Journal Title: Calculus of a single variable /

ISSN:

Volume:
Issue:
Month/Year:
Pages: 747-783

Article Author:

Article Title: Chapter 8: Infinite Series (Part B)

Imprint:

Deliver to Middlebury College Patron:

-Save to C:/Ariel Scan as a PDF
-Run Odyssey Helper
-Switch to Process Type: Document Delivery
-Process
-Switch back to Lending before closing.

Michael Olinick (molinick)
Department of Mathematic s
Warner Hall
Middlebury, VT 05753

We now have a variety of tests that can be used to investigate a series for convergence or divergence. Considerable skill is needed to determine which test is best suited for a particular series. This skill can be obtained by working many exercises involving different types of series. The following summary may be helpful in deciding which test to apply; however, some series cannot be investigated by any of these tests. In those cases, it may be necessary to use results from advanced mathematics courses.

Summary of Convergence and Divergence Tests for Series

Test	Series	Convergence or divergence	Comments
nth-term	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim_{n\to\infty} a_n = 0$
Geometric series	$\sum_{n=1}^{\infty} ar^{n-1}$	(i) Converges with sum $S = \dfrac{a}{1-r}$ if $\lvert r\rvert < 1$ (ii) Diverges if $\lvert r\rvert \geq 1$	Useful for comparison tests if the nth term a_n of a series is *similar* to ar^{n-1}
p-series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	(i) Converges if $p > 1$ (ii) Diverges if $p \leq 1$	Useful for comparison tests if the nth term a_n of a series is *similar* to $1/n^p$
Integral	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$	(i) Converges if $\int_1^{\infty} f(x)\,dx$ converges (ii) Diverges if $\int_1^{\infty} f(x)\,dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing, and readily integrable.
Comparison	$\sum a_n, \sum b_n$ $a_n > 0, b_n > 0$	(i) If $\sum b_n$ converges and $a_n \leq b_n$ for every n, then $\sum a_n$ converges. (ii) If $\sum b_n$ diverges and $a_n \geq b_n$ for every n, then $\sum a_n$ diverges. (iii) If $\lim_{n\to\infty}(a_n/b_n) = c$ for some positive real number c, then both series converge or both diverge.	The comparison series $\sum b_n$ is often a geometric series or a p-series. To find b_n in (iii), consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$	If $\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert = L$ (or ∞), the series (i) converges (absolutely) if $L < 1$ (ii) diverges if $L > 1$ (or ∞)	Inconclusive if $L = 1$ Useful if a_n involves factorials or nth powers If $a_n > 0$ for every n, the absolute value sign may be disregarded.
Root	$\sum a_n$	If $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$ (or ∞), the series (i) converges (absolutely) if $L < 1$ (ii) diverges if $L > 1$ (or ∞)	Inconclusive if $L = 1$ Useful if a_n involves nth powers If $a_n > 0$ for every n, the absolute value sign may be disregarded.
Alternating series	$\sum(-1)^n a_n$ $a_n > 0$	Converges if $a_k \geq a_{k+1}$ for every k and $\lim_{n\to\infty} a_n = 0$	Applicable only to an alternating series
$\sum \lvert a_n\rvert$	$\sum a_n$	If $\sum \lvert a_n\rvert$ converges, then $\sum a_n$ converges.	Useful for series that contain both positive and negative terms

EXERCISES 8.5

Exer. 1–4: Determine whether the series (a) satisfies conditions (i) and (ii) of the alternating series test (8.30) and (b) converges or diverges.

1 $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1}{n^2+7}$

2 $\sum_{n=1}^{\infty} (-1)^{n-1} n5^{-n}$

3 $\sum_{n=1}^{\infty} (-1)^{n} (1+e^{-n})$

4 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{e^{2n}+1}{e^{2n}-1}$

Exer. 5–32: Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

5 $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1}{\sqrt{2n+1}}$

6 $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1}{n^{2/3}}$

7 $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{\ln(n+1)}$

8 $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n}{n^2+4}$

9 $\sum_{n=2}^{\infty} (-1)^{n} \dfrac{n}{\ln n}$

10 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{\ln n}{n}$

11 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{5}{n^3+1}$

12 $\sum_{n=1}^{\infty} (-1)^{n} e^{-n}$

13 $\sum_{n=1}^{\infty} \dfrac{(-10)^n}{n!}$

14 $\sum_{n=1}^{\infty} \dfrac{n!}{(-5)^n}$

15 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{n^2+3}{(2n-5)^2}$

16 $\sum_{n=1}^{\infty} \dfrac{\sin\sqrt{n}}{\sqrt{n^3+4}}$

17 $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{\sqrt[3]{n}}{n+1}$

18 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{(n+1)^2}{n^5+1}$

19 $\sum_{n=1}^{\infty} \dfrac{\cos\frac{1}{6}\pi n}{n^2}$

20 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{\ln n}{(1.5)^n}$

21 $\sum_{n=1}^{\infty} (-1)^{n} n \sin\dfrac{1}{n}$

22 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{\arctan n}{n^2}$

23 $\sum_{n=2}^{\infty} (-1)^{n} \dfrac{1}{n\sqrt{\ln n}}$

24 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{2^{1/n}}{n!}$

25 $\sum_{n=1}^{\infty} \dfrac{n^n}{(-5)^n}$

26 $\sum_{n=1}^{\infty} \dfrac{(n^2+1)^n}{(-n)^n}$

27 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1+4^n}{1+3^n}$

28 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{n^4}{e^n}$

29 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{\cos\pi n}{n}$

30 $\sum_{n=1}^{\infty} \dfrac{1}{n} \sin\dfrac{(2n-1)\pi}{2}$

31 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1}{(n-4)^2+5}$

32 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{\ln n}{\sqrt[3]{n}}$

c **Exer. 33–38: Approximate the sum of each series to three decimal places.**

33 $\sum_{n=0}^{\infty} (-1)^{n} \dfrac{1}{n!}$

34 $\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{1}{(2n)!}$

35 $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1}{n^3}$

36 $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{1}{n^5}$

37 $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{n+1}{5^n}$

38 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1}{n} \left(\dfrac{1}{2}\right)^n$

c **Exer. 39–42: Use Theorem (8.31) to find a positive integer n such that S_n approximates the sum of the series to four decimal places.**

39 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1}{n^2}$

40 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1}{\sqrt{n}}$

41 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1}{n^n}$

42 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1}{n^3+1}$

Exer. 43–44: Show that the alternating series converges for every positive integer k.

43 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{(\ln n)^k}{n}$

44 $\sum_{n=1}^{\infty} (-1)^{n} \dfrac{1}{\sqrt[k]{n}}$

45 If $\sum a_n$ and $\sum b_n$ are both convergent series, is $\sum a_n b_n$ convergent? Explain.

46 If $\sum a_n$ and $\sum b_n$ are both divergent series, is $\sum a_n b_n$ divergent? Explain.

8.6 POWER SERIES

The most important reason for developing the theory in the previous sections is to represent functions as *power series*—that is, as series whose terms contain powers of a variable x. To illustrate, if we use the formula

$S = a/(1-r)$ for the sum of a geometric series (see Theorem (8.15)(i)), we obtain

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x},$$

provided $|x| < 1$. If we let $f(x) = 1/(1-x)$ with $|x| < 1$, then

$$f(x) = 1 + x + x^2 + \cdots + x^n + \cdots.$$

We say that $f(x)$ is *represented* by this power series. To find a function value $f(c)$, we can let $x = c$ and find the sum of a series. For example,

$$f\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n + \cdots = \frac{1}{1-\frac{1}{2}} = 2.$$

Later we shall apply other techniques to express many different types of functions as series.

The following definition may be considered as a generalization of the notion of a polynomial to an infinite series.

Definition 8.36

Let x be a variable. A **power series in x** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

where each a_k is a real number.

If a number c is substituted for x in the power series $\sum_{n=0}^{\infty} a_n x^n$, we obtain

$$\sum_{n=0}^{\infty} a_n c^n = a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n + \cdots.$$

This series of constant terms may then be tested for convergence or divergence. To simplify the nth term, we assume that $x^0 = 1$, even if $x = 0$. The main objective of this section is to determine all values of x for which a power series converges. Every power series in x converges if $x = 0$, since

$$a_0 + a_1(0) + a_2(0)^2 + \cdots + a_n(0)^n + \cdots = a_0.$$

To find other values of x that produce convergent series, we often use the ratio test for absolute convergence (8.35), as illustrated in the following examples.

EXAMPLE 1 Find all values of x for which the following power series is absolutely convergent:

$$1 + \frac{1}{5}x + \frac{2}{5^2}x^2 + \cdots + \frac{n}{5^n}x^n + \cdots$$

SOLUTION If we let

$$u_n = \frac{n}{5^n}x^n = \frac{nx^n}{5^n},$$

then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{nx^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{5n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{5n}\right)|x| = \frac{1}{5}|x|.$$

By the ratio test (8.35), with $L = \frac{1}{5}|x|$, the series is absolutely convergent if the following equivalent inequalities are true:

$$\tfrac{1}{5}|x| < 1, \qquad |x| < 5, \qquad -5 < x < 5$$

The series diverges if $\frac{1}{5}|x| > 1$—that is, if $x > 5$ or $x < -5$.

If $\frac{1}{5}|x| = 1$, the ratio test is inconclusive, and hence the numbers 5 and -5 require special consideration. Substituting 5 for x in the power series, we obtain

$$1 + 1 + 2 + 3 + \cdots + n + \cdots,$$

which is divergent, by the nth-term test (8.17), because $\lim_{n\to\infty} a_n \neq 0$. If we let $x = -5$, we obtain

$$1 - 1 + 2 - 3 + \cdots + (-1)^n n + \cdots,$$

which is also divergent, by the nth-term test. Consequently, the power series is absolutely convergent for every x in the open interval $(-5, 5)$ and diverges elsewhere.

EXAMPLE 2 Find all values of x for which the following power series is absolutely convergent:

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

SOLUTION We shall employ the same technique as was used in Example 1. If we let

$$u_n = \frac{1}{n!}x^n = \frac{x^n}{n!},$$

then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0.$$

The limit 0 is less than 1 for every value of x, and hence, from the ratio test (8.35), the power series is absolutely convergent for *every* real number x.

EXAMPLE ▪ 3 Find all values of x for which $\sum n!\,x^n$ is convergent.

SOLUTION Let $u_n = n!\,x^n$. If $x \neq 0$, then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= \lim_{n\to\infty}|(n+1)x| = \lim_{n\to\infty}(n+1)\,|x| = \infty$$

and, by the ratio test (8.35), the series diverges. Hence, the power series is convergent only if $x = 0$.

Theorem (8.38) will show that the solutions of the preceding examples are typical in the sense that if a power series converges for nonzero values of x, then either it is absolutely convergent for every real number or it is absolutely convergent throughout some open interval $(-r, r)$ and diverges outside of the closed interval $[-r, r]$. The proof of this fact depends on the next theorem.

Theorem 8.37

(i) If a power series $\sum a_n x^n$ converges for a nonzero number c, then it is absolutely convergent whenever $|x| < |c|$.

(ii) If a power series $\sum a_n x^n$ diverges for a nonzero number d, then it diverges whenever $|x| > |d|$.

PROOF If $\sum a_n c^n$ converges and $c \neq 0$, then, by Theorem (8.16), $\lim_{n\to\infty} a_n c^n = 0$. Using Definition (8.3) with $\epsilon = 1$, we know that there is a positive integer N such that

$$|a_n c^n| < 1 \quad \text{whenever} \quad n \geq N.$$

Consequently,

$$|a_n x^n| = \left|\frac{a_n c^n x^n}{c^n}\right| = |a_n c^n|\left|\frac{x}{c}\right|^n < \left|\frac{x}{c}\right|^n,$$

provided $n \geq N$. If $|x| < |c|$, then $|x/c| < 1$ and $\sum |x/c|^n$ is a convergent geometric series. Hence, by the basic comparison test (8.26), the series obtained by deleting the first N terms of $\sum |a_n x^n|$ is convergent. It follows that the series $\sum |a_n x^n|$ is also convergent, which proves (i).

