Journal Title: Calculus of a single variable /

ISSN:

Volume:
Issue:
Month/Year:
Pages: 692-746

Article Author:

Article Title: Chapter 8: Infinite Series (Part A)

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INTRODUCTION

MATHEMATICAL MODELS of population growth—whether populations of people, bacteria in a petri dish, or radioactive atoms—all reflect the assumption that at least at some stage of growth or decay, the rate of change of population is proportional to the size of the population. When such assumptions are written in terms of differential equations, the solution invariably involves the natural exponential and logarithmic functions. For example, the simplest model $dP/dt = aP$ has the solution $P(t) = P_o e^{at}$. Predicting the population at various times from this solution requires the evaluation of the exponential function at specific values of the variables.

Models of other important applications also frequently involve the transcendental functions. If x is a real number, we generally find $\arcsin x$, e^x, $\ln x$, $\cosh x$, and other values of transcendental functions by using a calculator or a table. A more fundamental problem is determining *how* calculators compute these numbers or *how* a table is constructed. A principal goal of this chapter is to demonstrate how *infinite series* can be used to find function values.

We begin with a careful study of *sequences* in Section 8.1. These are basic to the definition (Section 8.2) of convergence or divergence of a *series*. We then develop various tests for the convergence of a series of positive constants in Sections 8.3 and 8.4. In Section 8.5, we again consider series of constants, but without restrictions on their signs.

We see in Section 8.6 how to use infinite series to find function values. Specifically, if a function f satisfies certain conditions, we develop techniques for *representing* $f(x)$ as an infinite series whose terms contain powers of x. Substituting a number c for x and then finding (or approximating) the resulting infinite sum gives us the value (or an approximation) of $f(c)$. This method is essentially the same as that which a calculator uses when it approximates function values. We explore these techniques further in Sections 8.7 and 8.8.

This new way of representing functions is the most important reason for developing the theory in the first five sections of the chapter. Infinite series representations for $\sin x$, e^x, and other expressions allow us to consider problems that cannot be solved by finite methods. For example, if x is suitably restricted, we can evaluate integrals such as $\int \sin\sqrt{x}\, dx$ and $\int e^{-x^2}\, dx$, something we could not do in Chapter 7. As another application, in Chapter 15 we use infinite series to extend the definitions of $\sin x$, e^x, and other expressions to the case where x is a *complex number* $a + bi$ with a and b real and $i^2 = -1$.

CHAPTER 8

Using calculus to predict future behavior or population often requires using infinite series to estimate the numerical value of transcendental functions.

Infinite Series

8.1 SEQUENCES

An arbitrary *infinite sequence* (or simply a *sequence*) is often denoted as follows.

Sequence Notation 8.1

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

We may regard (8.1) as a collection of real numbers that is in one-to-one correspondence with the positive integers. Each number a_k is a **term** of the sequence. The sequence is *ordered* in the sense that there is a **first term** a_1, a **second term** a_2, and, if n denotes an arbitrary positive integer, an **nth term** a_n.

We may also define a sequence as a function. Recall that a function f is a correspondence that associates with each number x in the domain exactly one number $f(x)$ in the range. If we restrict the domain to the positive integers $1, 2, 3, \ldots$, we obtain a sequence.

Definition 8.2

A **sequence** is a function f whose domain is the set of positive integers.

In this text, the range of a sequence will be a set of real numbers. If a function f is a sequence, then to each positive integer k there corresponds a real number $f(k)$. The numbers in the range of f may be denoted by

$$f(1), f(2), f(3), \ldots, f(n), \ldots$$

The three dots at the end indicate that the sequence does not terminate.

Note that Definition (8.2) leads to the subscript form (8.1) if we let $a_k = f(k)$ for each positive integer k. Conversely, given (8.1), we can obtain the function f in (8.2) by letting $f(k) = a_k$ for each k.

If we regard a sequence as a function f, then we may consider its graph in an xy-plane. Since the domain of f is the set of positive integers, the only points on the graph are

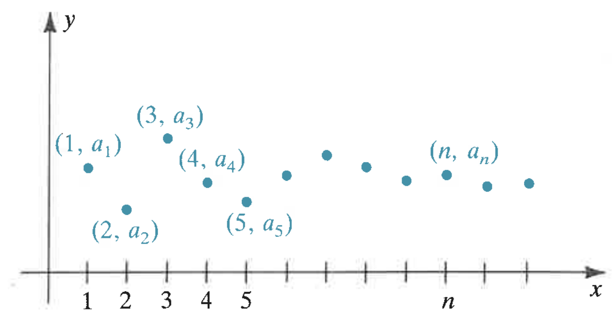
$$(1, a_1), (2, a_2), (3, a_3), \ldots, (n, a_n), \ldots,$$

where a_n is the nth term of the sequence (see Figure 8.1). We sometimes use the graph of a sequence to illustrate the behavior of the nth term a_n as n increases without bound.

Another notation for a sequence with nth term a_n is $\{a_n\}$. For example, the sequence $\{2^n\}$ has nth term 2^n. Using the notation in (8.1), we write this sequence as follows:

$$2^1, 2^2, 2^3, \ldots, 2^n, \ldots$$

Figure 8.1 Graph of a sequence



By Definition (8.2), the sequence $\{2^n\}$ is the function f with $f(n) = 2^n$ for every positive integer n.

