int \frac{\sin 4x}{\cos 2x}\,dx$ (*Hint:* $\sin 2\theta = 2\sin\theta\cos\theta$.)

35 $\int \sin x(1+\cos x)^2\,dx$ 36 $\int \sin^3 x\cos x\,dx$

37 $\int \frac{\sin x}{\cos^4 x}\,dx$ 38 $\int \sin 2x\sec^5 2x\,dx$

39 $\int \frac{\cos t}{(1-\sin t)^2}\,dt$ 40 $\int (2+5\cos t)^3\sin t\,dt$

41 $\int \sec^2(3x-4)\,dx$ 42 $\int \frac{\csc 2x}{\sin 2x}\,dx$

43 $\int \sec^2 3x\tan 3x\,dx$ 44 $\int \frac{1}{\tan 4x\sin 4x}\,dx$

45 $\int \frac{1}{\sin^2 5x}\,dx$ 46 $\int \frac{x}{\cos^2(x^2)}\,dx$

47 $\int x\cot(x^2)\csc(x^2)\,dx$ 48 $\int \sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)dx$

Exer. 49 – 52: Solve the differential equation subject to the given conditions.

49 $f'(x) = \sqrt[3]{3x+2};\quad f(2) = 9$

50 $\frac{dy}{dx} = x\sqrt{x^2+5};\quad y = 12$ if $x = 2$

51 $f''(x) = 16\cos 2x - 3\sin x;\quad f(0) = -2;\quad f'(0) = 4$

52 $f''(x) = 4\sin 2x + 16\cos 4x;\quad f(0) = 6;\quad f'(0) = 1$

Exer. 53 – 56: Evaluate the integral by (a) the method of substitution and (b) expanding the integrand. In what way do the constants of integration differ?

53 $\int (x+4)^2\,dx$ 54 $\int (x^2+4)^2 x\,dx$

55 $\int \frac{(\sqrt{x}+3)^2}{\sqrt{x}}\,dx$ 56 $\int \left(1+\frac{1}{x}\right)^2\frac{1}{x^2}\,dx$

57 A charged particle is moving on a coordinate line in a magnetic field such that its velocity (in centimeters per second) at time t is given by $v(t) = \frac{1}{2}\sin(3t - \frac{1}{4}\pi)$. Show that the motion is simple harmonic (see page 321).

58 The acceleration of a particle that is moving on a coordinate line is given by $a(t) = k\cos(\omega t + \phi)$ for constants k, ω, and ϕ and time t (in seconds). Show that the motion is simple harmonic (see page 321).

59 A reservoir supplies water to a community. In summer, the demand A for water (in cubic feet per day) changes according to the formula

$$dA/dt = 4000 + 2000\sin(\tfrac{1}{90}\pi t)$$

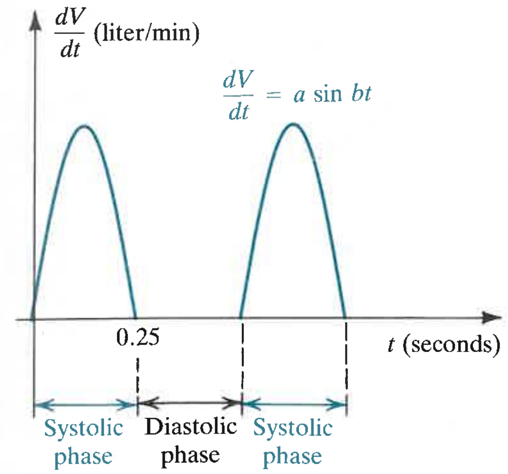
for time t (in days), with $t = 0$ corresponding to the beginning of summer. Estimate the total amount of water consumption during 90 days of summer.

60 The pumping action of the heart consists of the systolic phase, in which blood rushes from the left ventricle into the aorta, and the diastolic phase, during which the heart muscle relaxes. The graph shown in the figure on the following page is sometimes used to model one complete cycle of the process. For a particular individual, the systolic phase lasts $\frac{1}{4}$ sec and has a maximum flow rate dV/dt of 8 L/min, where V is the volume of blood in the heart at time t.

(a) Show that $dV/dt = 8\sin(240\pi t)$ L/min.

(b) Estimate the total amount of blood pumped into the aorta during a systolic phase.

Exercise 60



61 The rhythmic process of breathing consists of alternating periods of inhaling and exhaling. For an adult, one complete cycle normally takes place every 5 sec. If V denotes the volume of air in the lungs at time t, then dV/dt is the flow rate.

(a) If the maximum flow rate is 0.6 L/sec, find a formula $dV/dt = a\sin bt$ that fits the given information.

(b) Use part (a) to estimate the amount of air inhaled during one cycle.

62 Many animal populations fluctuate over 10-yr cycles. Suppose that the rate of growth of a rabbit population is given by $dN/dt = 1000\cos(\frac{1}{5}\pi t)$ rabbits/yr, where N denotes the number in the population at time t (in years) and $t = 0$ corresponds to the beginning of a cycle. If the population after 5 yr is estimated to be 3000 rabbits, find a formula for N at time t and estimate the maximum population.

63 Show, by evaluating in three different ways, that

$$\int \sin x\cos x\,dx = \tfrac{1}{2}\sin^2 x + C$$
$$= -\tfrac{1}{2}\cos^2 x + D$$
$$= -\tfrac{1}{4}\cos 2x + E.$$

How can all three answers be correct?

4.3 SUMMATION NOTATION AND AREA

In this section, we lay the foundation for the definition of the *definite integral.* At the outset, it is virtually impossible to see any connection between definite integrals and indefinite integrals. In Section 4.6, however, we show that there is a very close relationship: *Indefinite integrals can be used to evaluate definite integrals.*

In our development of the definite integral, we shall employ sums of many numbers. To express such sums compactly, it is convenient to use **summation notation.** Given a collection of numbers $\{a_1, a_2, \ldots, a_n\}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sum as follows.

Summation Notation 4.9

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

The Greek capital letter Σ (sigma) indicates a sum, and a_k represents the kth term of the sum. The letter k is the **index of summation,** or the **summation variable,** and assumes successive integer values. The integers 1 and n indicate the extreme values of the summation variable.

EXAMPLE ■ 1 Evaluate $\sum_{k=1}^{4} k^2(k-3)$.

SOLUTION Comparing the sum with (4.9), we have $a_k = k^2(k-3)$ and $n = 4$. To find the sum, we substitute 1, 2, 3, and 4 for k and add the resulting terms. Thus,

$$\begin{aligned}\sum_{k=1}^{4} k^2(k-3) &= 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3)\\ &= (-2) + (-4) + 0 + 16 = 10.\end{aligned}$$

Letters other than k can be used for the summation variable. To illustrate,

$$\sum_{k=1}^{4} k^2(k-3) = \sum_{i=1}^{4} i^2(i-3) = \sum_{j=1}^{4} j^2(j-3) = 10.$$

If $a_k = c$ for every k, then

$$\sum_{k=1}^{2} a_k = a_1 + a_2 = c + c = 2c = \sum_{k=1}^{2} c,$$

$$\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3 = c + c + c = 3c = \sum_{k=1}^{3} c.$$

In general, the following result is true for every positive integer n.

Theorem 4.10

$$\sum_{k=1}^{n} c = nc$$

The domain of the summation variable does not have to begin at 1. For example,

$$\sum_{k=4}^{8} a_k = a_4 + a_5 + a_6 + a_7 + a_8.$$

EXAMPLE ■ 2 Evaluate $\sum_{k=0}^{3} \frac{2^k}{(k+1)}$.

SOLUTION

$$\begin{aligned}\sum_{k=0}^{3} \frac{2^k}{(k+1)} &= \frac{2^0}{(0+1)} + \frac{2^1}{(1+1)} + \frac{2^2}{(2+1)} + \frac{2^3}{(3+1)}\\ &= 1 + 1 + \tfrac{4}{3} + 2 = \tfrac{16}{3}\end{aligned}$$

The next theorem states some elementary properties of summation.

Theorem 4.11

If n is any positive integer and $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are sets of real numbers, then

(i) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(ii) $\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$, for every real number c

(iii) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

PROOF To prove (i), we begin with

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n).$$

Rearranging terms on the right, we obtain

$$\begin{aligned}\sum_{k=1}^{n} (a_k + b_k) &= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)\\ &= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.\end{aligned}$$

For (ii),

$$\begin{aligned}\sum_{k=1}^{n} (ca_k) &= ca_1 + ca_2 + ca_3 + \cdots + ca_n\\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\left(\sum_{k=1}^{n} a_k\right).\end{aligned}$$

To prove (iii), we write $a_k - b_k = a_k + (-1)b_k$ and use (i) and (ii). ■

The formulas in the following theorem will be useful later in this section. They may be proved by mathematical induction.