To prove (ii), suppose the series diverges for $x = d \neq 0$. If the series converges for some number c_1 with $|c_1| > |d|$, then, by (i), it converges whenever $|x| < |c_1|$. In particular, the series converges for $x = d$, contrary to our supposition. Hence, the series diverges whenever $|x| > |d|$. ▬

We may now prove the following.

Theorem 8.38

If $\sum a_n x^n$ is a power series, then exactly one of the following is true:

(i) The series converges only if $x = 0$.

(ii) The series is absolutely convergent for every x.

(iii) There is a number $r > 0$ such that the series is absolutely convergent if x is in the open interval $(-r, r)$ and divergent if $x < -r$ or $x > r$.

PROOF If neither (i) nor (ii) is true, then there exist nonzero numbers c and d such that the series converges if $x = c$ and diverges if $x = d$. Let S denote the set of all real numbers for which the series is absolutely convergent. By Theorem (8.37), the series diverges if $|x| > |d|$, and hence every number in S is less than $|d|$. By the completeness property (8.10), S has a least upper bound r. It follows that the series is absolutely convergent if $|x| < r$ and diverges if $|x| > r$. ▬

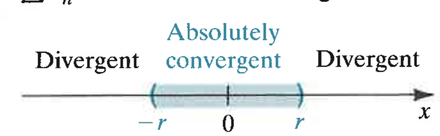


Figure 8.12
$\sum a_n x^n$ with radius of convergence r

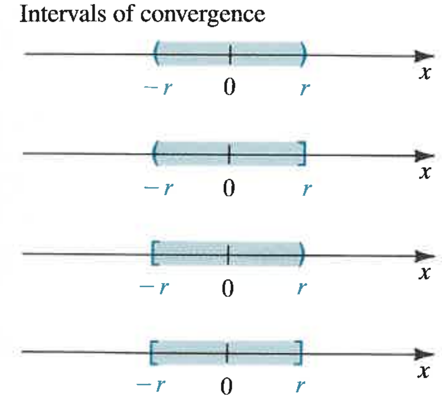


Figure 8.13
Intervals of convergence

Case (iii) of Theorem (8.38) is illustrated graphically in Figure 8.12. The number r is called the **radius of convergence** of the series. Either convergence or divergence may occur at $-r$ or r, depending on the nature of the series.

The totality of numbers for which a power series converges is called its **interval of convergence**. If the radius of convergence r is positive, then the interval of convergence is one of the following (see Figure 8.13):

$$(-r, r), \quad (-r, r], \quad [-r, r), \quad [-r, r]$$

To determine which of these intervals occurs, we must conduct separate investigations for the numbers $x = r$ and $x = -r$.

In (i) or (ii) of Theorem (8.38), the radius of convergence is denoted by 0 or ∞, respectively. In Example 1, the interval of the convergence is $(-5, 5)$ and the radius of convergence is 5. In Example 2, the interval of convergence is $(-\infty, \infty)$ and we write $r = \infty$. In Example 3, $r = 0$. The next example illustrates the case of a half-open interval of convergence.

EXAMPLE ▪ 4

(a) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n.$$

(b) Use a graphing utility to plot the polynomials

$$p_k(x) = \sum_{n=1}^{k} \frac{1}{\sqrt{n}} x^n$$

for $k = 3, 4, 5$, and 6.

SOLUTION

(a) Note that the coefficient of x^0 is 0, and the summation begins with $n = 1$. We let $u_n = x^n/\sqrt{n}$ and consider

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{n+1}}x\right|$$

$$= \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}\,|x| = (1)|x| = |x|.$$

It follows from the ratio test (8.35) that the power series is absolutely convergent if $|x| < 1$—that is, if x is in the open interval $(-1, 1)$. The series diverges if $x > 1$ or $x < -1$. The numbers 1 and -1 must be investigated separately by substitution in the power series.

If we substitute $x = 1$, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(1)^n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots,$$

which is a divergent p-series, with $p = \frac{1}{2}$. If we substitute $x = -1$, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(-1)^n = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \cdots + \frac{(-1)^n}{\sqrt{n}} + \cdots,$$

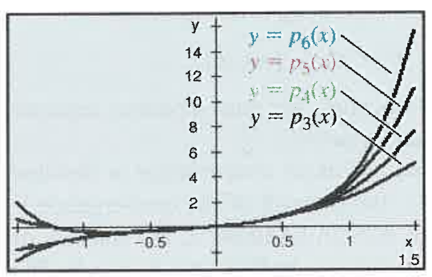
which converges, by the alternating series test. Thus, the power series converges if $-1 \le x < 1$.

(b) Using a computer, we plot the polynomials

$$p_k(x) = \sum_{n=1}^{k}\frac{1}{\sqrt{n}}x^n$$

for $k = 3$, 4, 5, and 6 on the same coordinate axes. These polynomials are defined for all real numbers, and we expect the graphs to approximate the power series for x in the interval of convergence. Figure 8.14 shows the convergence of the graphs on the interval $-1 \le x < 1$, where each successive polynomial provides a better approximation to the power series because it contains an additional term of the series.

Figure 8.14
$-1.5 \le x \le 1.5, \quad -3 \le y \le 16$



We next consider the following more general types of power series.

Definition 8.39

Let c be a real number and x a variable. A **power series in $x - c$** is a series of the form

$$\sum_{n=0}^{\infty}a_n(x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots,$$

where each a_k is a real number.

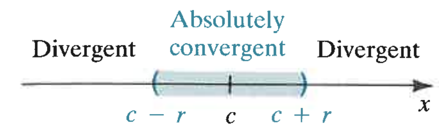
To simplify the nth term in (8.39), we assume that $(x - c)^0 = 1$ even if $x = c$. As in the proof of Theorem (8.38), but with x replaced by $x - c$, exactly one of the following cases is true:

(i) The series converges only if $x - c = 0$—that is, if $x = c$.

(ii) The series is absolutely convergent for every x.

(iii) There is a number $r > 0$ such that the series is absolutely convergent if x is in the open interval $(c - r, c + r)$ and divergent if $x < c - r$ or $x > c + r$.

Figure 8.15
$\sum a_n(x-c)^n$ with radius of convergence r



Case (iii) is illustrated in Figure 8.15. The endpoints $c - r$ and $c + r$ of the interval must be investigated separately. As before, the totality of numbers for which the series converges is called the *interval of convergence*, and r is the *radius of convergence*.

EXAMPLE 5 Find the interval of convergence of the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{3}(x-3)^2 + \cdots + (-1)^n\frac{1}{n+1}(x-3)^n + \cdots.$$

SOLUTION If we let

$$u_n = (-1)^n\frac{(x-3)^n}{n+1},$$

then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+2}\cdot\frac{n+1}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}(x-3)\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)|x-3|$$

$$= (1)|x-3| = |x-3|.$$

By the ratio test (8.35), the series is absolutely convergent if $|x - 3| < 1$; that is, if,

$$-1 < x - 3 < 1, \quad \text{or} \quad 2 < x < 4.$$

Thus, the series is absolutely convergent for every x in the open interval $(2, 4)$. The series diverges if $x < 2$ or $x > 4$. The numbers 2 and 4 each require separate investigation.

If we substitute $x = 4$ in the series, we obtain

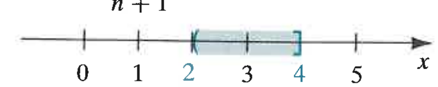
$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n\frac{1}{n+1} + \cdots,$$

which converges, by the alternating series test (8.30). Substituting $x = 2$ gives us

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + \cdots,$$

which is the divergent harmonic series. Hence, the interval of convergence is $(2, 4]$, as illustrated in Figure 8.16.

Figure 8.16
Interval of convergence of $\sum(-1)^n\frac{1}{n+1}(x-3)^n$



EXERCISES 8.6

Exer. 1–30: Find the interval of convergence of the power series.

1 $\sum_{n=0}^{\infty} \frac{1}{n+4} x^n$

2 $\sum_{n=0}^{\infty} \frac{1}{n^2+4} x^n$

3 $\sum_{n=0}^{\infty} \frac{n^2}{2^n} x^n$

4 $\sum_{n=1}^{\infty} \frac{(-3)^n}{n} x^{n+1}$

5 $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}} x^n$

6 $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)} x^n$

7 $\sum_{n=2}^{\infty} \frac{n}{n^2+1} x^n$

8 $\sum_{n=1}^{\infty} \frac{1}{4^n \sqrt{n}} x^n$

9 $\sum_{n=2}^{\infty} \frac{\ln n}{n^3} x^n$

10 $\sum_{n=0}^{\infty} \frac{10^{n+1}}{3^{2n}} x^n$

11 $\sum_{n=0}^{\infty} \frac{n+1}{10^n} (x-4)^n$

12 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} (x-2)^n$

13 $\sum_{n=0}^{\infty} \frac{n!}{100^n} x^n$

14 $\sum_{n=0}^{\infty} \frac{(3n)!}{(2n)!} x^n$

15 $\sum_{n=0}^{\infty} \frac{1}{(-4)^n} x^{2n+1}$

16 $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt[3]{n}\, 3^n} x^n$

17 $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^{2n}$

18 $\sum_{n=0}^{\infty} \frac{10^n}{n!} x^n$

19 $\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1} (x-2)^n$

20 $\sum_{n=1}^{\infty} \frac{1}{n5^n} (x-5)^n$

21 $\sum_{n=0}^{\infty} \frac{n^2}{2^{3n}} (x+4)^n$

22 $\sum_{n=0}^{\infty} \frac{1}{2n+1} (x+3)^n$

23 $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n+1} (x-3)^n$

24 $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^3} (x+2)^n$

25 $\sum_{n=1}^{\infty} \frac{\ln n}{e^n} (x-e)^n$

26 $\sum_{n=0}^{\infty} \frac{n}{3^{2n-1}} (x-1)^{2n}$

27 $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n6^n} (2x-1)^n$

28 $\sum_{n=0}^{\infty} \frac{1}{\sqrt{3n+4}} (3x+4)^n$

29 $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!} (x-4)^n$

30 $\sum_{n=1}^{\infty} (-1)^n \frac{e^{n+1}}{n^n} (x-1)^n$

c Exer. 31–34: (a) Find the radius of convergence of the power series. (b) Graph, on the same coordinate axes, the polynomials

$$p_k(x) = \sum_{n=1}^{k} a_n x^n$$

associated with the power series for $k = 3, 4$, and 5.

31 $\sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{3 \cdot 6 \cdot 9 \cdot \cdots \cdot (3n)} x^n$

32 $\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{4 \cdot 7 \cdot 10 \cdot \cdots \cdot (3n+1)} x^n$

33 $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$

34 $\sum_{n=0}^{\infty} \frac{(n+1)!}{10^n} (x-5)^n$

Exer. 35–36: Find the radius of convergence of the power series for positive integers c and d.

35 $\sum_{n=0}^{\infty} \frac{(n+c)!}{n!\,(n+d)!} x^n$

36 $\sum_{n=0}^{\infty} \frac{(cn)!}{(n!)^c} x^n$

37 *Bessel functions* are useful in the analysis of problems that involve oscillations. If α is a positive integer, the Bessel function $J_\alpha(x)$ *of the first kind of order* α is defined by the power series

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+\alpha)!} \left(\frac{x}{2}\right)^{2n+\alpha}$$

Show that this power series is convergent for every real number.

38 Refer to Exercise 37. The sixth-degree polynomial

$$1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

is sometimes used to approximate the Bessel function $J_0(x)$ of the first kind of order zero for $0 \le x \le 1$. Show that the error E involved in this approximation is less than 0.00001.

c Exer. 39–40: Refer to Exercise 37. For the given α, find the first four terms of the series for $J_\alpha(x)$ and graph J_α on the given interval.