EXAMPLE 1 List the first four terms and the tenth term of each sequence.

(a) $\left\{\dfrac{n}{n+1}\right\}$ **(b)** $\{2 + (0.1)^n\}$ **(c)** $\left\{(-1)^{n+1}\dfrac{n^2}{3n-1}\right\}$ **(d)** $\{4\}$

SOLUTION To find the first four terms, we substitute, successively, $n = 1, 2, 3,$ and 4 in the formula for a_n. The tenth term is found by substituting 10 for n. Doing this and simplifying gives us the following:

	Sequence	nth term a_n	First four terms	Tenth term
(a)	$\left\{\dfrac{n}{n+1}\right\}$	$\dfrac{n}{n+1}$	$\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}$	$\dfrac{10}{11}$
(b)	$\{2 + (0.1)^n\}$	$2 + (0.1)^n$	2.1, 2.01, 2.001, 2.0001	2.0000000001
(c)	$\left\{(-1)^{n+1}\dfrac{n^2}{3n-1}\right\}$	$(-1)^{n+1}\dfrac{n^2}{3n-1}$	$\dfrac{1}{2}, -\dfrac{4}{5}, \dfrac{9}{8}, -\dfrac{16}{11}$	$-\dfrac{100}{29}$
(d)	$\{4\}$	4	4, 4, 4, 4	4

For some sequences, we state the first term a_1, together with a rule for obtaining any term a_{k+1} from the preceding term a_k whenever $k \geq 1$. We call this a **recursive definition**, and the sequence is said to be defined **recursively**.

EXAMPLE 2 Find the first four terms and the nth term of the sequence defined recursively as follows:

$$a_1 = 3 \quad \text{and} \quad a_{k+1} = 2a_k \quad \text{for } k \geq 1$$

SOLUTION The sequence is defined recursively, since the first term is given, as well as a rule for finding a_{k+1} whenever a_k is known. Thus, the first four terms of the sequence are

$$a_1 = 3$$
$$a_2 = 2a_1 = 2 \cdot 3 = 6$$
$$a_3 = 2a_2 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3 = 12$$
$$a_4 = 2a_3 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3 = 24.$$

We have written the terms as products to gain insight into the nature of the nth term. Continuing, we obtain $a_5 = 2^4 \cdot 3$ and $a_6 = 2^5 \cdot 3$; it appears that $a_n = 2^{n-1} \cdot 3$. We can use mathematical induction to prove that this guess is correct. Using the notation in (8.1), we write the sequence as

$$3, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, \ldots, 2^{n-1} \cdot 3, \ldots .$$

A sequence $\{a_n\}$ may have the property that as n increases, a_n gets very close to some real number L—that is, $|a_n - L| \approx 0$ if n is large. As an illustration, suppose that

$$a_n = 2 + \left(-\tfrac{1}{2}\right)^n .$$

The first few terms of the sequence $\{a_n\}$ are

$$2 - \tfrac{1}{2}, 2 + \tfrac{1}{4}, 2 - \tfrac{1}{8}, 2 + \tfrac{1}{16}, 2 - \tfrac{1}{32}, 2 + \tfrac{1}{64}, \ldots ,$$

or, equivalently,

$$1.5, 2.25, 1.875, 2.0625, 1.96875, 2.015625, \ldots .$$

It appears that the terms get closer to 2 as n increases. Note that for every positive integer n,

$$|a_n - 2| = \left|2 + \left(-\frac{1}{2}\right)^n - 2\right| = \left|\left(-\frac{1}{2}\right)^n\right| = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}.$$

The number $1/2^n$, and hence $|a_n - 2|$, *can be made arbitrarily close to* 0 *by choosing n sufficiently large*. According to the next definition, the sequence *has the limit* 2, or *converges to* 2, and we write

$$\lim_{n\to\infty} \left[2 + \left(-\tfrac{1}{2}\right)^n\right] = 2.$$

This type of limit is almost the same as $\lim_{x\to\infty} f(x) = L$, given in Chapter 1. The only difference is that if $f(n) = a_n$, the domain of f is the set of positive integers and not an infinite interval of real numbers. As in Definition (1.16), but using a_n instead of $f(x)$, we state the following.

Definition 8.3

A sequence $\{a_n\}$ **has the limit** L, or **converges to** L, denoted by either

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty,$$

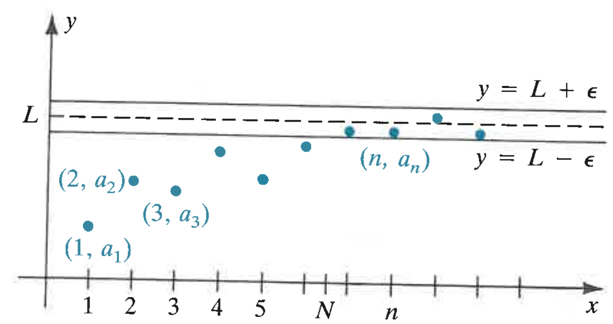
if for every $\epsilon > 0$ there exists a positive number N such that

$$|a_n - L| < \epsilon \quad \text{whenever} \quad n > N.$$

If such a number L does not exist, the sequence **has no limit**, or **diverges**.