Theorem 4.12

(i) $\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE ■ 3 Evaluate $\sum_{k=1}^{100} k$ and $\sum_{k=1}^{20} k^2$.

SOLUTION Using (i) and (ii) of Theorem (4.12), we obtain

$$\sum_{k=1}^{100} k = 1 + 2 + \cdots + 100 = \frac{100(101)}{2} = 5050$$

and

$$\sum_{k=1}^{20} k^2 = 1^2 + 2^2 + \cdots + 20^2 = \frac{20(21)(41)}{6} = 2870.$$

EXAMPLE ■ 4 Express $\sum_{k=1}^{n} (k^2 - 4k + 3)$ in terms of n.

SOLUTION We use Theorems (4.11), (4.12), and (4.10):

$$\begin{aligned}\sum_{k=1}^{n} (k^2 - 4k + 3) &= \sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3\\ &= \frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + 3n\\ &= \tfrac{1}{3}n^3 - \tfrac{3}{2}n^2 + \tfrac{7}{6}n\end{aligned}$$

We will be working quite extensively with sums of the form $\sum_{k=1}^{n} f(k)$ in our examination of areas of planar regions and in our study of the definite integral in this chapter. The definition of the definite integral (to be given in Section 4.4) is closely related to the areas of certain regions in a coordinate plane. We can easily calculate the area if the region is bounded by lines. For example, the area of a rectangle is the product of its length and width. The area of a triangle is one-half the product of an altitude and the corresponding base. The area of any polygon can be found by subdividing it into triangles.

In order to find areas of regions whose boundaries involve graphs of functions, however, we utilize a limiting process and then use methods of calculus. In particular, let us consider a region R in a coordinate plane, bounded by the vertical lines $x = a$ and $x = b$, by the x-axis, and by the graph of a function f that is continuous and nonnegative on the closed interval $[a, b]$. A region of this type is illustrated in Figure 4.2. Since $f(x) \geq 0$ for every x in $[a, b]$, no part of the graph lies below the x-axis. For convenience, we shall refer to R as **the region under the graph of f from a to b**. We wish to define the area A of R.

To arrive at a satisfactory definition of A, we shall consider many rectangles of equal width such that each rectangle lies completely under the graph of f and intersects the graph in at least one point, as illustrated in Figure 4.3. The boundary of the region formed by the totality of these

Figure 4.2 Region under the graph of f

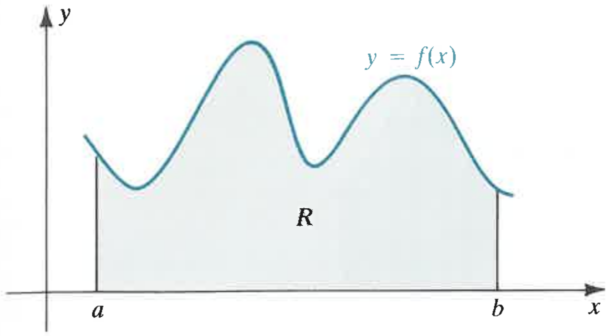
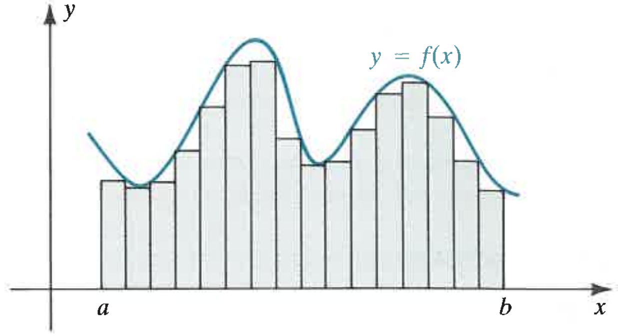


Figure 4.3 An inscribed rectangular polygon



rectangles is called an **inscribed rectangular polygon**. We shall use the following notation:

$$A_{IP} = \text{area of an inscribed rectangular polygon}$$

If the width of the rectangles in Figure 4.3 is small, then it appears that

$$A_{IP} \approx A.$$

This result suggests that we let the width of the rectangles approach zero and define A as a limiting value of the areas A_{IP} of the corresponding inscribed rectangular polygons. The notation discussed next will allow us to carry out this procedure rigorously.

If n is any positive integer, we divide the interval $[a, b]$ into n subintervals, all having the same length $\Delta x = (b - a)/n$. We choose the numbers $x_0, x_1, x_2, \ldots, x_n$, and let $a = x_0$, $b = x_n$, and

$$x_k - x_{k-1} = \frac{b-a}{n} = \Delta x$$

for $k = 1, 2, \ldots, n$, as indicated in Figure 4.4 on the following page. Note that

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \ldots$$
$$x_k = a + k\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = b.$$

The function f is continuous on each subinterval $[x_{k-1}, x_k]$, and hence, by the extreme value theorem (3.3), f takes on a minimum value at some number u_k in $[x_{k-1}, x_k]$. For each k, let us construct a rectangle of width $\Delta x = x_k - x_{k-1}$ and height equal to the minimum distance $f(u_k)$ from the x-axis to the graph of f (see Figure 4.4). The area of the kth rectangle is $f(u_k)\Delta x$. The area A_{IP} of the resulting inscribed rectangular polygon is the sum of the areas of the n rectangles—that is,

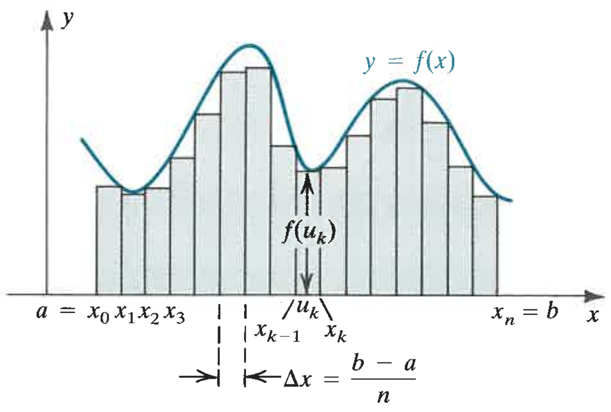
$$A_{IP} = f(u_1)\Delta x + f(u_2)\Delta x + \cdots + f(u_n)\Delta x.$$

Using summation notation, we may write

$$A_{IP} = \sum_{k=1}^{n} f(u_k)\Delta x,$$

where $f(u_k)$ is the minimum value of f on $[x_{k-1}, x_k]$.

Figure 4.4



If n is very large, or, equivalently, if Δx is very small, then the sum A_{IP} of the rectangular areas should approximate the area of the region R. Intuitively, we know that if there exists a number A such that $\sum_{k=1}^{n} f(u_k)\Delta x$ gets closer to A as Δx gets closer to 0 (but $\Delta x \neq 0$), we can call A the **area** of R and write

$$A = \lim_{\Delta x \to 0} A_{\text{IP}} = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(u_k)\Delta x.$$

The meaning of this *limit of sums* is not the same as that of the limit of a function, introduced in Chapter 1. To eliminate the word *closer* and arrive at a satisfactory definition of A, let us take a slightly different point of view. If A denotes the area of the region R, then the difference

$$A - \sum_{k=1}^{n} f(u_k)\Delta x$$

is the area of the portion in Figure 4.4 that lies *under* the graph of f and *over* the inscribed rectangular polygon. This number may be regarded as the error in using the area of the inscribed rectangular polygon to approximate A. We should be able to make this error as small as desired by choosing the width Δx of the rectangles sufficiently small. This procedure is the motivation for the following definition of the area A of R. The notation is the same as that used in the preceding discussion.

Definition 4.13

Let f be continuous and nonnegative on $[a, b]$. Let A be a real number, and let $f(u_k)$ be the minimum value of f on $[x_{k-1}, x_k]$. The notation

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(u_k)\Delta x$$

means that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < \Delta x < \delta$, then

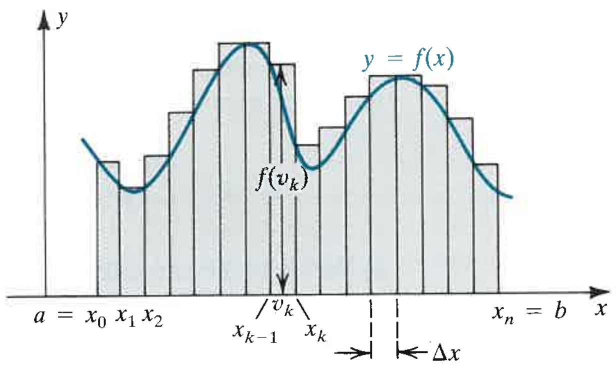
$$A - \sum_{k=1}^{n} f(u_k)\Delta x < \epsilon.$$

If A is the indicated limit and we let $\epsilon = 10^{-9}$, then Definition (4.13) states that by using rectangles of sufficiently small width Δx, we can make the difference between A and the area of the inscribed polygon less than one-billionth of a square unit. Similarly, if $\epsilon = 10^{-12}$, we can make this difference less than one-trillionth of a square unit. In general, the difference can be made less than *any* preassigned ϵ.