39 $\alpha = 0$; $[0, 2]$

40 $\alpha = 1$; $[0, 4]$

41 If $\lim_{n\to\infty} |a_{n+1}/a_n| = k$ and $k \ne 0$, prove that the radius of convergence of $\sum a_n x^n$ is $1/k$.

42 If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = k$ and $k \ne 0$, prove that the radius of convergence of $\sum a_n x^n$ is $1/k$.

43 If $\sum a_n x^n$ has radius of convergence r, prove that $\sum a_n x^{2n}$ has radius of convergence $\sqrt{r}$.

44 If $\sum a_n$ is absolutely convergent, prove that $\sum a_n x^n$ is absolutely convergent for every x in the interval $[-1, 1]$.

45 If the interval of convergence of $\sum a_n x^n$ is $(-r, r]$, prove that the series is conditionally convergent at r.

46 If $\sum a_n x^n$ is absolutely convergent at one endpoint of its interval of convergence, prove that it is also absolutely convergent at the other endpoint.

8.7 POWER SERIES REPRESENTATIONS OF FUNCTIONS

A power series $\sum a_n x^n$ determines a function f whose domain is the interval of convergence of the series. Specifically, for each x in this interval, we let $f(x)$ equal the sum of the series—that is,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

If a function f is defined in this way, we say that $\sum a_n x^n$ is a **power series representation for** $f(x)$ (or *of* $f(x)$). We also use the phrase **f is represented by the power series.**

Numerical computations using power series provide the basis for the design of calculators and the construction of mathematical tables. In addition to this use, power series representations for functions have far-reaching consequences in advanced mathematics and applications. The proof of Theorem (8.41) will show that e^x may be represented as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

We can thus consider e^x as a *series* instead of as the inverse of the natural logarithmic function. As we shall see, algebraic manipulations, differentiation, and integration can be performed by using the series for e^x, instead of previous methods. The same will be true for trigonometric, inverse trigonometric, logarithmic, and hyperbolic functions. In the next example, we consider a power series representation for a simple algebraic function.

EXAMPLE 1 Find a function f that is represented by the power series

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots.$$

SOLUTION If $|x| < 1$, then, by Theorem (8.15)(i), the given geometric series converges and has the sum

$$S = \frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}.$$

Hence, we may write

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots.$$

This result is a power series representation for $f(x) = 1/(1+x)$ on the interval $(-1, 1)$.

If a function f is represented by a power series in x, then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for every x in the interval of convergence of the series. Since a polynomial in x is a *finite* sum of terms of the form $a_n x^n$, it may not be surprising that f has properties similar to those for polynomial functions. In particular, in the next theorem (stated without proof), we see that f has a derivative f' whose power series representation can be found by differentiating each term of the series for f. Similarly, definite integrals of $f(x)$ may be obtained by integrating each term of the series $\sum a_n x^n$. In the statement of the theorem, note that for the nth term $a_n x^n$ of the series, we have

$$\frac{d}{dx}(a_n x^n) = n a_n x^{n-1} \quad \text{and} \quad \int_0^x a_n t^n \, dt = \left[a_n \frac{t^{n+1}}{n+1} \right]_0^x = a_n \frac{x^{n+1}}{n+1}.$$

Theorem 8.40

Suppose that a power series $\sum a_n x^n$ has a radius of convergence $r > 0$, and let f be defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

for every x in the interval of convergence. If $-r < x < r$, then

(i) $$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

(ii) $$\int_0^x f(t)\, dt = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \cdots + a_n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

The series obtained by differentiation in (i) or integration in (ii) of Theorem (8.40) has the same radius of convergence as $\sum a_n x^n$. However, convergence at the endpoints $x = r$ and $x = -r$ of the interval may change. As usual, these numbers require separate investigation.

As a corollary of Theorem (8.40)(i), *a function that is represented by a power series in an interval $(-r, r)$ is continuous throughout $(-r, r)$* (see Theorem (2.12)). Similar results are true for functions represented by power series of the form $\sum a_n (x-c)^n$.

EXAMPLE 2 Use a power series representation for $1/(1+x)$ to obtain a power series representation for

$$\frac{1}{(1+x)^2} \quad \text{if} \quad |x| < 1.$$

SOLUTION From Example 1,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \quad \text{if} \quad |x| < 1.$$

If we differentiate each term of this series, then, by Theorem (8.40)(i),

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 - \cdots + (-1)^n n x^{n-1} + \cdots.$$

By Theorem (8.20)(ii), we may multiply both sides by -1, obtaining

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 + \cdots + (-1)^{n+1} n x^{n-1} + \cdots$$

if $|x| < 1$.

EXAMPLE 3 Find a power series representation for $\ln(1+x)$ if $|x| < 1$.

SOLUTION If $|x| < 1$, then

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt = \int_0^x [1 - t + t^2 - \cdots + (-1)^n t^n + \cdots]\, dt,$$

where the last equality follows from Example 1. By Theorem (8.40)(ii), we may integrate each term of the series as follows:

$$\ln(1+x) = \int_0^x 1\, dt - \int_0^x t\, dt + \int_0^x t^2\, dt - \cdots + (-1)^n \int_0^x t^n\, dt + \cdots$$

$$= \Big[t\Big]_0^x - \left[\frac{t^2}{2}\right]_0^x + \left[\frac{t^3}{3}\right]_0^x - \cdots + (-1)^n \left[\frac{t^{n+1}}{n+1}\right]_0^x + \cdots$$

Hence,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots$$

if $|x| < 1$.

EXAMPLE 4 Use the results of Example 3 to calculate $\ln(1.1)$ to five decimal places.

SOLUTION In Example 3, we found a series representation for $\ln(1+x)$ if $|x| < 1$. Substituting 0.1 for x in that series gives us the alternating series

$$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5} - \cdots$$
$$\approx 0.1 - 0.005 + 0.000333 - 0.000025 + 0.000002 - \cdots.$$

If we sum the first four terms on the right and round off to five decimal places, we obtain

$$\ln(1.1) \approx 0.09531.$$

By Theorem (8.31), the error E is less than or equal to the absolute value 0.000002 of the fifth term of the series, and therefore the number 0.09531 is accurate to five decimal places.

EXAMPLE 5 Find a power series representation for $\arctan x$.

SOLUTION We first observe that

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt.$$

Next, we note that if $|t| < 1$, then, by Theorem (8.15)(i) with $a = 1$ and $r = -t^2$,

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + \cdots.$$

By Theorem (8.40)(ii), we may integrate each term of the series from 0 to x to obtain

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots,$$

provided $|x| < 1$. It can be proved that this series representation is also valid if $|x| = 1$.

In the next theorem, we find a power series representation for e^x.

Theorem 8.41

If x is any real number,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

PROOF We considered the indicated power series in Example 2 of the preceding section and found that it is absolutely convergent for every real number x. If we let f denote the function represented by the series, then

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Applying Theorem (8.40)(i) gives us

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

That is,

$$f'(x) = f(x) \quad \text{for every } x.$$

If, in Theorem (6.33), we let $y = f(t)$, $t = x$, and $c = 1$, we obtain

$$f(x) = f(0)e^x.$$

However,

$$f(0) = 1 + 0 + \frac{0^2}{2!} + \cdots + \frac{0^n}{n!} + \cdots = 1$$

and hence

$$f(x) = e^x,$$

which is what we wished to prove. ■

Note that Theorem (8.41) allows us to express the number e as the sum of a convergent positive-term series, namely,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$$

We can use a power series representation for a function to obtain representations for other related functions by making algebraic substitutions. Thus, by Theorem (8.41), if x is any real number,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

To obtain a power series representation for e^{-x}, we need only substitute $-x$ for x:

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots + \frac{(-x)^n}{n!} + \cdots,$$

or

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots$$

By Theorem (8.20)(i), we may add corresponding terms of the series for e^x and e^{-x}, obtaining

$$e^x + e^{-x} = 2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + \cdots + 2\cdot\frac{x^{2n}}{(2n)!} + \cdots.$$

(Note that odd powers of x cancel.) We can now find a power series for $\cosh x = \frac{1}{2}(e^x + e^{-x})$ by multiplying each term of the last series by $\frac{1}{2}$ (see Theorem (8.20)(ii)). Thus,

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots.$$

We could find a power series representation for $\sinh x$ either by using $\frac{1}{2}(e^x - e^{-x})$ or by differentiating each term of the series for $\cosh x$. It is left as an exercise to show that

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots.$$

EXAMPLE 6 Find a power series representation for xe^{-2x}.

SOLUTION First we substitute $-2x$ for x in Theorem (8.41):

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots + \frac{(-2x)^n}{n!} + \cdots,$$

or

$$e^{-2x} = 1 - 2x + (2^2)\frac{x^2}{2!} - (2^3)\frac{x^3}{3!} + \cdots + (-2)^n\frac{x^n}{n!} + \cdots$$

Multiplying both sides by x gives us

$$xe^{-2x} = x - 2x^2 + (2^2)\frac{x^3}{2!} - (2^3)\frac{x^4}{3!} + \cdots + (-2)^n\frac{x^{n+1}}{n!} + \cdots,$$

which may be written as

$$xe^{-2x} = \sum_{n=0}^{\infty}(-2)^n\frac{x^{n+1}}{n!}.$$

EXAMPLE 7

(a) If g is the function defined by

$$g(t) = \begin{cases} \dfrac{e^t - 1}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

show that g is continuous at 0.

(b) Find a power series representation $\sum_{n=1}^{\infty} a_n x^n$ for the function represented by $\int_0^x g(t)\,dt$.

(c) Plot the graphs of the polynomials $p_k(x) = \sum_{n=1}^{k} a_n x^n$ associated with the power series in part (a) for $k = 3$, 4, and 5.

SOLUTION

(a) To show that g is continuous at 0, we need to show that $\lim_{t\to 0} g(t) = g(0) = 1$. Since $\lim_{t\to 0}(e^t - 1) = \lim_{t\to 0} t = 0$, we have the indeterminate form 0/0. By l'Hôpital's rule,

$$\lim_{t\to 0}\frac{e^t - 1}{t} = \lim_{t\to 0}\frac{(e^t - 1)'}{(t)'} = \lim_{t\to 0}\frac{e^t}{1} = \frac{1}{1} = 1.$$

(b) From Theorem (8.41), we have

$$e^t - 1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!} + \cdots.$$

Dividing through by t gives

$$\frac{e^t - 1}{t} = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots + \frac{t^{n-1}}{n!} + \cdots.$$

Since the power series in this equation has value 1 when $t = 0$ and $g(0) = 1$, we have

$$g(t) = 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots + \frac{t^{n-1}}{n!} + \cdots$$

for all t.

Applying Theorem (8.40)(ii) yields

$$\int_0^x g(t)\,dt = \Big[t\Big]_0^x + \left[\frac{t^2}{2\cdot 2!}\right]_0^x + \left[\frac{t^3}{3\cdot 3!}\right]_0^x + \cdots + \left[\frac{t^n}{n\cdot n!}\right]_0^x + \cdots$$

$$= x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \cdots + \frac{x^n}{n\cdot n!} + \cdots.$$

Thus, the power series representation of $\int_0^x g(t)\,dt$ is $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = 1/(n\cdot n!)$.

(c) The polynomials are

$$p_3(x) = x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!}$$

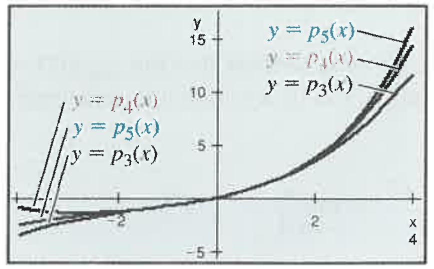
$$p_4(x) = x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!}$$

$$p_5(x) = x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!} + \frac{x^5}{5\cdot 5!}.$$

Using a graphing utility and an x-range restricted to the interval $[-4, 4]$, we plot the graphs of each polynomial on the same coordinate axes to

Figure 8.17

$-4 \le x \le 4, -5 \le y \le 18$



obtain the results shown in Figure 8.17. For $-2 < x < 2$, we see that the graphs converge to approximate the function represented by the polynomials in the power series.