A graphical interpretation similar to that shown for the limit of a function in Figure 1.34 can be given for the limit of a sequence. The only difference is that the x-coordinate of each point on the graph is a positive integer. Figure 8.2 is the graph of a sequence $\{a_n\}$ for a specific case in which $\lim_{n\to\infty} a_n = L$. Note that for any $\epsilon > 0$, the points (n, a_n) lie between the lines $y = L \pm \epsilon$, provided n is sufficiently large. Of course, the approach to L may vary from that illustrated in the figure (see, for example, Figures 8.3 and 8.6).

Figure 8.2



If we can make a_n as large as desired by choosing n sufficiently large, then the sequence $\{a_n\}$ diverges, but we still use the limit notation and write $\lim_{n\to\infty} a_n = \infty$. A more precise definition follows.

Definition 8.4

The notation

$$\lim_{n\to\infty} a_n = \infty$$

means that for every positive real number P there exists a number N such that $a_n > P$ whenever $n > N$.

As was the case for functions in Section 1.4, $\lim_{n\to\infty} a_n = \infty$ does *not* mean that the limit exists, but rather that the number a_n increases without bound as n increases. Similarly, $\lim_{n\to\infty} a_n = -\infty$ means that a_n decreases without bound as n increases.

The next theorem is important because it allows us to use results from Chapter 1 to investigate convergence or divergence of sequences. The proof follows from Definitions (8.3) and (1.16).

Theorem 8.5

Let $\{a_n\}$ be a sequence, let $f(n) = a_n$, and suppose that $f(x)$ exists for every real number $x \geq 1$.

(i) If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

(ii) If $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then $\lim_{n\to\infty} f(n) = \infty$ (or $-\infty$).

The following example illustrates the use of Theorem (8.5).

EXAMPLE ■ 3 If $a_n = 1 + (1/n)$, determine whether $\{a_n\}$ converges or diverges.

SOLUTION We let $f(n) = 1 + (1/n)$ and consider

$$f(x) = 1 + \frac{1}{x} \quad \text{for every real number } x \geq 1.$$

From our work in Section 1.4,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x} = 1 + 0 = 1.$$

Hence, by Theorem (8.5),

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1.$$

Thus, the sequence $\{a_n\}$ converges to 1.

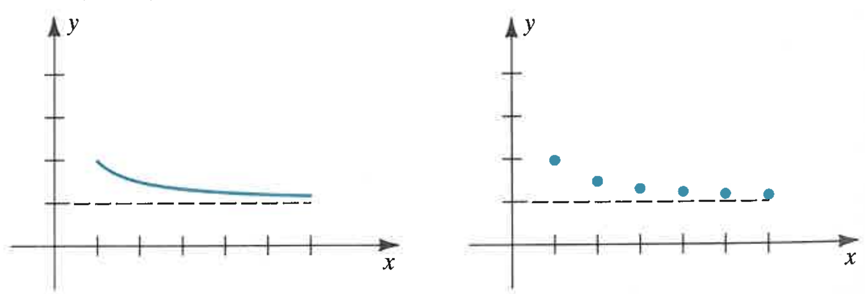
The difference between

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$$

is illustrated in Figure 8.3. Note that for $1 + (1/x)$, the function f is continuous if $x \geq 1$, and the graph has a horizontal asymptote $y = 1$. For $1 + (1/n)$, we consider only the points whose x-coordinates are positive integers.

Figure 8.3

$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right) = 1$ $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$



EXAMPLE ■ 4 Determine whether the sequence converges or diverges.

(a) $\left\{\frac{1}{4}n^2 - 1\right\}$ **(b)** $\left\{(-1)^{n-1}\right\}$

SOLUTION

(a) If we let $f(x) = \frac{1}{4}x^2 - 1$, then $f(x)$ exists for every $x \geq 1$ and

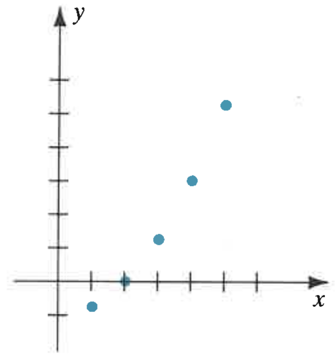
$$\lim_{x\to\infty}\left(\tfrac{1}{4}x^2 - 1\right) = \infty.$$

Hence, by Theorem (8.5),

$$\lim_{n\to\infty}\left(\tfrac{1}{4}n^2 - 1\right) = \infty.$$

Since the limit does not exist, the sequence diverges. The graph in Figure 8.4 illustrates the manner in which the sequence diverges.