If f is continuous on $[a, b]$, it is shown in more advanced texts that a number A satisfying Definition (4.13) actually exists. We shall call A **the area under the graph of f from a to b.**

The area A may also be obtained by means of **circumscribed rectangular polygons** of the type illustrated in Figure 4.5. In this case, we select the number v_k in each interval $[x_{k-1}, x_k]$ such that $f(v_k)$ is the *maximum* value of f on $[x_{k-1}, x_k]$.

Figure 4.5 A circumscribed rectangular polygon



Let

$$A_{\text{CP}} = \text{area of a circumscribed rectangular polygon.}$$

Using summation notation, we have

$$A_{\text{CP}} = \sum_{k=1}^{n} f(v_k)\Delta x,$$

where $f(v_k)$ is the maximum value of f on $[x_{k-1}, x_k]$. Note that

$$\sum_{k=1}^{n} f(u_k)\Delta x \le A \le \sum_{k=1}^{n} f(v_k)\Delta x.$$

The limit of A_{CP} as $\Delta x \to 0$ is defined as in (4.13). The only change is that we use

$$\sum_{k=1}^{n} f(v_k)\Delta x - A < \epsilon,$$

since we want this difference to be nonnegative. It can be proved that the same number A is obtained using either inscribed or circumscribed rectangles.

The next example illustrates how close the areas of the inscribed and circumscribed rectangles become if we use a small value for Δx.

EXAMPLE ■ 5 Let $f(x) = \sqrt{x}$, and let R be the region under the graph of f from 1 to 5. Approximate the area A of R using

(a) an inscribed rectangular polygon with $\Delta x = 0.1$

(b) a circumscribed rectangular polygon with $\Delta x = 0.1$

SOLUTION With $\Delta x = 0.1 = 1/10$, the interval $[1, 5]$ is divided into 40 subintervals. Since $x_0 = 1$ and $x_n = 5$ with $n = 40$, we have $x_k = 1 + k\,\Delta x = 1 + k/10$.

(a) Since f is increasing on the interval $[1, 5]$, we obtain inscribed rectangles by selecting $u_k = x_{k-1}$, the left-hand endpoint of each subinterval $[x_{k-1}, x_k]$. Thus,

$$u_k = 1 + \frac{k-1}{10} \quad \text{and} \quad f(u_k) = \sqrt{1 + \frac{k-1}{10}}.$$

Using a computational device to sum the 40 terms, we find that the inscribed rectangular polygon has area

$$A_{\text{IP}} = \sum_{k=1}^{40} \sqrt{1 + \frac{k-1}{10}}\left(\frac{1}{10}\right) \approx 6.72485958283.$$

(b) We obtain circumscribed rectangles over $[1, 5]$ by selecting $u_k = x_k$, the right-hand endpoint of each subinterval $[x_{k-1}, x_k]$. Hence, the area of the circumscribed rectangular polygon is

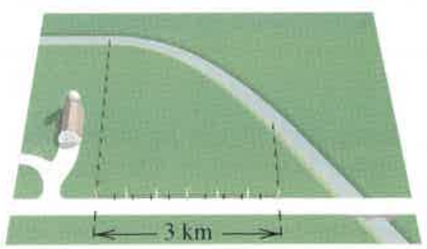
$$A_{\text{CP}} = \sum_{k=1}^{40} \sqrt{1 + \frac{k}{10}}\left(\frac{1}{10}\right) \approx 6.84846638058.$$

Thus, we can conclude that the area A satisfies

$$6.72485958283 < A < 6.84846638058.$$

For larger values of Δx, there are fewer rectangles, and we may be able to compute the areas of the inscribed and circumscribed polygons by hand. The next example illustrates this approach and also shows that there may be a considerable gap between the numbers A_{IP} and A_{CP}.

Figure 4.6



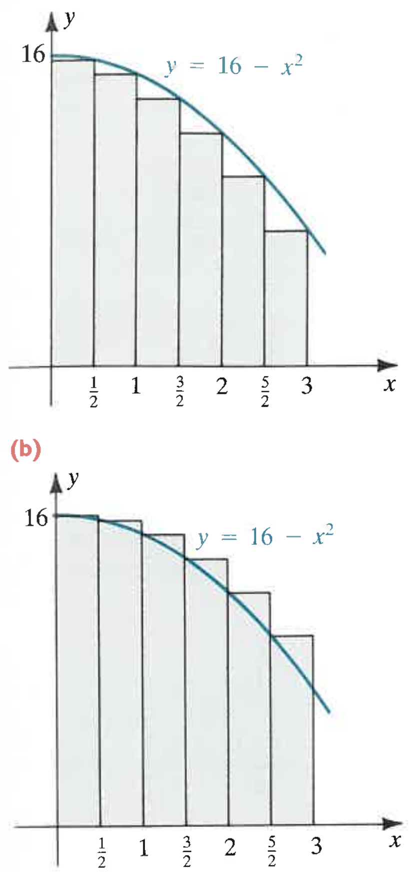
EXAMPLE ■ 6 One side of a farmer's field is bordered by a straight stretch of highway. The opposite side is bordered by a river whose path traces a curve that is modeled by the function $f(x) = 16 - x^2$. The farmer measures the side of the field along the highway, placing markers at every $\frac{1}{2}$ km for a total of 3 km, as shown in Figure 4.6. Approximate the area A of the field using

(a) an inscribed rectangular polygon with $\Delta x = \frac{1}{2}$

(b) a circumscribed rectangular polygon with $\Delta x = \frac{1}{2}$

Figure 4.7

(a) (b)



SOLUTION

(a) We model the farmer's field by graphing the function $f(x) = 16 - x^2$ and considering the area of the region under the graph from 0 to 3 on the x-axis, which is the side of the field measured in units of $\frac{1}{2}$ km. The graph of f and the inscribed rectangular polygon with $\Delta x = \frac{1}{2}$ are sketched in Figure 4.7(a) (with different scales on the x- and y-axes). Note that f is decreasing on $[0, 3]$, and hence the minimum value $f(u_k)$ on the kth subinterval occurs at the right-hand endpoint of the subinterval. Since there are six rectangles to consider, the formula for A_{IP} is

$$\begin{aligned} A_{\text{IP}} &= \sum_{k=1}^{6} f(u_k)\Delta x \\ &= f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2} \\ &= \tfrac{63}{4}\cdot\tfrac{1}{2} + 15\cdot\tfrac{1}{2} + \tfrac{55}{4}\cdot\tfrac{1}{2} + 12\cdot\tfrac{1}{2} + \tfrac{39}{4}\cdot\tfrac{1}{2} + 7\cdot\tfrac{1}{2} \\ &= \tfrac{293}{8} = 36.625. \end{aligned}$$

Using inscribed rectangles, we find that the area of the field is approximately 36.6 km^2.

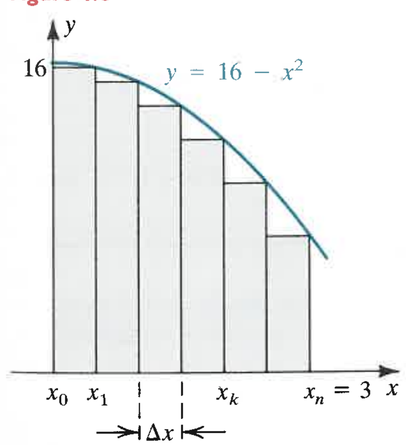
(b) The graph of f and the circumscribed rectangular polygon are sketched in Figure 4.7(b). Since f is decreasing on $[0, 3]$, the maximum value $f(v_k)$ occurs at the left-hand endpoint of the kth subinterval. Hence,

$$\begin{aligned} A_{\text{CP}} &= \sum_{k=1}^{6} f(v_k)\Delta x \\ &= f(0)\cdot\tfrac{1}{2} + f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} \\ &= 16\cdot\tfrac{1}{2} + \tfrac{63}{4}\cdot\tfrac{1}{2} + 15\cdot\tfrac{1}{2} + \tfrac{55}{4}\cdot\tfrac{1}{2} + 12\cdot\tfrac{1}{2} + \tfrac{39}{4}\cdot\tfrac{1}{2} \\ &= \tfrac{329}{8} = 41.125. \end{aligned}$$

Using circumscribed rectangles, we find that the area of the field is approximately 41.1 km^2.