EXAMPLE 8 Approximate $\int_0^{0.1} e^{-x^2}\,dx$.

SOLUTION We cannot use the fundamental theorem of calculus to evaluate the integral, because we do not know of an antiderivative for e^{-x^2}. Although we could use the trapezoidal rule (4.37) or Simpson's rule (4.38), the following method is simpler and, in addition, produces a high degree of accuracy using a sum of only two terms. Letting $x = -t^2$ in Theorem (8.41), we obtain

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots$$

for every t. Applying Theorem (8.40)(ii) yields

$$\begin{aligned}\int_0^{0.1} e^{-x^2}\,dx &= \int_0^{0.1} e^{-t^2}\,dt \\ &= \Big[t\Big]_0^{0.1} - \left[\frac{t^3}{3}\right]_0^{0.1} + \left[\frac{t^5}{10}\right]_0^{0.1} - \cdots \\ &= 0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{10} - \cdots.\end{aligned}$$

If we use the first two terms to approximate the sum of this convergent alternating series, then, by Theorem (8.31), the error is less than the third term $(0.1)^5/10 = 0.000001$. Hence,

$$\begin{aligned}\int_0^{0.1} e^{-x^2}\,dx &\approx 0.1 - \frac{0.001}{3} \\ &\approx 0.09967,\end{aligned}$$

which is accurate to five decimal places.

The method used in Example 8 is accurate because the numbers in the interval $[0, 0.1]$ are close to 0. The method would be much less accurate (for the same number of terms of the series) if, for example, the limits of integration were 3 and 3.1. Recall also Example 5 of Section 7.7.

Thus far, the methods we have used to obtain power series representations of functions are *indirect* in the sense that we started with known series and then differentiated or integrated. In the next section, we shall discuss a *direct* method that can be used to find power series representations for a large variety of functions.

EXERCISES 8.7

Exer. 1–4: (a) Find a power series representation for $f(x)$. (b) Use Theorem (8.40) to find power series representations for $f'(x)$ and $\int_0^x f(t)\,dt$.

1 $f(x) = \dfrac{1}{1-3x}$; $|x| < \frac{1}{3}$

2 $f(x) = \dfrac{1}{1+5x}$; $|x| < \frac{1}{5}$

3 $f(x) = \dfrac{1}{2+7x}$; $|x| < \frac{2}{7}$

4 $f(x) = \dfrac{1}{3-2x}$; $|x| < \frac{3}{2}$

Exer. 5–10: Find a power series in x that has the given sum, and specify the radius of convergence. (*Hint:* Use (8.15), (8.40), or long division, as necessary.)

5 $\dfrac{x^2}{1-x^2}$ 6 $\dfrac{x}{1-x^4}$ 7 $\dfrac{x}{2-3x}$

8 $\dfrac{x^3}{4-x^3}$ 9 $\dfrac{x^2+1}{x-1}$ 10 $\dfrac{x^2-3}{x-2}$

11 (a) Prove that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{if} \quad |x| < 1.$$

(b) Use the series in part (a) to approximate $\ln(1.2)$ to three decimal places, and compare the approximation with that obtained using a calculator.

c 12 Use the first three terms of the series in Exercise 11(a) to approximate $\ln(0.9)$, and compare the approximation with that obtained using a calculator.

13 Use Example 5 to prove that

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n(2n+1)}.$$

c 14 (a) Use the first five terms of the series in Example 5 to approximate $\pi/4$.

(b) Estimate the error in the approximation obtained in part (a).

Exer. 15–26: Use a power series representation obtained in this section to find a power series representation for $f(x)$.

15 $f(x) = xe^{3x}$ 16 $f(x) = x^2 e^{(x^2)}$

17 $f(x) = x^3 e^{-x}$ 18 $f(x) = xe^{-3x}$

19 $f(x) = x^2 \ln(1+x^2)$; $|x| < 1$

20 $f(x) = x \ln(1-x)$; $|x| < 1$

21 $f(x) = \arctan \sqrt{x}$; $|x| < 1$

22 $f(x) = x^4 \arctan(x^4)$; $|x| < 1$

23 $f(x) = \sinh(-5x)$ 24 $f(x) = \sinh(x^2)$

25 $f(x) = x^2 \cosh(x^3)$ 26 $f(x) = \cosh(-2x)$

c **Exer. 27–32: Use an infinite series to approximate the integral to four decimal places.**

27 $\displaystyle\int_0^{1/3} \frac{1}{1+x^6}\,dx$ 28 $\displaystyle\int_0^{1/2} \arctan x^2\,dx$

29 $\displaystyle\int_{0.1}^{0.2} \frac{\arctan x}{x}\,dx$ 30 $\displaystyle\int_0^{0.2} \frac{x^3}{1+x^5}\,dx$

31 $\displaystyle\int_0^1 e^{-x^2/10}\,dx$ 32 $\displaystyle\int_0^{0.5} e^{-x^3}\,dx$

33 Use the power series representation for $(1-x^2)^{-1}$ to find a power series representation for $2x(1-x^2)^{-2}$.

34 Use the method of Example 3 to find a power series representation for $\ln(3+2x)$.

35 Refer to Exercise 37 of Section 8.6. Use Theorem (8.40) to prove the following.

(a) If $J_0(x)$ and $J_1(x)$ are Bessel functions of the first kind of orders 0 and 1, respectively, then

$$\frac{d}{dx}(J_0(x)) = -J_1(x).$$

(b) If $J_2(x)$ and $J_3(x)$ are Bessel functions of the first kind of orders 2 and 3, respectively, then

$$\int x^3 J_2(x)\,dx = x^3 J_3(x) + C.$$

36 Light is absorbed by rods and cones in the retina of the eye. The number of photons absorbed by a photoreceptor during a given flash of light is governed by the *Poisson distribution.* More precisely, the probability p_n that a photoreceptor absorbs exactly n photons is given by the formula $p_n = e^{-\lambda}\lambda^n/n!$ for some $\lambda > 0$.

(a) Show that $\sum_{n=0}^{\infty} p_n = 1$.

(b) Sight usually occurs when two or more photons are absorbed by a photoreceptor. Show that the probability that this will occur is $1 - e^{-\lambda}(\lambda + 1)$.

Exer. 37–38: Find a power series representation for $f(x)$. (If the integrand is denoted by $g(t)$, assume that the value of $g(0)$ is $\lim_{t\to 0} g(t)$.)

37 $f(x) = \int_0^x \frac{\ln(1-t)}{t}\,dt$ **38** $f(x) = \int_0^x \frac{\sin t}{t}\,dt$

c Exer. 39–42: **(a)** Find a power series to represent the function. **(b)** Plot the graphs of the polynomials $p_k(x) = \sum_{n=1}^{k} a_n x^n$ associated with the power series in part (a) for $k = 3$, 4, and 5.

39 $\int_0^x \frac{1}{1+t^4}\,dt$ **40** $\int_0^x \frac{t}{(1+t)^3}\,dt$

41 $\int_0^x \frac{1-e^{-t}}{t}\,dt$ **42** $\int_0^x e^{-t^2/4}\,dt$

8.8 MACLAURIN AND TAYLOR SERIES

In the preceding section, we considered power series representations for several special functions, including those where $f(x)$ has the form

$$\frac{1}{1+x}, \quad \ln(1+x), \quad \arctan x, \quad e^x, \quad \text{or} \quad \cosh x,$$

provided x is suitably restricted. We now wish to consider the following two general questions.

Question 1: If a function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{or} \quad f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n,$$

what is the form of a_n?

Question 2: What conditions are sufficient for a function f to have a power series representation?

Let us begin with question 1. Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

and the radius of convergence of the series is $r > 0$. By Theorem (8.40)(i), a power series representation for $f'(x)$ may be obtained by differentiating each term of the series for $f(x)$. We may then find a series for $f''(x)$ by differentiating the terms of the series for $f'(x)$. Series for $f'''(x)$, $f^{(4)}(x)$, and so on, can be found in similar fashion. Thus,

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$f''(x) = 2a_2 + (3\cdot 2)a_3 x + (4\cdot 3)a_4 x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$f'''(x) = (3\cdot 2)a_3 + (4\cdot 3\cdot 2)a_4 x + \cdots = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3},$$

and for every positive integer k,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k}.$$

Each series obtained by differentiation has the same radius of convergence r as the series for $f(x)$. Substituting 0 for x in each of these series representations, we obtain

$$f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2a_2, \quad f'''(0) = (3\cdot 2)a_3,$$

and for every positive integer k,

$$f^{(k)}(0) = k(k-1)(k-2)\cdots(1)a_k.$$

If we let $k = n$, then

$$f^{(n)}(0) = n!\,a_n.$$

Solving the preceding equations for $a_0, a_1, a_2, \ldots$, we see that

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2}, \quad a_3 = \frac{f'''(0)}{3\cdot 2},$$

and, in general,

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

We have proved that the power series for $f(x)$ has the form stated in the next theorem. It is called a *Maclaurin series for* $f(x)$—named after the Scottish mathematician Colin Maclaurin (1698–1746).

Maclaurin Series for $f(x)$ 8.42

If a function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with radius of convergence $r > 0$, then $f^{(k)}(0)$ exists for every positive integer k and $a_n = f^{(n)}(0)/n!$. Thus,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

Employing the type of proof used for (8.42) gives us the next theorem. If $c \neq 0$, we call the series a *Taylor series for* $f(x)$ *at* c—named after the English mathematician Brook Taylor (1685–1731).

Taylor Series for $f(x)$ 8.43

If a function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

with radius of convergence $r > 0$, then $f^{(k)}(c)$ exists for every positive integer k and $a_n = f^{(n)}(c)/n!$. Thus,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots.$$

Note that the special Taylor series with $c = 0$ is the Maclaurin series (8.42). If we use the convention $f^{(0)}(c) = f(c)$, then the Maclaurin and Taylor series for f may be written in the following summation forms:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

EXAMPLE ■ 1 By Theorem (8.41), e^x has the following power series representation:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

Verify that this is a Maclaurin series.

SOLUTION If $f(x) = e^x$, then the nth derivative of f is $f^{(n)}(x) = e^x$ and

$$f^{(n)}(0) = e^0 = 1 \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence, the Maclaurin series (8.42) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots,$$

which is the same as the given series.

Theorems (8.42) and (8.43) imply that *if* a function f is represented by a power series in x or in $x - c$, then the series *must* be a Maclaurin or Taylor series, respectively. However, the theorems do *not* answer question 2 posed at the beginning of this section: What conditions on a function guarantee that a power series representation *exists*? We shall next obtain such conditions for any series in $x - c$ (including $c = 0$). Let us begin with the following definition.

Definition 8.44

Let c be a real number, and let f be a function that has n derivatives at c: $f'(c), f''(c), \ldots, f^{(n)}(c)$. The **$n$th-degree Taylor polynomial $P_n(x)$** of f at c is

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

In summation notation,

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

If $c = 0$ in (8.44), we call $P_n(x)$ the **nth-degree Maclaurin polynomial of f**. Note that $P_n(x)$ in (8.44) is the $(n+1)$st partial sum of the Taylor series (8.43). If we let $c = 0$, then $P_n(x)$ is the $(n+1)$st partial sum of the Maclaurin series (8.42). The next result will lead to an answer to question 2.

Taylor's Formula with Remainder 8.45

Let f have $n+1$ derivatives throughout an interval containing c. If x is any number in the interval that is different from c, then there is a number z between c and x such that

$$f(x) = P_n(x) + R_n(x), \quad \text{where} \quad R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

PROOF If x is any number in the interval that is different from c, let us *define* $R_n(x)$ as follows:

$$R_n(x) = f(x) - P_n(x)$$

This equation may be rewritten as

$$f(x) = P_n(x) + R_n(x).$$

All we need to show is that for a suitable number z, $R_n(x)$ has the form stated in the conclusion of the theorem.