Figure 8.4



(b) Letting $n = 1, 2, 3, \ldots$, we see that the terms of $(-1)^{n-1}$ oscillate between 1 and -1 as follows:

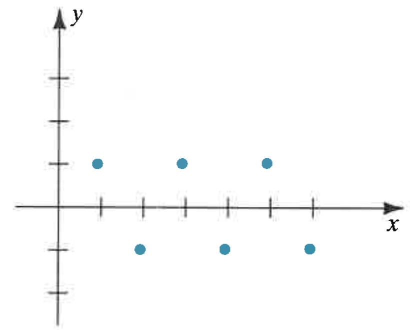
$$1, -1, 1, -1, 1, -1, \ldots$$

This result is illustrated graphically in Figure 8.5. Thus, since

$$\lim_{n\to\infty}(-1)^{n-1}$$

does not exist, the sequence diverges.

Figure 8.5



The next example shows how we may use l'Hôpital's rule (6.51) to find limits of certain sequences.

EXAMPLE ■ 5 Determine whether the sequence $\left\{5n/e^{2n}\right\}$ converges or diverges.

SOLUTION Let $f(x) = 5x/e^{2x}$ for every real number x. Since f takes on the indeterminate form ∞/∞ as $x \to \infty$, we may use l'Hôpital's rule, obtaining

$$\lim_{x\to\infty}\frac{5x}{e^{2x}} = \lim_{x\to\infty}\frac{5}{2e^{2x}} = 0.$$

Hence, by Theorem (8.5), $\lim_{n\to\infty}(5n/e^{2n}) = 0$. Thus, the sequence converges to 0.

The proof of the next theorem illustrates the use of Definition (8.3).

Theorem 8.6

(i) $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$

(ii) $\lim_{n\to\infty} |r^n| = \infty$ if $|r| > 1$

PROOF If $r = 0$, it follows trivially that the limit is 0. Let us assume that $0 < |r| < 1$. To prove (i) by means of Definition (8.3), we must show that for every $\epsilon > 0$, there exists a positive number N such that

$$\text{if} \quad n > N, \quad \text{then} \quad |r^n - 0| < \epsilon.$$

The inequality $|r^n - 0| < \epsilon$ is equivalent to each inequality in the following list:

$$|r|^n < \epsilon, \qquad \ln|r|^n < \ln\epsilon, \qquad n\ln|r| < \ln\epsilon, \qquad n > \frac{\ln\epsilon}{\ln|r|}$$

The final inequality sign is reversed because $\ln|r|$ is negative if $0 < |r| < 1$. The last inequality in the list provides a clue to the choice of N. Let us consider the two cases $\epsilon < 1$ and $\epsilon \geq 1$ separately. If $\epsilon < 1$, then $\ln\epsilon < 0$ and we let $N = \ln\epsilon/\ln|r| > 0$. In this event, if $n > N$, then the last inequality in the list is true and hence so is the first, which is what we wished to prove. If $\epsilon \geq 1$, then $\ln\epsilon \geq 0$ and hence $\ln\epsilon/\ln|r| \leq 0$. In this case, if N is *any* positive number, then whenever $n > N$, the last inequality in the list is again true.

To prove (ii), let $|r| > 1$ and consider any positive real number P. The following inequalities are equivalent:

$$|r|^n > P, \qquad \ln|r|^n > \ln P, \qquad n\ln|r| > \ln P, \qquad n > \frac{\ln P}{\ln|r|}$$

If we choose $N = \ln P/\ln|r|$, then whenever $n > N$, the last inequality is true and hence so is the first—that is, $|r|^n > P$. By Definition (8.4), this means that $\lim_{n\to\infty} |r|^n = \infty$. ■

EXAMPLE 6 List the first four terms of the sequence, and determine whether the sequence converges or diverges.

(a) $\left\{\left(-\frac{2}{3}\right)^n\right\}$ **(b)** $\{(1.01)^n\}$

SOLUTION

(a) The first four terms of $\left\{\left(-\frac{2}{3}\right)^n\right\}$ are

$$-\tfrac{2}{3}, \tfrac{4}{9}, -\tfrac{8}{27}, \tfrac{16}{81}.$$

If we let $r = -\frac{2}{3}$, then, by Theorem (8.6)(i), with $|r| = \frac{2}{3} < 1$,

$$\lim_{n\to\infty} \left(-\tfrac{2}{3}\right)^n = 0.$$

Hence, the sequence converges to 0.

(b) The first four terms of $\{(1.01)^n\}$ are

$$1.01, 1.0201, 1.030301, 1.04060401.$$

If we let $r = 1.01$, then, by Theorem (8.6)(ii),

$$\lim_{n\to\infty} (1.01)^n = \infty.$$

Since the limit does not exist, the sequence diverges.

Limit theorems that are analogous to those stated in Chapter 1 for sums, differences, products, and quotients of functions can be established for sequences. For example, if $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right),$$

and so on.

If $a_n = c$ for every n, the sequence $\{a_n\}$ is $c, c, \ldots, c, \ldots$ and

$$\lim_{n\to\infty} c = c.$$

Similarly, if c is a real number and k is a positive rational number, then, as in Theorem (1.18),

$$\lim_{n\to\infty} \frac{c}{n^k} = 0.$$

EXAMPLE 7 Find the limit of the sequence $\left\{\dfrac{2n^2}{5n^2 - 3}\right\}$.