It follows that $36.625 < A < 41.125$. In the next example, we prove that $A = 39$.

Figure 4.8



EXAMPLE ■ 7 Referring to Example 6, determine the area of the farmer's field, which is the area of the region under the graph of f from 0 to 3.

SOLUTION The graph of $f(x) = 16 - x^2$ and the inscribed rectangular polygon in Figure 4.7(a) is resketched in Figure 4.8. If the interval $[0, 3]$ is divided into n equal subintervals, then the length Δx of each subinterval is $3/n$. Employing the notation used in Figure 4.4, with $a = 0$ and $b = 3$, we have

$$x_0 = 0,\quad x_1 = \Delta x,\quad x_2 = 2\Delta x,\quad \ldots,\quad x_k = k\Delta x,\quad \ldots,\quad x_n = n\Delta x = 3.$$

Since $\Delta x = 3/n$,

$$x_k = k\Delta x = k\frac{3}{n} = \frac{3k}{n}.$$

Since f is decreasing on $[0, 3]$, the number u_k in $[x_{k-1}, x_k]$ at which f takes on its minimum value is always the right-hand endpoint x_k of the subinterval; that is, $u_k = x_k = 3k/n$. Thus,

$$f(u_k) = f\left(\frac{3k}{n}\right) = 16 - \left(\frac{3k}{n}\right)^2 = 16 - \frac{9k^2}{n^2},$$

and the summation in Definition (4.13) is

$$\sum_{k=1}^{n} f(u_k)\Delta x = \sum_{k=1}^{n}\left[\left(16 - \frac{9k^2}{n^2}\right)\cdot\frac{3}{n}\right]$$
$$= \frac{3}{n}\sum_{k=1}^{n}\left(16 - \frac{9k^2}{n^2}\right),$$

where the last equality follows from (ii) of Theorem (4.11). (Note that $3/n$ does not contain the summation variable k.) We next use Theorems (4.11), (4.10), and (4.12) to obtain

$$\sum_{k=1}^{n} f(u_k)\Delta x = \frac{3}{n}\left(\sum_{k=1}^{n} 16 - \frac{9}{n^2}\sum_{k=1}^{n} k^2\right)$$
$$= \frac{3}{n}\left[n\cdot 16 - \frac{9}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right]$$
$$= 48 - \frac{9}{2}\,\frac{(n+1)(2n+1)}{n^2}.$$

To find the area of the region, we let Δx approach 0. Since $\Delta x = 3/n$, we can accomplish this by letting n increase without bound. Although our discussion of limits involving infinity in Section 1.4 was concerned with a real variable x, a similar discussion can be given if the variable is an integer n. Assuming that it is and that we can replace $\Delta x \to 0$ by $n \to \infty$, we obtain

$$\lim_{\Delta x\to 0}\sum_{k=1}^{n} f(u_k)\Delta x = \lim_{n\to\infty}\left[48 - \frac{9}{2}\,\frac{(n+1)(2n+1)}{n^2}\right]$$
$$= 48 - \tfrac{9}{2}\cdot 2 = 39.$$

Thus the area of the region under the graph of f from 0 to 3 is 39, which means that the area of the farmer's field is 39 km^2.

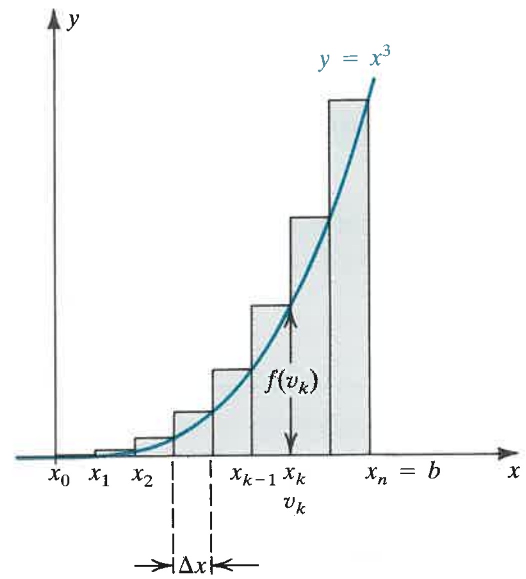
The area of a region under the graph of f may also be found by using circumscribed rectangular polygons. In this case, we select, in each subinterval $[x_{k-1}, x_k]$, the number $v_k = (k-1)(3/n)$ at which f takes on its maximum value.

The next example illustrates the use of circumscribed rectangles in finding an area.

EXAMPLE ■ 8 If $f(x) = x^3$, find the area under the graph of f from 0 to b for any $b > 0$.

SOLUTION Subdividing the interval $[0, b]$ into n equal parts (see Figure 4.9), we obtain a circumscribed rectangular polygon such that $\Delta x = b/n$ and $x_k = k\,\Delta x$.

Figure 4.9



Since f is an increasing function, the maximum value $f(v_k)$ in the interval $[x_{k-1}, x_k]$ occurs at the right-hand endpoint—that is,

$$v_k = x_k = k\Delta x = k\frac{b}{n} = \frac{bk}{n}.$$

The sum of the areas of the circumscribed rectangles is

$$\sum_{k=1}^{n} f(v_k)\Delta x = \sum_{k=1}^{n}\left[\left(\frac{bk}{n}\right)^3\cdot\frac{b}{n}\right] = \sum_{k=1}^{n}\frac{b^4}{n^4}k^3$$
$$= \frac{b^4}{n^4}\sum_{k=1}^{n}k^3 = \frac{b^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$
$$= \frac{b^4}{4}\cdot\frac{n^2(n+1)^2}{n^4},$$

where we have used Theorem (4.12)(iii). If we let Δx approach 0, then n increases without bound and the expression involving n approaches 1. It follows that the area under the graph is

$$\lim_{\Delta x\to 0}\sum_{k=1}^{n} f(v_k)\Delta x = \frac{b^4}{4}.$$

The last example shows how we can use knowledge of the area under the graph of f on the interval $[0, b]$ for any $b > 0$ to find the area under the graph for a subinterval of $[0, b]$.

EXAMPLE 9 If $f(x) = x^3$, find the area A of the region under the graph of f from $\frac{1}{2}$ to 2.

SOLUTION The region is sketched in Figure 4.10. If we let

$A_1 =$ area under the graph of f from 0 to $\frac{1}{2}$

and $A_2 =$ area under the graph of f from 0 to 2,

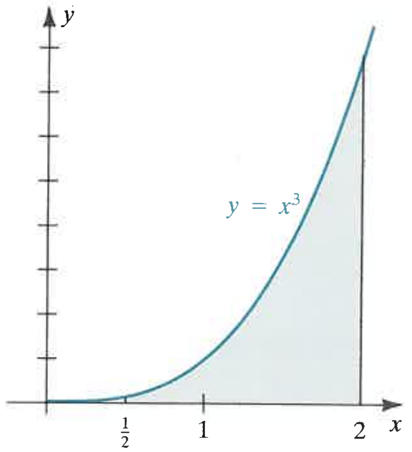
the area A can be found by subtracting A_1 from A_2:

$$A = A_2 - A_1$$

In Example 8, we found that the area under the graph of $y = x^3$ from 0 to b is $\frac{1}{4}b^4$. Hence, using $b = \frac{1}{2}$ for A_1 and $b = 2$ for A_2 yields

$$A = \frac{2^4}{4} - \frac{(\frac{1}{2})^4}{4} = 4 - \frac{1}{64} \approx 3.98.$$

Figure 4.10



EXERCISES 4.3

Exer. 1–8: Evaluate the sum.

1 $\sum_{j=1}^{4} (j^2 + 1)$

2 $\sum_{j=1}^{4} (2^j + 1)$

3 $\sum_{k=0}^{5} k(k-1)$

4 $\sum_{k=0}^{4} (k-2)(k-3)$

5 $\sum_{n=1}^{10} [1 + (-1)^n]$

6 $\sum_{n=1}^{4} (-1)^n \left(\frac{1}{n}\right)$

7 $\sum_{i=1}^{50} 10$

8 $\sum_{k=1}^{1000} 2$

Exer. 9–12: Express the sum in terms of n (see Example 4).

9 $\sum_{k=1}^{n} (k^2 + 3k + 5)$

10 $\sum_{k=1}^{n} (3k^2 - 2k + 1)$

11 $\sum_{k=1}^{n} (k^3 + 2k^2 - k + 4)$

12 $\sum_{k=1}^{n} (3k^3 + k)$

Exer. 13–18: Express in summation notation.