If t is any number in the interval, let g be the function defined by

$$g(t) = f(x) - \left[f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n \right] - R_n(x)\frac{(x-t)^{n+1}}{(x-c)^{n+1}}.$$

If we differentiate each side of the equation *with respect to t* (regarding x as a constant), then many terms on the right-hand side cancel. You may verify that

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + R_n(x)\cdot(n+1)\frac{(x-t)^n}{(x-c)^{n+1}}.$$

By referring to the formula for $g(t)$, we can verify that $g(x) = 0$. We also see that

$$\begin{aligned} g(c) &= f(x) - [P_n(x)] - R_n(x)\frac{(x-c)^{n+1}}{(x-c)^{n+1}} \\ &= f(x) - P_n(x) - R_n(x) \\ &= f(x) - [P_n(x) + R_n(x)] \\ &= f(x) - f(x) = 0. \end{aligned}$$

Hence, by Rolle's theorem (3.10), there is a number z between c and x such that $g'(z) = 0$—that is,

$$-\frac{f^{(n+1)}(z)}{n!}(x-z)^n + R_n(x)\cdot(n+1)\frac{(x-z)^n}{(x-c)^{n+1}} = 0.$$

Solving for $R_n(x)$, we obtain

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1},$$

which is what we wished to prove. ■

If $c = 0$, we refer to (8.45) as **Maclaurin's formula with remainder**. The expression $R_n(x)$ obtained in Theorem (8.45) is called the **Taylor remainder of f at c**. If $c = 0$, $R_n(x)$ is the **Maclaurin remainder of f**. In the next theorem, we use the Taylor remainder to obtain sufficient conditions for the existence of power series representations for a function f.

Theorem 8.46

Let f have derivatives of all orders throughout an interval containing c, and let $R_n(x)$ be the Taylor remainder of f at c. If

$$\lim_{n\to\infty} R_n(x) = 0$$

for every x in the interval, then $f(x)$ is represented by the Taylor series for $f(x)$ at c.

PROOF The Taylor polynomial $P_n(x)$ is the $(n+1)$st term for the sequence of partial sums of the Taylor series for $f(x)$ at c. By Theorem (8.45), $P_n(x) = f(x) - R_n(x)$, and hence

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} f(x) - \lim_{n\to\infty} R_n(x) = f(x) - 0 = f(x).$$

Thus, the sequence of partial sums converges to $f(x)$, which proves the theorem. ■

In Example 2 of Section 8.6, we proved that the power series $\sum x^n/n!$ is absolutely convergent for every real number x. Since the nth term of a convergent series must approach 0 as $n \to \infty$ (see Theorem (8.16)), we obtain the following result.

Theorem 8.47

If x is any real number,

$$\lim_{n\to\infty}\frac{|x|^n}{n!} = 0.$$

We shall use Theorem (8.47) in the solution of the following example.

EXAMPLE 2 Find the Maclaurin series for $\sin x$, and prove that it represents $\sin x$ for every real number x.

SOLUTION Let us arrange our work as follows:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \end{aligned}$$

Successive derivatives follow this pattern. Substitution in (8.42) gives us the following Maclaurin series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$$

At this stage, all we know is that *if* $\sin x$ is represented by a power series in x, then it is given by the preceding series. To prove that $\sin x$ *is* actually represented by this Maclaurin series, let us use Theorem (8.46) with $c = 0$. If n is a positive integer, then either

$$|f^{(n+1)}(x)| = |\cos x| \quad \text{or} \quad |f^{(n+1)}(x)| = |\sin x|.$$

Hence, $|f^{(n+1)}(z)| \le 1$ for every number z. Using the formula for $R_n(x)$ in Theorem (8.45), with $c = 0$, we obtain

$$|R_n(x)| = \frac{|f^{(n+1)}(z)|}{(n+1)!}|x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}.$$

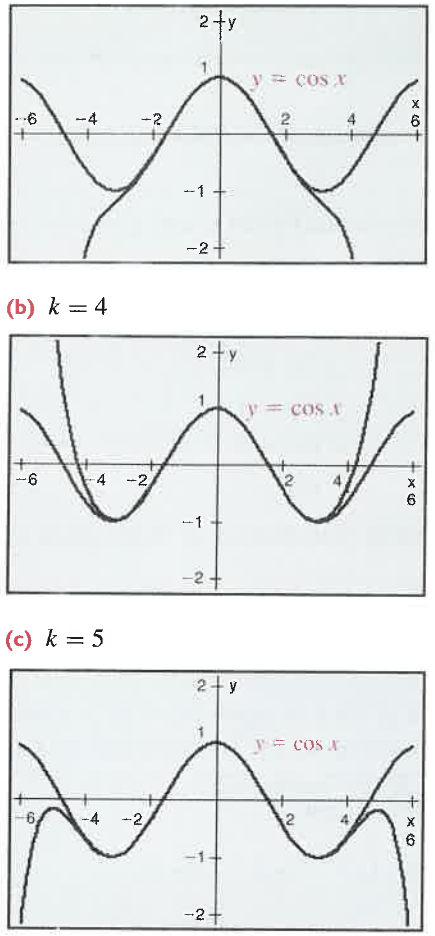
It follows from Theorem (8.47) and the sandwich theorem (8.7) that $\lim_{n\to\infty}|R_n(x)| = 0$. Consequently, $\lim_{n\to\infty} R_n(x) = 0$, and the Maclaurin series representation for $\sin x$ is true for every x.

EXAMPLE 3

(a) Find the Maclaurin series for $\cos x$.

(b) Plot the graphs of several polynomial approximations to the Maclaurin series of part (a).

Figure 8.18
$-6 \le x \le 6, -2 \le y \le 2$
(a) $k = 3$
(b) $k = 4$
(c) $k = 5$



SOLUTION

(a) We could proceed directly, as in Example 2; however, let us obtain the series for $\cos x$ by differentiating the series for $\sin x$ obtained in Example 2:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

(b) We use the polynomials

$$P_{2k}(x) = \sum_{n=0}^{k} (-1)^n \frac{x^{2n}}{(2n)!}$$

for $k = 3, 4$, and 5. Although we can plot the graphs of $\cos x$ and the polynomials on the same coordinate axes, we see the graphs more distinctly if we plot $\cos x$ and each of the polynomials separately, as in Figure 8.18. From the three views shown in the figure, we note that each successive polynomial approximates $\cos x$ over a larger interval of x, and we easily see how rapidly the Maclaurin series approaches the cosine function that it represents.

The Maclaurin series for e^x was obtained in Theorem (8.41) by an indirect technique (see also Example 1 of this section). We next give a direct derivation of this important formula.

EXAMPLE 4 Find a Maclaurin series that represents e^x for every real number x.

SOLUTION If $f(x) = e^x$, then $f^{(k)}(x) = e^x$ for every positive integer k. Hence, $f^{(k)}(0) = 1$, and substitution in (8.42) gives us

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

As in the solution of Example 2, we now use Theorem (8.46) to prove that this power series representation of e^x is true for every real number x. Using the formula for $R_n(x)$ with $c = 0$, we obtain

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} = \frac{e^z}{(n+1)!} x^{n+1},$$

where z is a number between 0 and x. If $0 < x$, then $e^z < e^x$, since the natural exponential function is increasing, and hence for every positive integer n,

$$0 < R_n(x) < \frac{e^x}{(n+1)!} x^{n+1}.$$

By Theorem (8.47),

$$\lim_{n\to\infty} \frac{e^x}{(n+1)!} x^{n+1} = e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0,$$

and by the sandwich theorem (8.7),

$$\lim_{n\to\infty} R_n(x) = 0.$$

If $x < 0$, then $z < 0$, and hence $e^z < e^0 = 1$. Consequently,

$$0 < |R_n(x)| < \left| \frac{x^{n+1}}{(n+1)!} \right|,$$

and we again see that $R_n(x)$ has the limit 0 as $n \to \infty$. It follows from Theorem (8.46) that the power series representation for e^x is valid for all nonzero x. Finally, note that if $x = 0$, then the series reduces to $e^0 = 1$.

EXAMPLE 5 Find the Taylor series for the function $f(x) = \sin x$ in powers of $x - (\pi/6)$.

SOLUTION The derivatives of $f(x) = \sin x$ are listed in Example 2. If we evaluate them at $c = \pi/6$, we obtain

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}, \quad f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2},$$

and this pattern of four numbers repeats itself indefinitely. Substitution in (8.43) gives us

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3 + \cdots.$$

The nth term u_n of this series is given by

$$u_n = \begin{cases} (-1)^{n/2} \dfrac{1}{2(n!)} \left(x - \dfrac{\pi}{6}\right)^n & \text{if } n = 0, 2, 4, 6, \ldots \\ (-1)^{(n-1)/2} \dfrac{\sqrt{3}}{2(n!)} \left(x - \dfrac{\pi}{6}\right)^n & \text{if } n = 1, 3, 5, 7, \ldots \end{cases}$$

The proof that the series represents $\sin x$ for every x is similar to that given in Example 2 and is therefore omitted.

The next example brings out the fact that a function f may have derivatives of all orders at some number c, but may not have a Taylor series representation at that number. This shows that an additional condition, such as $\lim_{n\to\infty} R_n(x) = 0$, is required to guarantee the existence of a Taylor series.

EXAMPLE 6 Let f be the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $f(x)$ cannot be represented by a Maclaurin series.

SOLUTION By Definition (2.6), the derivative of f at 0 is

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \lim_{x\to 0} \frac{(1/x)}{e^{1/x^2}}.$$

The last expression has the indeterminate form ∞/∞. Applying l'Hôpital's rule (6.51), we see that

$$f'(0) = \lim_{x\to 0} \frac{(-1/x^2)}{(-2/x^3)e^{1/x^2}} = \lim_{x\to 0} \frac{x}{2e^{1/x^2}} = 0.$$

It can be proved that $f''(0) = 0$, $f'''(0) = 0$, and, in general, $f^{(n)}(0) = 0$ for every positive integer n. According to Theorem (8.42), *if* $f(x)$ has a Maclaurin series representation, then it is given by

$$f(x) = 0 + 0x + \frac{0}{2!}x^2 + \cdots + \frac{0}{n!}x^n + \cdots,$$

which implies that $f(x) = 0$ throughout an interval containing 0. However, this contradicts the definition of f. Consequently, $f(x)$ does not have a Maclaurin series representation.

As a by-product of Example 6, it follows from Theorem (8.46) that for the given function f, $\lim_{n\to\infty} R_n(x) \neq 0$ at $c = 0$.

We next list, for reference, Maclaurin series that have been obtained in examples in this section and Section 8.7. These series are important because of their uses in advanced mathematics and applications.

Important Maclaurin Series 8.48

Maclaurin series	Interval of convergence
(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$	$(-\infty, \infty)$
(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$	$(-\infty, \infty)$
(c) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$	$(-\infty, \infty)$
(d) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots$	$(-1, 1]$
(e) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$	$[-1, 1]$
(f) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$	$(-\infty, \infty)$
(g) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$	$(-\infty, \infty)$

We can use Maclaurin or Taylor series to approximate values of functions and definite integrals, as illustrated in the next two examples.

EXAMPLE 7 Use the first two nonzero terms of a Maclaurin series to approximate the following, and estimate the error in the approximation.

(a) $\sin(0.1)$ **(b)** $\sin x$ for any nonzero real number x in $[-1, 1]$

SOLUTION

(a) Letting $x = 0.1$ in the Maclaurin series for $\sin x$ (see (8.48)(a)) yields

$$\sin(0.1) = 0.1 - \frac{0.001}{6} + \frac{0.00001}{120} - \cdots.$$

By Theorem (8.31), the error involved in approximating $\sin(0.1)$ by using the first two terms of this alternating series is less than the third term, $0.00001/120 \approx 0.00000008$. To six decimal places,

$$\sin(0.1) \approx 0.1 - \frac{0.001}{6} \approx 0.099833.$$

(b) Using the first two terms of (8.48)(a) gives us the approximation formula

$$\sin x \approx x - \frac{x^3}{6}.$$

By Theorem (8.31), the error involved in using this formula for a real number x in $[-1, 1]$ is less than $|x^5|/5!$.