SOLUTION To find $\lim_{n\to\infty} a_n$, where $a_n = 2n^2/(5n^2 - 3)$, we divide both the numerator and the denominator of a_n by n^2 and apply limit theorems to obtain

$$\lim_{n\to\infty} \frac{2n^2}{5n^2 - 3} = \lim_{n\to\infty} \frac{2}{5 - (3/n^2)} = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} [5 - (3/n^2)]}$$

$$= \frac{2}{\lim_{n\to\infty} 5 - \lim_{n\to\infty} (3/n^2)} = \frac{2}{5 - 0} = \frac{2}{5}.$$

Hence, the sequence has the limit $\frac{2}{5}$. We can also prove this by applying l'Hôpital's rule to $2x^2/(5x^2 - 3)$.

The next theorem, which is similar to Theorem (1.15), states that if the terms of a sequence are always sandwiched between corresponding terms of two sequences that have the same limit L, then the given sequence also has the limit L.

Sandwich Theorem for Sequences 8.7

If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences and $a_n \le b_n \le c_n$ for every n and if

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then

$$\lim_{n\to\infty} b_n = L.$$

EXAMPLE ■ 8 Find the limit of the sequence $\left\{\dfrac{\cos^2 n}{3^n}\right\}$.

SOLUTION Since $0 < \cos^2 n < 1$ for every positive integer n,

$$0 < \frac{\cos^2 n}{3^n} < \frac{1}{3^n}.$$

Applying Theorem (8.6)(i) with $r = \frac{1}{3}$, we have

$$\lim_{n\to\infty} \frac{1}{3^n} = \lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0.$$

Moreover, $\lim_{n\to\infty} 0 = 0$. It follows from the sandwich theorem (8.7), with $a_n = 0$, $b_n = (\cos^2 n)/3^n$, and $c_n = \left(\frac{1}{3}\right)^n$, that

$$\lim_{n\to\infty} \frac{\cos^2 n}{3^n} = 0.$$

Hence, the limit of the sequence is 0.

The next theorem can be proved using Definition (8.3).

Theorem 8.8

Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE ■ 9 Suppose the nth term of a sequence is

$$a_n = (-1)^{n+1}\frac{1}{n}.$$

Prove that $\lim_{n\to\infty} a_n = 0$.

SOLUTION The terms of the sequence are alternately positive and negative. For example, the first seven terms are

$$1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \tfrac{1}{5}, -\tfrac{1}{6}, \tfrac{1}{7}.$$

Since

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

it follows from Theorem (8.8) that $\lim_{n\to\infty} a_n = 0$.

A sequence is **monotonic** if successive terms are nondecreasing:

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots;$$

or if they are nonincreasing:

$$a_1 \ge a_2 \ge \cdots \ge a_n \ge \cdots.$$

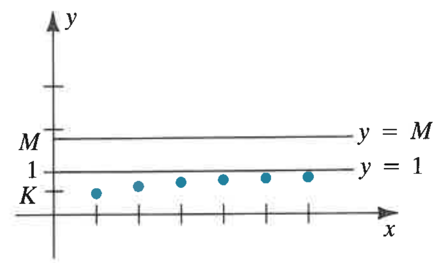
A sequence is **bounded** if there is a positive real number M such that $|a_k| \le M$ for every k. To illustrate, the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$$

is both monotonic (the terms are increasing) and bounded (since we have $k/(k+1) < 1$ for every k). The graph of the sequence is illustrated in Figure 8.6. Note that any number $M \ge 1$ is a bound for the sequence; however, if $K < 1$, then K is not a bound, since $K < k/(k+1)$ when k is sufficiently large.

The next theorem is fundamental for later developments.

Figure 8.6 $\left\{\dfrac{n}{n+1}\right\}$



Theorem 8.9

A bounded, monotonic sequence has a limit.

To prove Theorem (8.9), it is necessary to use an important property of real numbers. Let us first state several definitions. If S is a nonempty set of real numbers, then a real number u is an **upper bound** of S if $x \le u$ for every x in S. A number v is a **least upper bound** of S if v is an upper bound and no number less than v is an upper bound of S. Thus, *the least upper bound is the smallest real number that is greater than or equal to every number in S.* To illustrate, if S is the open interval (a, b), then any number greater than b is an upper bound of S; however, the least upper bound of S is unique and equals b. The monotonic sequence $\{n/(n+1)\}$ illustrated in Figure 8.6 has the least upper bound (and limit) 1.

The following statement is an axiom for the real number system.

Completeness Property 8.10

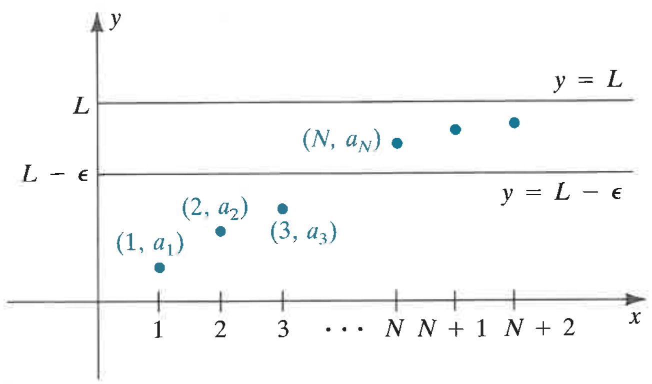
If a nonempty set S of real numbers has an upper bound, then S has a least upper bound.