13 $1 + 5 + 9 + 13 + 17$

14 $2 + 5 + 8 + 11 + 14$

15 $\frac{1}{2} + \frac{2}{5} + \frac{3}{8} + \frac{4}{11}$

16 $\frac{1}{4} + \frac{2}{9} + \frac{3}{14} + \frac{4}{19}$

17 $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots + (-1)^n \frac{x^{2n}}{2n}$

18 $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}$

c **Exer. 19–26: Approximate the sum using a calculator or a computer. Write a short program or use a built-in summation procedure in your calculator or CAS.**

19 $\sum_{k=1}^{20} \frac{2^k}{k}$

20 $\sum_{k=1}^{15} \frac{3^k}{k}$

21 $\sum_{k=1}^{1000} \frac{1}{k}$

22 $\sum_{k=1}^{50} \frac{1}{k^2}$

23 $\sum_{k=1}^{40} \frac{\sin(k/40)}{k}$

24 $\sum_{k=1}^{50} \frac{\cos[-1 + (k/25)]}{k}$

25 $\sum_{k=1}^{40} k^4$

26 $\sum_{k=1}^{30} k^5$

Exer. 27–32: Let A be the area under the graph of the given function f from a to b. Approximate A by dividing $[a, b]$ into subintervals of equal length Δx and using (a) A_{IP} and (b) A_{CP}.

27 $f(x) = 3 - x$; $a = -2$, $b = 2$; $\Delta x = 1$

28 $f(x) = x + 2$; $a = -1$, $b = 4$; $\Delta x = 1$

29 $f(x) = x^2 + 1$; $a = 1$, $b = 3$; $\Delta x = \frac{1}{2}$

30 $f(x) = 4 - x^2$; $a = 0$, $b = 2$; $\Delta x = \frac{1}{2}$

c 31 $f(x) = \sqrt{\sin x}$; $a = 0$, $b = 1.5$; $\Delta x = 0.15$

c 32 $f(x) = \frac{1}{\sqrt{x^3 + 1}}$; $a = 0$, $b = 3$; $\Delta x = 0.3$

Exer. 33–38: Refer to Examples 7 and 8. Find the area under the graph of the given function f from 0 to b using (a) inscribed rectangles and (b) circumscribed rectangles.

33 $f(x) = 2x + 3$; $b = 4$

34 $f(x) = 8 - 3x$; $b = 2$

35 $f(x) = 9 - x^2$; $b = 3$

36 $f(x) = x^2$; $b = 5$

37 $f(x) = x^3 + 1$; $b = 2$

38 $f(x) = 4x + x^3$; $b = 2$

Exer. 39–40: Refer to Example 8. Find the area under the graph of f corresponding to the interval (a) [1, 3] and (b) $[a, b]$.

39 $f(x) = x^3$

40 $f(x) = x^3 + 2$

4.4 THE DEFINITE INTEGRAL

Our objective in this section is a careful definition, using Riemann sums, of the definite integral of a function on a closed interval. We examine the concept of an *integrable function* and discuss the relationship between continuous functions and integrable functions.

In Section 4.3, we defined the area under the graph of a function f from a to b as a limit of the form

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(w_k)\Delta x.$$

In our discussion, we restricted f and Δx as follows:

1. The function f is continuous on the closed interval $[a, b]$.
2. $f(x)$ is nonnegative for every x in $[a, b]$.
3. All the subintervals $[x_{k-1}, x_k]$ have the same length Δx.
4. The number w_k is chosen such that $f(w_k)$ is always the minimum (or maximum) value of f on $[x_{k-1}, x_k]$.

There are many applications involving this type of limit in which one or more of these conditions is not satisfied. Thus it is desirable to allow the following changes in (1)–(4):

1′ The function f may be discontinuous at some number in $[a, b]$.

2′ $f(x)$ may be negative for some x in $[a, b]$.

3′ The lengths of the subintervals $[x_{k-1}, x_k]$ may be different.

4′ The number w_k may be *any* number in $[x_{k-1}, x_k]$.

Note that if (2′) is true, part of the graph lies under the x-axis, and therefore the limit is no longer the area under the graph of f.

Let us introduce some new terminology and notation. A **partition** P of a closed interval $[a, b]$ is any decomposition of $[a, b]$ into subintervals of the form

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

for a positive integer n and numbers x_k such that

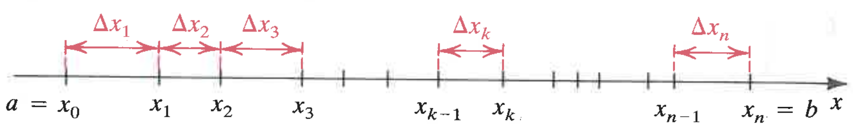
$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

The length of the kth subinterval $[x_{k-1}, x_k]$ will be denoted by Δx_k—that is,

$$\Delta x_k = x_k - x_{k-1}.$$

A typical partition of $[a, b]$ is illustrated in Figure 4.11. The largest of the numbers $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ is the **norm** of the partition P and is denoted by $\|P\|$.

Figure 4.11 A partition of $[a, b]$



EXAMPLE ■ 1 The numbers $\{1, 1.7, 2.2, 3.3, 4.1, 4.5, 5, 6\}$ determine a partition P of the interval $[1, 6]$. Find the lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ of the subintervals in P and the norm of the partition.

SOLUTION The lengths Δx_k of the subintervals are found by subtracting successive numbers in P. Thus,

$$\Delta x_1 = 0.7, \quad \Delta x_2 = 0.5, \quad \Delta x_3 = 1.1, \quad \Delta x_4 = 0.8,$$
$$\Delta x_5 = 0.4, \quad \Delta x_6 = 0.5, \quad \Delta x_7 = 1.0.$$

The norm of P is the largest of these numbers. Hence,

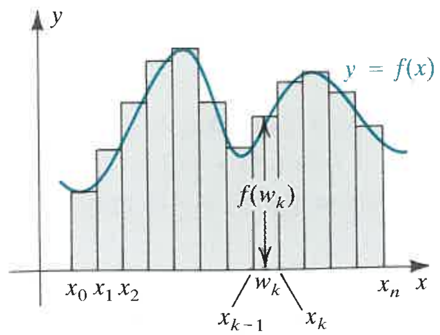
$$\|P\| = \Delta x_3 = 1.1.$$

The following concept, named after the nineteenth-century mathematician G. F. B. Riemann (see *Mathematicians and Their Times,* Chapter 11), is fundamental to the definition of the definite integral.

Definition 4.14

> Let f be defined on a closed interval $[a, b]$, and let P be a partition of $[a, b]$. A **Riemann sum** of f (or $f(x)$) for P is any expression R_P of the form
>
> $$R_P = \sum_{k=1}^{n} f(w_k)\Delta x_k,$$
>
> where w_k is in $[x_{k-1}, x_k]$ and $k = 1, 2, \ldots, n$.

Figure 4.12



In Definition (4.14), $f(w_k)$ is not necessarily a maximum or minimum value of f on $[x_{k-1}, x_k]$. If we construct a rectangle of length $|f(w_k)|$ and width Δx_k, as illustrated in Figure 4.12, the rectangle may be neither inscribed nor circumscribed. Moreover, since $f(x)$ may be negative, certain terms of the Riemann sum R_P may be negative. Consequently, R_P does not always represent a sum of areas of rectangles.

We may interpret the Riemann sum R_P in (4.14) geometrically, as follows. For each subinterval $[x_{k-1}, x_k]$, we construct a vertical line segment through the point $(w_k, f(w_k))$, thereby obtaining a collection of rectangles. If $f(w_k)$ is positive, the rectangle lies above the x-axis, as illustrated by the lighter rectangles in Figure 4.13, and the product $f(w_k)\Delta x_k$ is the area of this rectangle. If $f(w_k)$ is negative, then the rectangle lies below the x-axis, as illustrated by the darker rectangles in Figure 4.13. In this case, the product $f(w_k)\Delta x_k$ is the *negative* of the area of a rectangle. It follows that R_P is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis.