EXAMPLE 8 Approximate $\int_0^1 \sin(x^2)\,dx$ to four decimal places.

SOLUTION Substituting x^2 for x in (8.48)(a) gives us

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots.$$

Integrating each term of this series, we obtain

$$\int_0^1 \sin(x^2)\,dx = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75{,}600} + \cdots.$$

Summing the first three terms yields

$$\int_0^1 \sin(x^2)\,dx \approx 0.31028.$$

By Theorem (8.31), the error is less than $\frac{1}{75{,}600} \approx 0.00001$.

Note that in the preceding example we achieved accuracy to four decimal places by summing only *three* terms of the integrated series for

$\sin(x^2)$. To obtain this degree of accuracy by means of the trapezoidal rule or Simpson's rule, it would be necessary to use a large value of n for the interval $[0, 1]$. However, if the interval were $[10, 11]$, the efficiency of each method would be quite different. An important point for numerical applications is that in addition to analyzing a given problem, we should also strive to find the most efficient method for computing the answer.

To obtain a Taylor or Maclaurin series representation for a function f, it is necessary to find a general formula for $f^{(n)}(x)$ and, in addition, to investigate $\lim_{n\to\infty} R_n(x)$. For this reason, our examples have been restricted to expressions such as $\sin x$, $\cos x$, and e^x. The method cannot be used if, for example, $f(x)$ equals $\tan x$ or $\sin^{-1} x$, because $f^{(n)}(x)$ becomes very complicated as n increases. Most of the exercises that follow are based on functions whose nth derivatives can be determined easily or on series representations that we have already established. In more complicated cases, we shall restrict our attention to only the first few terms of a Taylor or Maclaurin series representation.

EXERCISES 8.8

Exer. 1–6: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find a_n by using the formula for a_n in (8.42).

1 $f(x) = e^{3x}$ 2 $f(x) = e^{-2x}$

3 $f(x) = \sin 2x$ 4 $f(x) = \cos 3x$

5 $f(x) = \dfrac{1}{1+3x}$ 6 $f(x) = \dfrac{1}{1-2x}$

7 Let $f(x) = \cos x$.

(a) Use the method of Example 2 to prove that $\lim_{n\to\infty} R_n(x) = 0$.

(b) Use (8.42) to find a Maclaurin series for $f(x)$.

8 Let $f(x) = e^{-x}$.

(a) Use the method of Example 4 to prove that $\lim_{n\to\infty} R_n(x) = 0$.

(b) Use (8.42) to find a Maclaurin series for $f(x)$.

Exer. 9–14: Use a Maclaurin series obtained in this section to obtain a Maclaurin series for $f(x)$.

9 $f(x) = x \sin 3x$ 10 $f(x) = x^2 \sin x$

11 $f(x) = \cos(-2x)$ 12 $f(x) = \cos(x^2)$

13 $f(x) = \cos^2 x$ (*Hint:* Use a half-angle formula.)

14 $f(x) = \sin^2 x$

Exer. 15–16: Find a Maclaurin series for $f(x)$. (Do not verify that $\lim_{n\to\infty} R_n(x) = 0$.)

15 $f(x) = 10^x$ 16 $f(x) = \ln(3+x)$

Exer. 17–20: Find a Taylor series for $f(x)$ at c. (Do not verify that $\lim_{n\to\infty} R_n(x) = 0$.)

17 $f(x) = \sin x$; $c = \pi/4$ 18 $f(x) = \cos x$; $c = \pi/3$

19 $f(x) = 1/x$; $c = 2$ 20 $f(x) = e^x$; $c = -3$

21 Find a series representation for e^{2x} in powers of $x+1$.

22 Find a series representation of $\ln x$ in powers of $x-1$.

Exer. 23–28: Find the first three terms of the Taylor series for $f(x)$ at c.

23 $f(x) = \sec x$; $c = \pi/3$

24 $f(x) = \tan x$; $c = \pi/4$

25 $f(x) = \sin^{-1} x$; $c = \frac{1}{2}$

26 $f(x) = \tan^{-1} x$; $c = 1$

27 $f(x) = xe^x$; $c = -1$

28 $f(x) = \csc x$; $c = 2\pi/3$

Exer. 29–38: Use the first two nonzero terms of a Maclaurin series to approximate the number, and estimate the error in the approximation.

29 $\dfrac{1}{\sqrt{e}}$ 30 $\dfrac{1}{e}$ 31 $\cos 3^\circ$

32 $\sin 1^\circ$ 33 $\tan^{-1} 0.1$ 34 $\ln 1.5$

35 $\displaystyle\int_0^1 e^{-x^2}\,dx$ 36 $\displaystyle\int_0^{1/2} x\cos(x^3)\,dx$

37 $\displaystyle\int_0^{0.5} \cos(x^2)\,dx$ 38 $\displaystyle\int_0^{0.1} \tan^{-1}(x^2)\,dx$

c **Exer. 39–42: Approximate the improper integral to four decimal places. (Assume that if the integrand is $f(x)$, then $f(0) = \lim_{x\to 0} f(x)$.)**

39 $\displaystyle\int_0^1 \frac{1-\cos x}{x^2}\,dx$ 40 $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$

41 $\displaystyle\int_0^{1/2} \frac{\ln(1+x)}{x}\,dx$ 42 $\displaystyle\int_0^1 \frac{1-e^{-x}}{x}\,dx$

c **Exer. 43–44: (a) Let $g(x)$ be the sum of the first two nonzero terms of the Maclaurin series for $f(x)$. Use $g(x)$ to approximate $\int_0^1 f(x)\,dx$ and $\int_1^2 f(x)\,dx$. (b) First sketch the graphs, on the same coordinate axes, of f and g for $0 \le x \le 2$, and then use the graphs to compare the accuracy of the approximations in part (a).**

43 $f(x) = \sin(x^2)$ 44 $f(x) = \sinh x$

45 Use (8.48)(d) to find the Maclaurin series for

$$f(x) = \ln\frac{1+x}{1-x}.$$

c 46 Use the first five terms of the series in Exercise 45 to calculate $\ln 2$, and compare your answer to the value obtained using a calculator.

47 (a) Use (8.48)(e) with $x = 1$ to represent π as the sum of an infinite series.

(b) What accuracy is obtained by using the first five terms of the series to approximate π?

(c) Approximately how many terms of the series are required to obtain four-decimal-place accuracy for π?

48 (a) Use the identity

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$$

to express π as the sum of two infinite series.

(b) Use the first five terms of each series in part (a) to approximate π, and compare the result with that obtained in Exercise 47.

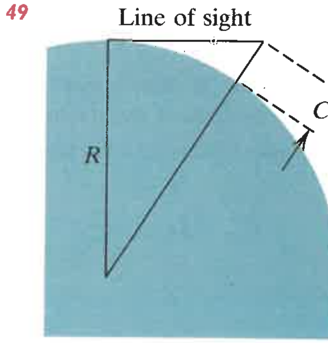
49 In planning a highway across a desert, a surveyor must make compensations for the curvature of the earth when measuring differences in elevation (see figure).

(a) If s is the length of the highway and R is the radius of the earth, show that the correction C is given by $C = R[\sec(s/R) - 1]$.

(b) Use the Maclaurin series for $\sec x$ to show that C is approximately $s^2/(2R) + (5s^4)/(24R^3)$.

(c) The average radius of the earth is 3959 mi. Estimate the correction, to the nearest 0.1 ft, for a stretch of highway 5 mi long.

Exercise 49



50 The velocity v of a water wave is related to its length L and the depth h of the water by

$$v^2 = \frac{gL}{2\pi}\tanh\frac{2\pi h}{L},$$

where g is a gravitational constant.

(a) Show that $\tanh x \approx x - \frac{1}{3}x^3$ if $x \approx 0$.

(b) Use the approximation $\tanh x \approx x$ to show that $v^2 \approx gh$ if h/L is small.

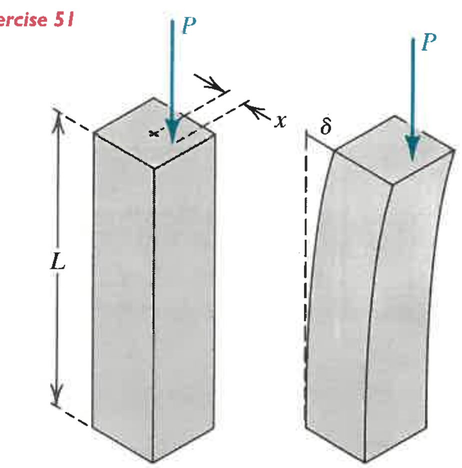
(c) Use part (a) and the fact that the Maclaurin series for $\tanh x$ is an alternating series to show that if $L > 20h$, then the error involved in using $v^2 \approx gh$ is less than $0.002gL$.

51 If too large a downward force of P pounds is applied to a cantilever column of length L at a point x units to the right of center (see figure), the column will buckle. The horizontal deflection δ can be expressed as

$$\delta = x(\sec kL - 1) \quad \text{with} \quad k = \sqrt{P/R},$$

where R is a constant called the *flexural rigidity* of the material and $0 \le kL < \pi/2$. Use Exercise 49(b) to show that $\delta \approx \frac{1}{2}PxL^2/R$ if PL^2 is small compared to $2R$.

Exercise 51



52 Show that $\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$ is accurate to five decimal places if $0 \le x \le \pi/4$.

c **Exer. 53–60: (a) Find a Maclaurin series for the function. (b) Plot the graphs of the polynomials $p_k(x) = \sum_{n=0}^{k} a_n x^n$ associated with the series in part (a) for $k = 3$, 4, and 5.**

53 $\sin(0.6x)$

54 $\cosh 2x$

55 $\cos(x^2)$

56 $\tan^{-1}(0.4x)$

57 $\int_0^x \frac{\sin t}{t}\,dt$

58 $\int_0^x \frac{1 - \cos t}{t^2}\,dt$

59 $\int_0^x \sin(t^3)\,dt$

60 $\int_0^x e^{-t^4/81}\,dt$

8.9 APPLICATIONS OF TAYLOR POLYNOMIALS

In this section, we consider how to use Taylor polynomials to approximate transcendental functions. In particular, we investigate how accurately a Taylor polynomial of particular degree is in estimating values such as $\sin x$, $\cos x$, e^x, or $\ln x$.

In Example 7(b) of Section 8.8, we used the first two nonzero terms of the Maclaurin series for $\sin x$ to obtain the approximation formula

$$\sin x \approx x - \frac{x^3}{6}.$$

By (8.44), the expression on the right-hand side of this formula is the third-degree Taylor polynomial $P_3(x)$ of $\sin x$ at $c = 0$. Thus, we could write

$$\sin x \approx P_3(x).$$

Using additional terms of the Maclaurin series for $\sin x$ would give us other approximation formulas. To illustrate,

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} = P_5(x).$$

By Theorem (8.31), the error involved in using this formula is less than $|x^7|/7!$. Thus, the approximation is very accurate if x is close to 0.

We can use this procedure for any function f that has a sufficient number of derivatives. Specifically, if f satisfies the hypotheses of Taylor's formula with remainder (8.45), then

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ is the nth-degree Taylor polynomial of f at c and $R_n(x)$ is the Taylor remainder. If $\lim_{n\to\infty} R_n(x) = 0$, then, as n increases, we have $P_n(x) \to f(x)$; hence the approximation formula $f(x) \approx P_n(x)$ improves as n gets larger. Thus, we can approximate values of many different transcendental functions by using *polynomial* functions. This is a very important fact, because polynomial functions are the simplest functions to use for calculations—their values can be found by employing only additions and multiplications of real numbers.