PROOF OF THEOREM (8.9) Let $\{a_n\}$ be a bounded, monotonic sequence with nondecreasing terms. Thus,

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots,$$

and there is a number M such that $a_k \le M$ for every positive integer k. Since M is an upper bound for the set S of all numbers in the sequence, it follows from the completeness property (8.10) that S has a least upper bound L such that $L \le M$ (see Figure 8.7).

Figure 8.7



If $\epsilon > 0$, then $L - \epsilon$ is not an upper bound of S, and hence at least one term of $\{a_n\}$ is greater than $L - \epsilon$; that is,

$$L - \epsilon < a_N \quad \text{for some positive integer } N,$$

as shown in Figure 8.7. Since the terms of $\{a_n\}$ are nondecreasing,

$$a_N \le a_{N+1} \le a_{N+2} \le \cdots$$

and, therefore,

$$L - \epsilon < a_n \quad \text{for every } n \ge N.$$

It follows that if $n > N$, then

$$0 \le L - a_n < \epsilon \quad \text{or} \quad |L - a_n| < \epsilon.$$

By Definition (8.3), this result means that

$$\lim_{n\to\infty} a_n = L \le M.$$

That is, $\{a_n\}$ has a limit.

We may obtain the proof for a sequence $\{a_n\}$ of nonincreasing terms in a similar fashion or by considering the sequence $\{-a_n\}$. ■

Programmable calculators and computers have features that allow us to easily investigate sequences that are defined recursively.

EXAMPLE ■ 10 If a sequence is defined recursively by $a_1 = 5$, $a_{k+1} = f(a_k)$, where $f(x) = e^{x/4} - 2$, find the first five terms and discuss what happens to the terms of the sequence as k increases.

SOLUTION On a programmable or graphing calculator that permits storage of variables, we can easily compute many terms of this sequence.

For example, on most graphing calculators, the first term can be stored in the variable memory by the command.

```
5 → X.
```

The command line

```
[e^x](X/4) − 2 → X [ENTER]
```

calculates the second term. Repeatedly pressing [ENTER] causes the previous command to execute again, using the most recently stored value in X. This repetition gives the successive terms in the sequence, which are approximately

$$5, \quad 1.490343, \quad -0.548517, \quad -1.128142, \quad -1.245753$$

Since the range of the function $f(x) = e^{x/4} - 2$ is $(-2, \infty)$, this sequence is bounded below by the number -2. So long as the sequence continues to be monotone decreasing, it must converge. With this assurance, we look at more terms to approximate the limit:

$$a_{10} \approx -1.27248018104, \quad a_{15} \approx -1.27248552218,$$
$$a_{20} \approx -1.27248552324$$

The sequence appears to converge to a number that is approximately equal to -1.27248552.

There are many applications of sequences. In particular, sequences may be applied to the investigation of the time course of an S → I → S epidemic.* Suppose that physicians issue daily reports indicating the number of persons who have become infected with a particular disease and those who have been cured. We shall label the reporting days as $1, 2, \ldots, n, \ldots$ and let N denote the total population. In addition, let

$$I_n = \text{number of persons who have the disease on day } n$$
$$F_n = \text{number of newly infected persons on day } n$$
$$C_n = \text{number of persons cured on day } n.$$

It follows that for every $n \ge 1$,

$$I_{n+1} = I_n + F_{n+1} - C_{n+1}.$$

Suppose health officials decide that the number of new cases on a given day is directly proportional to the product of the number ill and the number not infected on the previous day. (This is known as the *law of mass action* and is typical of a population of students on a college campus.) Moreover, suppose that the number cured each day is directly proportional to the number ill the previous day. Hence,

$$F_{n+1} = aI_n(N - I_n) \quad \text{and} \quad C_{n+1} = bI_n,$$

where a and b are positive constants that can be approximated from early data. Substituting in the preceding formula for I_{n+1}, we have

$$I_{n+1} = I_n + aI_n(N - I_n) - bI_n.$$

*The notation S → I → S is an abbreviation for *Susceptible* → *Infected* → *Susceptible* and signifies that an infected person who becomes cured is not immune to the disease, but may contract it again. Examples of such diseases are gonorrhea and strep throat. Recall the discussions of epidemics in Section 3.8.