Figure 4.13

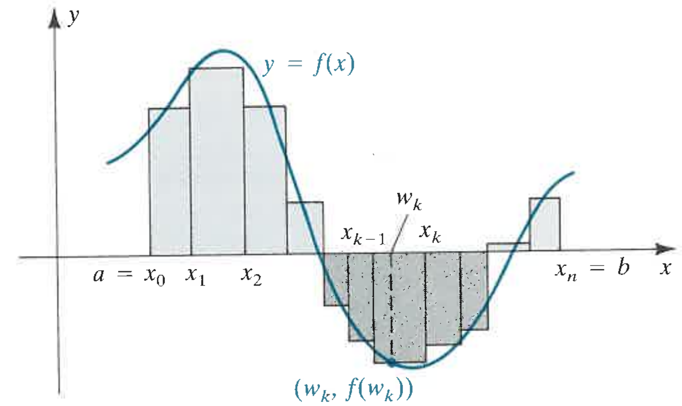
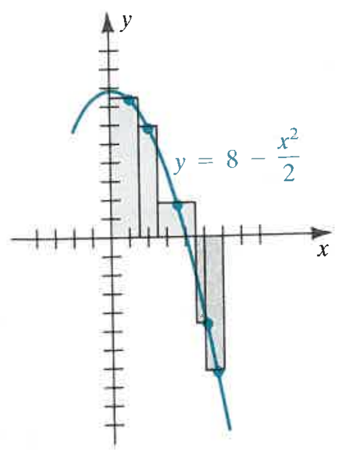


Figure 4.14



EXAMPLE ■ 2 Let $f(x) = 8 - \frac{1}{2}x^2$, and let P be the partition of $[0, 6]$ into the five subintervals determined by

$$x_0 = 0, \quad x_1 = 1.5, \quad x_2 = 2.5, \quad x_3 = 4.5, \quad x_4 = 5, \quad x_5 = 6.$$

Find the norm of the partition and the Riemann sum R_P if

$$w_1 = 1, \quad w_2 = 2, \quad w_3 = 3.5, \quad w_4 = 5, \quad w_5 = 5.5.$$

SOLUTION The graph of f is sketched in Figure 4.14, where we have also shown the points that correspond to w_k and the rectangles of lengths $|f(w_k)|$ for $k = 1, 2, 3, 4,$ and 5. Thus,

$$\Delta x_1 = 1.5, \quad \Delta x_2 = 1, \quad \Delta x_3 = 2, \quad \Delta x_4 = 0.5, \quad \Delta x_5 = 1.$$

The norm $\|P\|$ of the partition is Δx_3, or 2.

Using Definition (4.14) with $n = 5$, we have

$$\begin{aligned} R_P &= \sum_{k=1}^{5} f(w_k)\Delta x_k \\ &= f(w_1)\Delta x_1 + f(w_2)\Delta x_2 + f(w_3)\Delta x_3 + f(w_4)\Delta x_4 + f(w_5)\Delta x_5 \\ &= f(1)(1.5) + f(2)(1) + f(3.5)(2) + f(5)(0.5) + f(5.5)(1) \\ &= (7.5)(1.5) + (6)(1) + (1.875)(2) + (-4.5)(0.5) + (-7.125)(1) \\ &= 11.625. \end{aligned}$$

We shall not always specify the number n of subintervals in a partition P of $[a, b]$. A Riemann sum (4.14) will then be written

$$R_P = \sum_k f(w_k)\Delta x_k,$$

and we will assume that terms of the form $f(w_k)\Delta x_k$ are to be summed over all subintervals $[x_{k-1}, x_k]$ of the partition P.

Using the same approach as in Definition (4.13), we next define

$$\lim_{\|P\|\to 0} \sum_k f(w_k)\Delta x_k = L$$

for a real number L.

Definition 4.15

Let f be defined on a closed interval $[a, b]$, and let L be a real number. The statement

$$\lim_{\|P\|\to 0} \sum_k f(w_k)\Delta x_k = L$$

means that for every $\epsilon > 0$, there is a $\delta > 0$ such that if P is a partition of $[a, b]$ with $\|P\| < \delta$, then

$$\left|\sum_k f(w_k)\Delta x_k - L\right| < \epsilon$$

for any choice of numbers w_k in the subintervals $[x_{k-1}, x_k]$ of P. The number L is a **limit of (Riemann) sums.**

For every $\delta > 0$, there are infinitely many partitions P of $[a, b]$ with $\|P\| < \delta$. Moreover, for each such partition P, there are infinitely many ways of choosing the number w_k in $[x_{k-1}, x_k]$. Consequently, an infinite number of different Riemann sums may be associated with *each* partition P. However, if the limit L exists, then for any $\epsilon > 0$, every Riemann sum is within ϵ units of L, provided a small enough norm is chosen. Although Definition (4.15) differs from the definition of the limit of a function, we may use a proof similar to that given for the uniqueness theorem in Appendix I to show that if the limit L exists, then it is unique.

We next define the definite integral as a limit of a sum, where w_k and Δx_k have the same meanings as in Definition (4.15).

Definition 4.16

Let f be defined on a closed interval $[a, b]$. The **definite integral of f from a to b,** denoted by $\int_a^b f(x)\,dx$, is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_k f(w_k)\Delta x_k,$$

provided the limit exists.

If the limit in Definition (4.16) exists, then f is **integrable** on $[a, b]$, and we say that the definite integral $\int_a^b f(x)\,dx$ **exists**. The process of finding the limit is called **evaluating the integral**. Note that the value of a definite integral is a *real number*, not a family of antiderivatives, as was the case for indefinite integrals.

The integral sign in Definition (4.16), which may be thought of as an elongated letter S (the first letter of the word *sum*), is used to indicate the connection between definite integrals and Riemann sums. The numbers a and b are the **limits of integration**, a being the **lower limit** and b the **upper limit**. In this context, *limit* refers to the smallest or largest number in the interval $[a, b]$ and is not related to definitions of limits given earlier in the text. The expression $f(x)$, which appears to the right of the integral sign, is the *integrand*, as it is with indefinite integrals. The differential symbol dx that follows $f(x)$ may be associated with the increment Δx_k of a Riemann sum of f. This association will be useful in later applications.

EXAMPLE 3 Express the following limit of sums as a definite integral on the interval $[3, 8]$:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (5w_k^3 + \sqrt{w_k} - 4\sin w_k)\Delta x_k,$$

where w_k and Δx_k are as in Definition (4.15).

SOLUTION The given limit of sums has the form stated in Definition (4.16), with

$$f(x) = 5x^3 + \sqrt{x} - 4\sin x.$$

Hence the limit can be expressed as the definite integral

$$\int_3^8 (5x^3 + \sqrt{x} - 4\sin x)\,dx.$$

Letters other than x may be used in the notation for the definite integral. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b f(s)\,ds = \int_a^b f(t)\,dt$$

and so on. For this reason, the letter x in Definition (4.16) is called a **dummy variable.**

Whenever an interval $[a, b]$ is used, we assume that $a < b$. Consequently, Definition (4.16) does not take into account the cases in which the lower limit of integration is greater than or equal to the upper limit. The definition may be extended to include the case where the lower limit is greater than the upper limit, as follows.

Definition 4.17

If $c > d$, then $\int_c^d f(x)\,dx = -\int_d^c f(x)\,dx$.

Definition (4.17) may be phrased as follows: *Interchanging the limits of integration changes the sign of the integral.*

The case in which the lower and upper limits of integration are equal is covered by the next definition.

Definition 4.18

If $f(a)$ exists, then $\int_a^a f(x)\,dx = 0$.

If f is integrable, then the limit in Definition (4.16) exists no matter how w_k is chosen in the subinterval $[x_{k-1}, x_k]$ and no matter what type of partition is used (provided the norm of the partition gets small). Thus we may choose the partition and the w_k's in a computationally or theoretically convenient manner.

For numerical approximations, it is convenient to select w_k as either the left-hand endpoint x_{k-1}, the right-hand endpoint x_k, or the midpoint of every subinterval. For theoretical considerations, we may want to select each w_k so that $f(w_k)$ is the minimum value for f on $[x_{k-1}, x_k]$ as with inscribed rectangles for area, or so that $f(w_k)$ is the maximum value for f on $[x_{k-1}, x_k]$ as with circumscribed rectangles for area. For both numerical and theoretical work, it is convenient to select partitions in which all the subintervals $[x_{k-1}, x_k]$ are of equal length. Such a partition is called a **regular partition**.

If a regular partition of $[a, b]$ contains n subintervals, then $\Delta x = (b-a)/n$. The requirement that $\|P\| \to 0$ is equivalent to $\Delta x \to 0$ or $n \to \infty$. For a regular partition, Definition (4.16) takes the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\Delta x.$$

If we specialize Definition (4.16) to **left-hand endpoints for a regular partition,** for example, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\Delta x.$$

We can use this result to obtain a numerical approximation to the definite integral. The **left endpoint approximation L_n**, where $w_k = x_{k-1}$ for all k, has the form

$$\int_a^b f(x)\,dx \approx L_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x.$$

Similarly, if we let $w_k = x_k$ for all k, then we obtain a **right endpoint approximation R_n,**

$$\int_a^b f(x)\,dx \approx R_n = \sum_{k=1}^{n} f(x_k)\Delta x.$$

Often one of these endpoint approximations tends to be a lower estimate (as with inscribed rectangles) for the area under the graph, while the other endpoint approximation gives a higher estimate (as with circumscribed rectangles).