As another illustration, consider the exponential function given by $f(x) = e^x$. From (8.48)(c), the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Figure 8.19

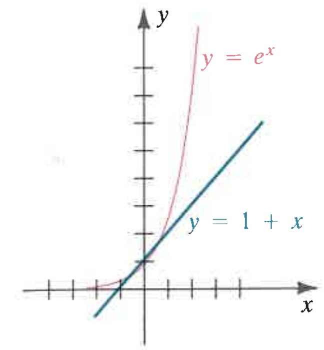


Figure 8.20

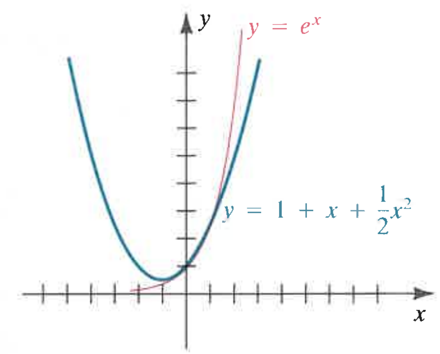
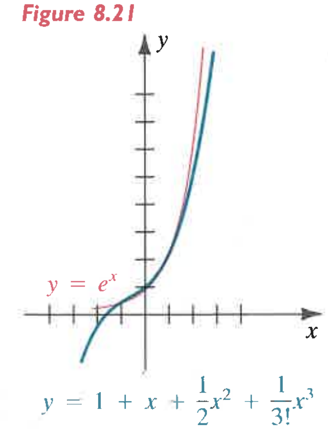


Figure 8.21



If we approximate e^x by means of Taylor polynomials (with $c = 0$), we obtain

$$e^x \approx P_1(x) = 1 + x$$

$$e^x \approx P_2(x) = 1 + x + \frac{x^2}{2}$$

$$e^x \approx P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

and so on. These approximation formulas are accurate only if x is close to 0. To approximate e^x for larger values of x, we may use Taylor polynomials with $c \ne 0$.

The accuracy of the preceding three approximation formulas for e^x is illustrated by the graphs of the functions P_1, P_2, and P_3 in Figures 8.19–8.21. It is of interest to note that the graph of $y = P_1(x) = 1 + x$ in Figure 8.19 is the tangent line to the graph of $y = e^x$ at the point (0, 1). You may verify that the parabola $y = P_2(x) = 1 + x + \frac{1}{2}x^2$ in Figure 8.20 has the same tangent line and the same concavity as the graph of $y = e^x$ at (0, 1). The graph of $y = P_3(x)$ in Figure 8.21 has the same tangent line and concavity at (0, 1) and also the same *rate of change of concavity,* since $P_3'''(0) = (d^3/dx)\,[e^x]_{x=0}$. In general, we can show that for any positive integer n, $P_n^{(n)}(0) = (d^n/dx)\,[e^x]_{x=0}$. As n increases without bound, the graph of the equation $y = P_n(x)$ more closely resembles the graph of $y = e^x$.

The following table indicates the accuracy of the approximation formula $e^x \approx P_3(x)$ for several values of x, approximated to the nearest hundredth.

x	−1.5	−1.0	−0.5	0	0.5	1.0	1.5
e^x	0.22	0.37	0.61	1	1.65	2.72	4.48
$P_3(x)$	0.06	0.33	0.60	1	1.65	2.67	4.19

If more accuracy is desired, we can use any larger positive integer n to obtain

$$e^x \approx P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

We can use this remarkably simple formula to approximate e^x to any degree of accuracy.

In the remainder of this section, we shall use Taylor polynomials to approximate values of functions that satisfy the hypotheses of Taylor's formula with remainder (8.45). Using the conclusion $f(x) = P_n(x) + R_n(x)$ of that theorem, we see that the error involved in approximating $f(x)$ by $P_n(x)$ is

$$|f(x) - P_n(x)| = |R_n(x)|.$$

The complete statement of this result is given in the next theorem.

Theorem 8.49

> Let f have $n+1$ derivatives throughout an interval containing c. If x is any number in the interval and $x \neq c$, then the error in approximating $f(x)$ by the nth-degree Taylor polynomial of f at c,
>
> $$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n,$$
>
> is equal to $|R_n(x)|$, where
>
> $$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$
>
> and z is the number between c and x given by (8.45).

In the next two examples, Taylor polynomials are used to approximate function values. If we are interested in k-decimal-place accuracy in the approximation of a sum, we often approximate each term of the sum to $k+1$ decimal places and then round off the final result to k decimal places. In certain cases, this may fail to produce the required degree of accuracy; however, it is customary to proceed in this way for elementary approximations. More precise techniques may be found in texts on *numerical analysis.*

EXAMPLE ■ 1 Let $f(x) = \ln x$.

(a) Find $P_3(x)$ and $R_3(x)$ at $c = 1$.

(b) Approximate $\ln 1.1$ to four decimal places by means of $P_3(1.1)$, and use $R_3(1.1)$ to estimate the error in this approximation.

SOLUTION

(a) As in Theorem (8.49), the general Taylor polynomial $P_3(x)$ and Taylor remainder $R_3(x)$ at $c = 1$ are

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

and $$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-1)^4,$$

where z is a number between 1 and x. Thus, we need the first four derivatives of f. It is convenient to arrange our work as follows:

$$\begin{aligned} f(x) &= \ln x & f(1) &= 0 \\ f'(x) &= x^{-1} & f'(1) &= 1 \\ f''(x) &= -x^{-2} & f''(1) &= -1 \\ f'''(x) &= 2x^{-3} & f'''(1) &= 2 \\ f^{(4)}(x) &= -6x^{-4} & f^{(4)}(z) &= -6z^{-4} \end{aligned}$$

Substituting in $P_3(x)$ and $R_3(x)$, we obtain

$$P_3(x) = 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

and $$R_3(x) = \frac{-6z^{-4}}{4!}(x-1)^4 = -\frac{1}{4z^4}(x-1)^4,$$

where z is between 1 and x.

(b) From part (a),

$$\ln 1.1 \approx P_3(1.1) = 0.1 - \tfrac{1}{2}(0.1)^2 + \tfrac{1}{3}(0.1)^3,$$

or $$\ln 1.1 \approx 0.0953.$$

To estimate the error in this approximation, we consider

$$|R_3(1.1)| = \left|-\frac{(0.1)^4}{4z^4}\right|, \quad \text{where} \quad 1 < z < 1.1.$$

Since $z > 1$, $1/z < 1$ and therefore $1/z^4 < 1$. Consequently,

$$|R_3(1.1)| = \left|-\frac{(0.1)^4}{4z^4}\right| < \left|-\frac{0.0001}{4}\right| = 0.000025.$$

Because $0.000025 < 0.00005$, it follows from Theorem (8.49) that the approximation $\ln 1.1 \approx 0.0953$ is accurate to four decimal places.

If we wish to approximate a function value $f(x)$ for some x, it is desirable to choose the number c in Theorem (8.49) such that the remainder $R_n(x)$ is very close to 0 when n is relatively small (say, $n = 3$ or $n = 4$). We obtain this result if we choose c close to x. In addition, we should choose c so that values of the first $n+1$ derivatives of f at c are easy to calculate, as was done in Example 1, where to approximate $\ln x$ for $x = 1.1$ we selected $c = 1$. The next example provides another illustration of a suitable choice of c.

EXAMPLE ■ 2 Use a Taylor polynomial to approximate $\cos 61°$, and estimate the accuracy of the approximation.

SOLUTION We wish to approximate $f(x) = \cos x$ if $x = 61°$. Let us begin by observing that $61°$ is close to $60°$, or $\pi/3$ radians, and that it is easy to calculate values of trigonometric functions at $\pi/3$. This suggests that we choose $c = \pi/3$ in (8.49). The choice of n will depend on the degree of accuracy we wish to attain. Let us try $n = 2$. In this case, the first three derivatives of f are required and we arrange our work as follows:

$$\begin{aligned} f(x) &= \cos x & f(\pi/3) &= 1/2 \\ f'(x) &= -\sin x & f'(\pi/3) &= -\sqrt{3}/2 \\ f''(x) &= -\cos x & f''(\pi/3) &= -1/2 \\ f'''(x) &= \sin x & f'''(z) &= \sin z \end{aligned}$$

As in (8.49), the second-degree Taylor polynomial of f at $c = \pi/3$ is

$$P_2(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1/2}{2!}\left(x - \frac{\pi}{3}\right)^2.$$

Since x represents a real number, we must convert $61°$ to radian measure before substituting into $P_2(x)$. Writing

$$61° = 60° + 1° = \frac{\pi}{3} + \frac{\pi}{180}$$

and substituting, we obtain

$$P_2\left(\frac{\pi}{3} + \frac{\pi}{180}\right) = \frac{1}{2} - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 \approx 0.48481,$$

and hence $\qquad \cos 61° \approx 0.48481.$

To estimate the accuracy of this approximation, we consider

$$|R_2(x)| = \left|\frac{f'''(z)}{3!}\left(x - \frac{\pi}{3}\right)^3\right| = \left|\frac{\sin z}{3!}\left(x - \frac{\pi}{3}\right)^3\right|,$$

where z is between $\pi/3$ and x. Substituting $x = (\pi/3) + (\pi/180)$ and using the fact that $|\sin z| \le 1$, we obtain

$$\left|R_2\left(\frac{\pi}{3} + \frac{\pi}{180}\right)\right| = \left|\frac{\sin z}{3!}\left(\frac{\pi}{180}\right)^3\right| \le \left|\frac{1}{3!}\left(\frac{\pi}{180}\right)^3\right| \le 0.000001.$$

Thus, by (8.49), the approximation $\cos 61° \approx 0.48481$ is accurate to five decimal places. For greater accuracy, we must find a value of n such that the maximum value of $|R_n[(\pi/3) + (\pi/180)]|$ is within the desired range.

EXAMPLE 3 If $f(x) = e^x$, use the Taylor polynomial $P_9(x)$ of f at $c = 0$ to approximate e, and estimate the error in the approximation.

SOLUTION For every positive integer k, $f^{(k)}(x) = e^x$, and hence $f^{(k)}(0) = e^0 = 1$. Thus, using $n = 9$ and $c = 0$ in Theorem (8.49) yields

$$P_9(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^9}{9!},$$

and therefore $\quad e \approx P_9(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{9!}.$

This result gives us $e \approx 2.71828153$.

To estimate the error, we consider

$$R_9(x) = \frac{e^z}{10!}x^{10}.$$

If $x = 1$, then $0 < z < 1$. Using results about e^x from Chapter 6, we have $e^z < e^1 < 3$, and

$$|R_9(1)| = \left|\frac{e^z}{10!}(1)\right| < \frac{3}{10!} < 0.000001.$$

Hence, the approximation $e \approx 2.71828$ is accurate to five decimal places.

EXERCISES 8.9

Exer. 1–4: (a) Find the Maclaurin polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ for $f(x)$. (b) Sketch, on the same coordinate axes, the graphs of P_1, P_2, P_3, and f. (c) Approximate $f(a)$ to four decimal places by means of $P_3(a)$, and use $R_3(a)$ to estimate the error in this approximation.

1 $f(x) = \sin x$; $\quad a = 0.05$

2 $f(x) = \cos x$; $\quad a = 0.2$

3 $f(x) = \ln(x + 1)$; $\quad a = 0.9$

4 $f(x) = \tan^{-1} x$; $\quad a = 0.1$

c **Exer. 5–6: Graph, on the same coordinate axes, f, P_1, P_3, and P_5 for $-3 \le x \le 3$.**

5 $f(x) = \sinh x$ $\qquad$ 6 $f(x) = \cosh x$

Exer. 7–18: Find Taylor's formula with remainder (8.45) for the given $f(x)$, c, and n.