In the early stages of an epidemic, I_n will be very small compared to N, and from the point of view of public health, it is better to *overestimate* the number ill than to underestimate and be unprepared for the spread of the disease. With this in mind, we drop the term $-a_n I_n^2$ in the formula for I_{n+1} and investigate the early dynamics of the epidemic by examining the equation

$$I_{n+1} = I_n + aNI_n - bI_n = (1 + aN - b)I_n.$$

If we let $r = 1 + aN - b$, then $I_{n+1} = rI_n$ and, therefore,

$$I_2 = rI_1, \quad I_3 = rI_2 = r^2 I_1, \quad I_4 = rI_3 = r^3 I_1, \quad \ldots, \quad I_n = r^{n-1} I_1, \quad \ldots.$$

This gives us the following sequence of numbers of infected individuals:

$$I_1, rI_1, r^2 I_1, \ldots, r^{n-1} I_1, \ldots$$

The number $r = 1 + aN - b$ is of critical import. If $r > 1$, then, by Theorem (8.6)(ii), $\lim_{n\to\infty} I_n = \infty$ and an epidemic is in progress. In this case, when n is large, I_n is no longer small compared to N, and the formula for I_{n+1} becomes invalid. If $r < 1$, then, by Theorem (8.6)(i), $\lim_{n\to\infty} I_n = 0$ and health officials need not be concerned. The case $r = 1$ results in the constant sequence $I_1, I_1, \ldots, I_1, \ldots$

EXERCISES 8.1

Exer. 1–16: The expression is the nth term a_n of a sequence $\{a_n\}$. Find the first four terms and $\lim_{n\to\infty} a_n$, if it exists.

1 $\dfrac{n}{3n+2}$

2 $\dfrac{6n-5}{5n+1}$

3 $\dfrac{7-4n^2}{3+2n^2}$

4 $\dfrac{4}{8-7n}$

5 -5

6 $\sqrt{2}$

7 $\dfrac{(2n-1)(3n+1)}{n^3+1}$

8 $8n+1$

9 $\dfrac{2}{\sqrt{n^2+9}}$

10 $\dfrac{100n}{n^{3/2}+4}$

11 $(-1)^{n+1}\dfrac{3n}{n^2+4n+5}$

12 $(-1)^{n+1}\dfrac{\sqrt{n}}{n+1}$

13 $1+(0.1)^n$

14 $1-\dfrac{1}{2^n}$

15 $1+(-1)^{n+1}$

16 $\dfrac{n+1}{\sqrt{n}}$

Exer. 17–42: Determine whether the sequence converges or diverges, and if it converges, find the limit.

17 $\left\{6\left(-\frac{5}{6}\right)^n\right\}$

18 $\left\{8-\left(\frac{7}{8}\right)^n\right\}$

19 $\{\arctan n\}$

20 $\left\{\dfrac{\tan^{-1} n}{n}\right\}$

21 $\{1000-n\}$

22 $\left\{\dfrac{(1.0001)^n}{1000}\right\}$

23 $\left\{(-1)^n\dfrac{\ln n}{n}\right\}$

24 $\left\{\dfrac{n^2}{\ln(n+1)}\right\}$

25 $\left\{\dfrac{4n^4+1}{2n^2-1}\right\}$

26 $\left\{\dfrac{\cos n}{n}\right\}$

27 $\left\{\dfrac{e^n}{n^4}\right\}$

28 $\{e^{-n}\ln n\}$

29 $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$

30 $\{(-1)^n n^3 3^{-n}\}$

31 $\{2^{-n}\sin n\}$

32 $\left\{\dfrac{4n^3+5n+1}{2n^3-n^2+5}\right\}$

33 $\left\{\dfrac{n^2}{2n-1}-\dfrac{n^2}{2n+1}\right\}$

34 $\left\{n\sin\dfrac{1}{n}\right\}$

35 $\{\cos \pi n\}$

36 $\left\{4+\sin\frac{1}{2}\pi n\right\}$

37 $\{n^{1/n}\}$

38 $\left\{\dfrac{n^2}{2^n}\right\}$

39 $\left\{\dfrac{n^{-10}}{\sec n}\right\}$

40 $\left\{(-1)^n\dfrac{n^2}{1+n^2}\right\}$

41 $\{\sqrt{n+1}-\sqrt{n}\}$

42 $\{\sqrt{n^2+n}-n\}$

43 A stable population of 35,000 birds lives on three islands. Each year, 10% of the population on island A migrates to island B, 20% of the population on island B migrates to island C, and 5% of the population on island C migrates to island A. Let A_n, B_n, and C_n denote the numbers of birds on islands A, B, and C, respectively, in year n before migration takes place.

(a) Show that

$$A_{n+1} = 0.9A_n + 0.05C_n$$
$$B_{n+1} = 0.1A_n + 0.80B_n$$

and

$$C_{n+1} = 0.95C_n + 0.20B_n.$$

(b) Assuming that $\lim_{n\to\infty} A_n$, $\lim_{n\to\infty} B_n$, and $\lim_{n\to\infty} C_n$ exist, approximate the number of birds on each island after many years.

44 A bobcat population is classified by age as kittens (less than one year old) and adults (at least one year old). All adult females, including those born the preceding year, have a litter each June, with an average litter size of three kittens. The survival rate of kittens is 50%, whereas that of adults is $66\frac{2}{3}\%$ per year. Let K_n be the number of newborn kittens in June of the nth year, let A_n be the number of adults, and assume that the ratio of males to females is always 1.