The **midpoint approximation**, which uses the midpoint $(x_{k-1} + x_k)/2$ as w_k for all k, often gives more accurate approximations. It has the form

$$\int_a^b f(x)\,dx \approx M_n = \sum_{k=1}^{n} f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x.$$

EXAMPLE 4 Use midpoint approximations with $n = 10, 20, 40$, and 80 to estimate $\int_0^\pi \cos(x^2)\,dx$.

SOLUTION For a regular partition, $\Delta x = (\pi - 0)/n = \pi/n$, so that $x_k = k\pi/n$. Thus,

$$\frac{x_{k-1} + x_k}{2} = \frac{\dfrac{(k-1)\pi}{n} + \dfrac{k\pi}{n}}{2} = \frac{k\pi + k\pi - \pi}{2n} = \frac{2k\pi - \pi}{2n} = \frac{k\pi}{n} - \frac{\pi}{2n}.$$

With $f(x) = \cos(x^2)$, we have

$$f\left(\frac{x_{k-1} + x_k}{2}\right) = \cos\left(\frac{k\pi}{n} - \frac{\pi}{2n}\right)^2.$$

We use a calculator to compute the desired summations for the midpoint approximations:

$$\int_0^\pi \cos(x^2)\,dx \approx M_n = \sum_{k=1}^{n}\left[\cos\left(\frac{k\pi}{n} - \frac{\pi}{2n}\right)^2\right]\frac{\pi}{n}$$

$$M_{10} = 0.553751825506$$
$$M_{20} = 0.562860413669$$
$$M_{40} = 0.564995257214$$
$$M_{80} = 0.565519579069$$

For an arbitrary partition of the interval $[a, b]$, evaluation of the sum $\sum_{k=1}^{n} f(w_k)\Delta x_k$ involves the addition of n terms, each of which is the product of two numbers $f(w_k)$ and Δx_k. Thus we have n additions and

n multiplications to perform. For a regular partition, $\Delta x = (b-a)/n$ is independent of k, and we have

$$\sum_{k=1}^{n} f(w_k)\Delta x = \sum_{k=1}^{n}\left[f(w_k)\left(\frac{b-a}{n}\right)\right] = \frac{b-a}{n}\sum_{k=1}^{n} f(w_k).$$

In this case, since evaluation of the sum involves n additions but only 1 multiplication, it is more efficient (computationally) to work with a regular partition than with an arbitrary partition. We shall give more careful consideration to numerical approximation for definite integrals in Section 4.7.

The following theorem is a first application of the special Riemann sums. Many other applications will be discussed in Chapter 5.

Theorem 4.19

If f is integrable and $f(x) \geq 0$ for every x in $[a, b]$, then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)\,dx.$$

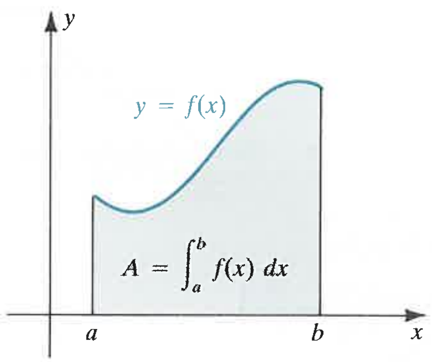


Figure 4.15

PROOF From the preceding section, we know that the area A is a limit of sums $\sum_k f(u_k)\Delta x$, where $f(u_k)$ is the minimum value of f on $[x_{k-1}, x_k]$. Since these are Riemann sums, the conclusion follows from Definition (4.16). ■

Theorem (4.19) is illustrated in Figure 4.15. It is important to keep in mind that area is merely our first application of the definite integral. *There are many instances where* $\int_a^b f(x)\,dx$ *does not represent the area of a region.* In fact, if $f(x) < 0$ for some x in $[a, b]$, then the definite integral may be negative or zero.

If f is continuous and $f(x) \geq 0$ on $[a, b]$, then Theorem (4.19) can be used to evaluate $\int_a^b f(x)\,dx$, provided we can find the area of the region under the graph of f from a to b. This will be true if the graph is a line or part of a circle, as in the following examples. (We consider more complicated definite integrals later in this chapter.) When evaluating a definite integral using this empirical technique, remember that the area of the region and the value of the integral are *numerically equal*; that is, if the area is A square units, the value of the integral is the real number A.

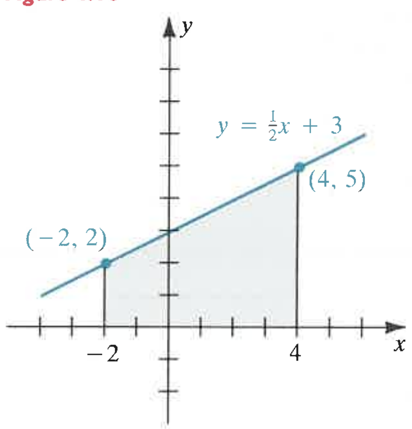


Figure 4.16

EXAMPLE 5 Evaluate $\int_{-2}^{4} (\frac{1}{2}x + 3)\,dx$.

SOLUTION If $f(x) = \frac{1}{2}x + 3$, then the graph of f is the line sketched in Figure 4.16. By Theorem (4.19), the value of the integral is numerically equal to the area of the region under this line from $x = -2$ to $x = 4$. The region is a trapezoid with bases parallel to the y-axis of lengths 2 and 5 and altitude on the x-axis of length 6. Using the formula for the area of a trapezoid, we obtain

$$\int_{-2}^{4} (\tfrac{1}{2}x + 3)\,dx = \tfrac{1}{2}(2+5)6 = 21.$$

EXAMPLE 6 Evaluate $\int_{-4}^{4} \sqrt{16 - x^2}\,dx$.

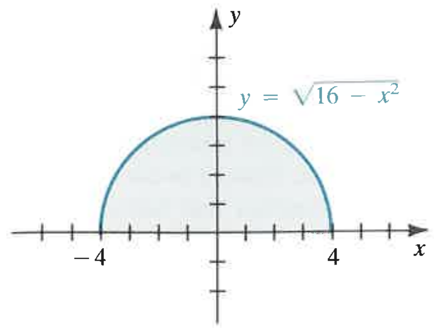


Figure 4.17

SOLUTION If $f(x) = \sqrt{16 - x^2}$, then the graph of f is the semicircle shown in Figure 4.17. By Theorem (4.19), the value of the integral is numerically equal to the area of the region under this semicircle from $x = -4$ to $x = 4$. Hence,

$$\int_{-4}^{4} \sqrt{16 - x^2}\,dx = \tfrac{1}{2} \cdot \pi(4)^2 = 8\pi.$$

EXAMPLE 7 Evaluate

(a) $\int_{4}^{-4} \sqrt{16 - x^2}\,dx$ **(b)** $\int_{4}^{4} \sqrt{16 - x^2}\,dx$

SOLUTION

(a) Using Definition (4.17) and Example 6, we have

$$\int_{4}^{-4} \sqrt{16 - x^2}\,dx = -\int_{-4}^{4} \sqrt{16 - x^2}\,dx = -8\pi.$$

(b) By Definition (4.18),

$$\int_{4}^{4} \sqrt{16 - x^2}\,dx = 0.$$

The next theorem states that functions that are continuous on closed intervals are integrable. This fact will play a crucial role in the proof of the fundamental theorem of calculus in Section 4.6.

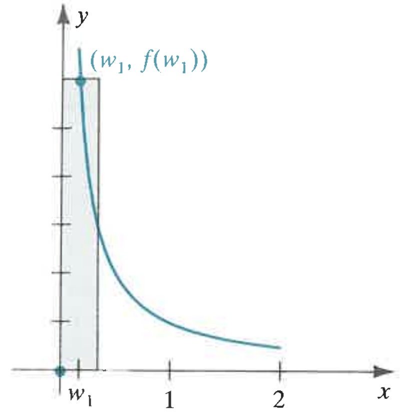
Theorem 4.20

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

A proof of Theorem (4.20) may be found in texts on advanced calculus.