7 $f(x) = \sin x$; $\quad c = \pi/2$, $\quad n = 3$

8 $f(x) = \cos x$; $\quad c = \pi/4$, $\quad n = 3$

9 $f(x) = \sqrt{x}$; $\quad c = 4$, $\quad n = 3$

10 $f(x) = e^{-x}$; $\quad c = 1$, $\quad n = 3$

11 $f(x) = \tan x$; $\quad c = \pi/4$, $\quad n = 2$

12 $f(x) = 1/(x - 1)^2$; $\quad c = 2$, $\quad n = 5$

13 $f(x) = 1/x$; $\quad c = -2$, $\quad n = 5$

14 $f(x) = \sqrt[3]{x}$; $\quad c = -8$, $\quad n = 3$

15 $f(x) = \tan^{-1} x$; $\quad c = 1$, $\quad n = 2$

16 $f(x) = \ln \sin x$; $\quad c = \pi/6$, $\quad n = 3$

17 $f(x) = xe^x$; $\quad c = -1$, $\quad n = 4$

18 $f(x) = \log x$; $\quad c = 10$, $\quad n = 2$

Exer. 19–30: Find Maclaurin's formula with remainder for the given $f(x)$ and n.

19 $f(x) = \ln(x + 1)$; $\quad n = 4$

20 $f(x) = \sin x$; $\quad n = 7$

21 $f(x) = \cos x$; $\quad n = 8$

22 $f(x) = \tan^{-1} x$; $\quad n = 3$

23 $f(x) = e^{2x}$; $\quad n = 5$

24 $f(x) = \sec x$; $\quad n = 3$

25 $f(x) = 1/(x - 1)^2$; $\quad n = 5$

26 $f(x) = \sqrt{4 - x}$; $\quad n = 3$

27 $f(x) = \arcsin x$; $\quad n = 2$

28 $f(x) = e^{-x^2}$; $\quad n = 3$

29 $f(x) = 2x^4 - 5x^3$; $\quad n = 4$ and $n = 5$

30 $f(x) = \cosh x$; $\quad n = 4$ and $n = 5$

c **Exer. 31–34: Approximate the number to four decimal places by using the indicated exercise and the fact that $\pi/180 \approx 0.0175$. Prove that your answer is correct by showing that $|R_n(x)| < 0.5 \times 10^{-4}$.**

31 $\sin 89°$ $\quad$ (Exercise 7)

32 $\cos 47°$ $\quad$ (Exercise 8)

33 $\sqrt{4.03}$ $\quad$ (Exercise 9)

34 $e^{-1.02}$ $\quad$ (Exercise 10)

c **Exer. 35–40: Approximate the number by using the indicated exercise, and estimate the error in the approximation by means of $R_n(x)$.**

35 $-1/(2.2)$ $\quad$ (Exercise 13)

36 $\sqrt[3]{-8.5}$ $\quad$ (Exercise 14)

37 $\ln 1.25$ $\quad$ (Exercise 19)

38 $\sin 0.1$ $\quad$ (Exercise 20)

39 $\cos 30°$ $\quad$ (Exercise 21)

40 $\log 10.01$ $\quad$ (Exercise 18)

Exer. 41–46: Use Maclaurin's formula with remainder to establish the approximation formula, and state, in terms of decimal places, the accuracy of the approximation if $|x| \le 0.1$.

41 $\cos x \approx 1 - \dfrac{x^2}{2}$ $\qquad$ 42 $\sqrt[3]{1 + x} \approx 1 + \dfrac{1}{3}x$

43 $e^x \approx 1 + x + \dfrac{x^2}{2}$ $\qquad$ 44 $\sin x \approx x - \dfrac{x^3}{6}$

45 $\ln(1 + x) \approx x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$

46 $\cosh x \approx 1 + \dfrac{x^2}{2}$

47 Let $P_n(x)$ be the nth-degree Maclaurin polynomial. If $f(x)$ is a polynomial of degree n, prove that $f(x) = P_n(x)$.

CHAPTER 8 REVIEW EXERCISES

Exer. 1–6: Determine whether the sequence converges or diverges; if it converges, find the limit.

1 $\left\{\dfrac{\ln(n^2+1)}{n}\right\}$ **2** $\{100(0.99)^n\}$

3 $\left\{\dfrac{10^n}{n^{10}}\right\}$ **4** $\left\{\dfrac{1}{n}+(-2)^n\right\}$

5 $\left\{\dfrac{n}{\sqrt{n+4}}-\dfrac{n}{\sqrt{n+9}}\right\}$ **6** $\left\{\left(1+\dfrac{2}{n}\right)^{2n}\right\}$

c **Exer. 7–8: For the recursively defined sequence, determine what happens to terms of the sequence as k increases.**

7 $a_1 = 1$ and $a_{k+1} = 0.5\cosh a_k$

8 $a_1 = 2$ and $a_{k+1} = \cosh^{-1} a_k + 1$

Exer. 9–34: If the series is positive-term, determine whether it is convergent or divergent; if the series contains negative terms, determine whether it is absolutely convergent, conditionally convergent, or divergent.

9 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n(n+1)(n+2)}}$ **10** $\displaystyle\sum_{n=0}^{\infty}\frac{(2n+3)^2}{(n+1)^3}$

11 $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^{n-1}$ **12** $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2+(\frac{1}{2})^n}$

13 $\displaystyle\sum_{n=1}^{\infty}\frac{3^{2n+1}}{n5^{n-1}}$ **14** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n+2}$

15 $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{\ln(n+1)}$ **16** $\displaystyle\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}$

17 $\displaystyle\sum_{n=1}^{\infty}(n^2+9)(-2)^{1-n}$ **18** $\displaystyle\sum_{n=1}^{\infty}\frac{n+\cos n}{n^3+1}$

19 $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^e}$ **20** $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$

21 $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt[5]{n}}$ **22** $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{(0.9)^n}{\ln n}$

23 $\displaystyle\sum_{n=1}^{\infty}\frac{\sin\frac{5\pi}{3}n}{n^{5\pi/3}}$ **24** $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\sqrt[3]{n}-1}{n^2-1}$

25 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$ **26** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2n+3}{n!}$

27 $\displaystyle\sum_{n=1}^{\infty}\frac{1-\cos n}{n^2}$

28 $\dfrac{2}{1!}-\dfrac{2\cdot 4}{2!}+\cdots+(-1)^{n-1}\dfrac{2\cdot 4\cdot\cdots\cdot(2n)}{n!}+\cdots$

29 $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)^n}{n^{2n}}$ **30** $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n-1}}{n^2+9}$

31 $\displaystyle\sum_{n=1}^{\infty}\frac{e^{2n}}{(2n-1)!}$ **32** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3^n}-\frac{5}{\sqrt{n}}\right)$

33 $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\sqrt{\ln n}}{n}$ **34** $\displaystyle\sum_{n=1}^{\infty}\frac{\tan^{-1} n}{\sqrt{1+n^2}}$

Exer. 35–40: Use the integral test (8.23) to determine the convergence or divergence of the series.

35 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3n+2)^3}$ **36** $\displaystyle\sum_{n=2}^{\infty}\frac{n}{\sqrt{n^2-1}}$

37 $\displaystyle\sum_{n=1}^{\infty}n^{-2}e^{1/n}$ **38** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$

39 $\displaystyle\sum_{n=1}^{\infty}\frac{10}{\sqrt[3]{n+8}}$ **40** $\displaystyle\sum_{n=5}^{\infty}\frac{1}{n^2-4n}$

c **Exer. 41–42: Approximate the sum of the series to three decimal places.**

41 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n+1)!}$ **42** $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2(n^2+1)}$

c **Exer. 43–44: Approximate the sum of the given series to four decimal places by using an integral of the form $\int_N^\infty f(x)\,dx$.**

43 $\displaystyle\sum_{n=1}^{\infty}n2^{-n^2}$ **44** $\displaystyle\sum_{n=1}^{\infty}\frac{2n+3}{(n^2+3n-1)^2}$

Exer. 45–48: Find the interval of convergence of the series.

45 $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{(-3)^n}x^n$ **46** $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{4^{2n}}{\sqrt{n+1}}x^n$

47 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\,2^n}(x+10)^n$ **48** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}(x-1)^n$

Exer. 49–50: Find the radius of convergence of the series.

49 $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}x^n$ **50** $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(n+5)!}(x+5)^n$

Exer. 51–54: Find the Maclaurin series for $f(x)$, and state the radius of convergence.

51 $f(x)=\begin{cases}\dfrac{1-\cos x}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$

52 $f(x)=xe^{-2x}$ **53** $f(x)=\sin x\cos x$

54 $f(x)=\ln(2+x)$

c **Exer. 55–58: (a) Find a Maclaurin series for the function. (b) Plot the graphs of the polynomials $P_k(x)=\sum_{n=0}^{k}a_nx^n$ associated with the power series in part (a) for $k=3, 4,$ and 5.**

55 $\sinh(0.6x)$ **56** $e^{0.5x}$

57 $\displaystyle\int_0^x \cosh t^2\,dt$ **58** $\displaystyle\int_0^x e^{-0.3t^2}\,dt$

59 Find a series representation for e^{-x} in powers of $x+2$.

60 Find a series representation for $\cos x$ in powers of $x-(\pi/2)$.

c **Exer. 61–64: Use an infinite series to approximate the number to three decimal places.**

61 $\displaystyle\int_0^1 x^2e^{-x^2}\,dx$ **62** $1/\sqrt[3]{e}$

63 $\displaystyle\int_0^1 f(x)\,dx$ with $f(x)=(\sin x)/\sqrt{x}$ if $x\neq 0$ and $f(0)=0$

64 $e^{-0.25}$

Exer. 65–66: Find Taylor's formula with remainder for the given $f(x)$, c, and n.

65 $f(x)=\ln\cos x$, $c=\pi/6$, $n=3$

66 $f(x)=\sqrt{x-1}$, $c=2$, $n=4$

Exer. 67–68: Find Maclaurin's formula with remainder for the given $f(x)$ and n.

67 $f(x)=e^{-x^2}$, $n=3$ **68** $f(x)=\dfrac{1}{1-x}$, $n=6$

c **69** Use Taylor's formula with remainder to approximate $\cos 43^\circ$ to four decimal places.

70 Use Taylor's formula with remainder to show that the approximation formula $\sin x\approx x-\frac{1}{6}x^3+\frac{1}{120}x^5$ is accurate to four decimal places for $0\le x\le \pi/4$.

EXTENDED PROBLEMS AND GROUP PROJECTS

1 **(a)** Define *lower bound* and *greatest lower bound* analogous to *upper bound* and *least upper bound*. Show that the completeness property is equivalent to the following statement: Every set of real numbers that has a lower bound has a greatest lower bound.

(b) A sequence $I_1, I_2, \ldots$ of sets is **nested** if I_{n+1} is a subset of I_n for each $n=1,2,\ldots$. Show that the completeness property is equivalent to the **nested interval property:** Every sequence of nested closed intervals has at least one real number that belongs to every interval.

(c) Does the nested interval property remain true if it is not required that each interval be closed?

2 Investigate the representation of the real numbers as decimals.

(a) Assuming that there is a one-to-one correspondence between points on the coordinate line and real numbers, show that every real number x can be written in the form $x=N.d_1d_2d_3\ldots d_k\ldots$, where N is an integer and each d_k is an integer between 0 and 9.

(b) Use the nested interval property to show that every decimal expression of the form $N.d_1d_2d_3\ldots d_k\ldots$ corresponds to a real number.

(c) Show that 0.5 and 0.499999... are both decimal representations of the number $\frac{1}{2}$ and that 0.9999... represents the same real number as 1.000....

(d) From part (c), we see that some real numbers have two different decimal representations. Characterize those numbers. Show that every other real number has a unique decimal representation.

(e) Show that a real number is rational if and only if it has decimal representation in which some block of digits repeats indefinitely; for example,

$$\frac{665}{3333}=0.1995199519951995\ldots.$$

3 There is a theory of **infinite products** analogous to the theory of infinite series. Given a sequence of nonzero numbers $\{a_n\}$, we can form the *kth partial product*:

$$P_k=\prod_{n=1}^{k}a_n=(a_1)(a_2)\cdots(a_k).$$

Formulate a definition for the convergence of an infinite product

$$\prod_{n=1}^{\infty}a_n$$

in terms of limits of partial products. Prove that if an infinite product converges, then a_n approaches 1. Find examples of convergent and divergent infinite products. Discover other tests for convergence of infinite products that may be analogous to tests for convergence of infinite series.