Definite integrals of discontinuous functions may or may not exist, depending on the types of discontinuities. In particular, *functions that have infinite discontinuities on a closed interval are not integrable on that in-*

Figure 4.18
Nonintegrable discontinuous function



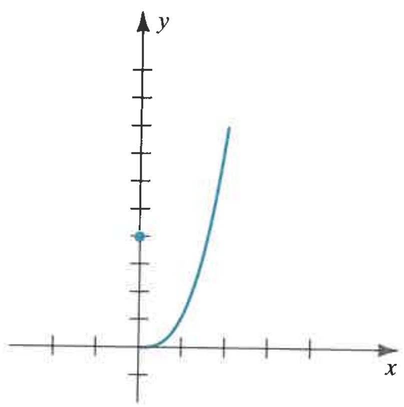
terval. To illustrate, we consider the piecewise-defined function f with domain [0, 2] such that

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{x} & \text{if } 0 < x \le 2 \end{cases}$$

The graph of f is sketched in Figure 4.18. Note that $\lim_{x\to 0^+} f(x) = \infty$. If M is any (large) positive number, then in the first subinterval $[x_0, x_1]$ of any partition P of $[a, b]$, we can find that there exists a number w_1 such that $f(w_1) > M/\Delta x_1$, or, equivalently, $f(w_1)\Delta x_1 > M$. It follows that there are Riemann sums $\sum_k f(w_k)\Delta x_k$ that are arbitrarily large, and hence the limit in Definition (4.16) cannot exist. Thus, f is not integrable. A similar argument can be given for *any* function that has an infinite discontinuity in $[a, b]$. Consequently, *if a function f is integrable on $[a, b]$, then it is bounded on $[a, b]$—that is, there is a real number M such that $|f(x)| \le M$ for every x in $[a, b]$.*

As an illustration of a discontinuous function that *is* integrable, we consider the piecewise-defined function f with domain [0, 2] such that

Figure 4.19
Integrable discontinuous function



$$f(x) = \begin{cases} 4 & \text{if } x = 0 \\ x^3 & \text{if } 0 < x \le 2 \end{cases}$$

The graph of f is sketched in Figure 4.19. Note that f has a jump discontinuity at $x = 0$. From Example 8 of Section 4.3, the area under the graph of $y = x^3$ from 0 to 2 is $2^4/4 = 4$. Thus, by Theorem (4.19), $\int_0^2 x^3\,dx = 4$. We can also show that $\int_0^2 f(x)\,dx = 4$. Hence, f is integrable.

We have shown that a function that is discontinuous on a closed interval may or may not be integrable. However, by Theorem (4.20), functions that are *continuous* on a closed interval are *always* integrable.

EXERCISES 4.4

Exer. 1–4: The given numbers determine a partition P of an interval. (a) Find the length of each subinterval of P. (b) Find the norm $\|P\|$ of the partition.

1 {0, 1.1, 2.6, 3.7, 4.1, 5}

2 {2, 3, 3.7, 4, 5.2, 6}

3 {−3, −2.7, −1, 0.4, 0.9, 1}

4 {1, 1.6, 2, 3.5, 4}

Exer. 5–10: Find the Riemann sum R_P for the given function f on the indicated partition P by choosing on each subinterval of P (a) the left-hand endpoint, (b) the right-hand endpoint, and (c) the midpoint.

5 $f(x) = 2x + 3$; $P = \{1, 3, 4, 5\}$, $n = 3$

6 $f(x) = 3 - 4x$; $P = \{-1, 0, 2, 4, 6\}$, $n = 4$

7 $f(x) = 8 - x^2$; $P = \{-1, -0.5, 0.3, 0.8, 1\}$, $n = 4$

8 $f(x) = 8 - \frac{1}{2}x^2$; $P = \{0, 1.5, 3, 4.5, 6\}$, $n = 4$

9 $f(x) = x^3$; $P = \{-2, 0, 1, 3, 4, 6\}$, $n = 5$

10 $f(x) = \sqrt{x}$; $P = \{1, 3, 4, 9, 12, 16\}$, $n = 5$

c **Exer. 11–14: Find the Riemann sum R_P for the given function f on the indicated interval with a regular partition P of size n by choosing on each subinterval of P (a) the left-hand endpoint, (b) the right-hand endpoint, and (c) the midpoint.**

11 $f(x) = x^3$; $[-2, 6]$, $n = 32$

12 $f(x) = \sqrt{x}$; $[1, 16]$, $n = 30$

13 $f(x) = x^2\sqrt{\cos x}$; $[0, 1]$, $n = 25$

14 $f(x) = \sin(\cos x)$; $[-1, 1]$, $n = 40$

Exer. 15–18: Use Definition (4.16) to express each limit as a definite integral on the given interval $[a, b]$.

15 $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (3w_k^2 - 2w_k + 5)\Delta x_k$; $[-1, 2]$

16 $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} \pi(w_k^2 - 4)\Delta x_k$; $[2, 3]$

17 $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi w_k(1 + w_k^3)\Delta x_k$; $[0, 4]$

18 $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sqrt[3]{w_k} + 4w_k)\Delta x_k$; $[-4, -3]$

Exer. 19–24: Given $\int_1^4 \sqrt{x}\,dx = \frac{14}{3}$, evaluate the integral.

19 $\displaystyle\int_4^1 \sqrt{x}\,dx$

20 $\displaystyle\int_1^4 \sqrt{s}\,ds$

21 $\displaystyle\int_1^4 \sqrt{t}\,dt$

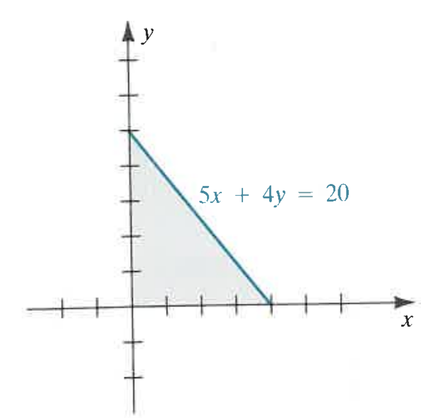
22 $\displaystyle\int_1^4 \sqrt{x}\,dx + \int_4^1 \sqrt{x}\,dx$

23 $\displaystyle\int_4^4 \sqrt{x}\,dx + \int_4^1 \sqrt{x}\,dx$

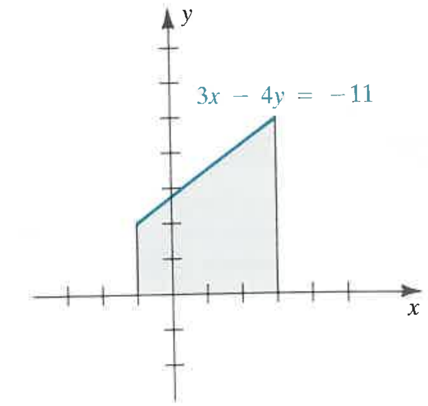
24 $\displaystyle\int_4^4 \sqrt{x}\,dx$

Exer. 25–28: Express the area of the region in the figure as a definite integral.

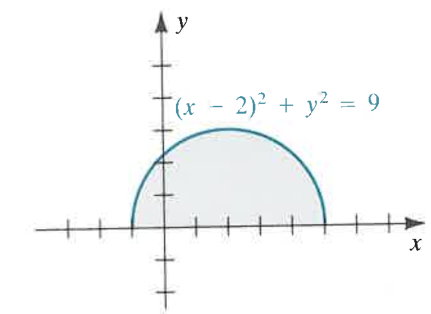
25



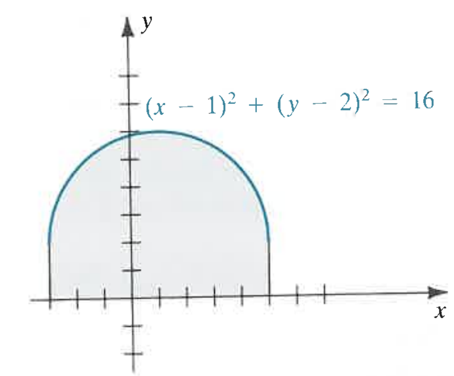
26



27



28



Exer. 29–38: Evaluate the definite integral by regarding it as the area under the graph of a function.

29 $\displaystyle\int_{-1}^{5} 6\,dx$

30 $\displaystyle\int_{-2}^{3} 4\,dx$

31 $\displaystyle\int_{-3}^{2} (2x + 6)\,dx$

32 $\displaystyle\int_{-1}^{2} (7 - 3x)\,dx$

33 $\displaystyle\int_0^3 |x - 1|\,dx$

34 $\displaystyle\int_{-1}^{4} |x|\,dx$

35 $\displaystyle\int_0^3 \sqrt{9 - x^2}\,dx$

36 $\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx$, $a > 0$

37 $\displaystyle\int_{-2}^{2} (3 + \sqrt{4 - x^2})\,dx$

38 $\displaystyle\int_{-2}^{2} (3 - \sqrt{4 - x^2})\,dx$