(a) Show that

$$K_{n+1} = \tfrac{3}{2}A_{n+1} \quad \text{and} \quad A_{n+1} = \tfrac{2}{3}A_n + \tfrac{1}{2}K_n.$$

(b) Conclude that $A_{n+1} = \frac{17}{12}A_n$ and $K_{n+1} = \frac{17}{12}K_n$, and that $A_n = \left(\frac{17}{12}\right)^{n-1} A_1$ and $K_n = \left(\frac{17}{12}\right)^{n-1} K_1$. What can you conclude about the population?

c 45 Terms of the sequence defined recursively by $a_1 = 5$ and $a_{k+1} = \sqrt{a_k}$ may be generated on a calculator. We enter 5 and then either repeatedly press [$\sqrt{x}$] for a scientific calculator or repeatedly use [$\sqrt{\ }$] [ANS] on a graphing calculator.

(a) Describe what happens to the terms of the sequence as k increases.

(b) Show that $a_n = 5^{1/2^n}$, and find $\lim_{n\to\infty} a_n$.

c 46 If a sequence is generated by entering a number and repeatedly performing the operation of [$1/x$], under what conditions does the sequence have a limit?

c 47 Terms of the sequence defined recursively by $a_1 = 1$ and $a_{k+1} = \cos a_k$ may be generated on a calculator. On most graphing calculators, we enter $1 \to$ A and then [cos] A $\to$ A [ENTER]. Repeatedly pressing [ENTER] will produce successive terms in the sequence.

(a) Describe what happens to the terms of the sequence as k increases.

(b) Assuming that $\lim_{n\to\infty} a_n = L$, prove that $L = \cos L$. (*Hint*: $\lim_{n\to\infty} a_{n+1} = L$.)

c 48 A sequence $\{x_n\}$ is defined recursively by the formula $x_{k+1} = x_k - \tan x_k$.

(a) If $x_1 = 3$, approximate the first five terms of the sequence. Predict $\lim_{n\to\infty} x_n$.

(b) If $x_1 = 6$, approximate the first five terms of the sequence. Predict $\lim_{n\to\infty} x_n$.

(c) Assuming that $\lim_{n\to\infty} x_n = L$, prove that $L = \pi n$ for some integer n.

c 49 Approximations to $\sqrt{N}$ may be generated from the sequence defined recursively by

$$x_1 = \frac{N}{2}, \qquad x_{k+1} = \frac{1}{2}\left(x_k + \frac{N}{x_k}\right).$$

(a) Approximate x_2, x_3, x_4, x_5, x_6 if $N = 10$.

(b) Assuming that $\lim_{n\to\infty} x_n = L$, prove that $L = \sqrt{N}$.

c 50 The famous *Fibonacci sequence* is defined recursively by $a_{k+1} = a_k + a_{k-1}$ with $a_1 = a_2 = 1$.

(a) Find the first ten terms of the sequence.

(b) The terms of the sequence $r_k = a_{k+1}/a_k$ give approximations to τ, the *golden ratio*. Approximate the first ten terms of this sequence.

(c) Assuming that $\lim_{n\to\infty} r_n = \tau$, prove that

$$\tau = \tfrac{1}{2}(1 + \sqrt{5}).$$

c **Exer. 51–52: If f is differentiable, then a sequence $\{a_n\}$ defined recursively by $a_{k+1} = f(a_k)$, for $k \geq 1$, will converge for any a_1 if the derivative f' is continuous and $|f'(x)| \leq B < 1$ for some positive constant B. (a) For the given f, verify that the sequence $\{a_n\}$ converges for any a_1 by finding a suitable B. (b) Approximate, to two decimal places, $\lim_{n\to\infty} a_n$, if $a_1 = 1$ and also if $a_1 = -100$.**

51 $f(a_k) = \frac{1}{4}\sin a_k \cos a_k + 1$

52 $f(a_k) = \dfrac{a_k^2}{a_k^2+1} + 2$

Mathematicians and Their Times

CARL FRIEDRICH GAUSS

BORN IN A HUMBLE COTTAGE in Germany, Carl Friedrich Gauss (1777–1855), arguably the greatest mathematician who ever lived, seemed at first destined for a life of poverty and hard physical labor. His father, Gerhard, worked as a gardener and bricklayer, and he expected his son to do likewise. Although he was scrupulously honest, Gerhard's harsh ways came close to brutality as he tried to prevent Carl from acquiring a suitable education. Fortunately, the boy's mother, Dorothea, recognized and encouraged Carl's talents.

Gauss's mental prowess was evident at an extremely young age. Before his third birthday, he found an error in his father's calculation of a weekly payroll. His schoolmaster confessed that at age 10, Gauss had mastered arithmetic so well that "I can teach him nothing more." Eventually, the Duke of Brunswick learned of Gauss's abilities and took responsibility for financing his education. At age 15, Gauss mastered infinite series (the subject of this chapter) and gave the first rigorous proof of the general binomial theorem, a result that had been conjectured and used by Newton.

Intellectual historians see Gauss as a transition figure. Felix Klein describes Gauss as "the point where historical epochs separate: he is the highest development of the past, which he closes, and the foundation of the new . . . Gauss is like the highest peak among our Bavarian mountains . . . the gradually ascending foothills culminate in the one gigantic Colossus, which falls away steeply into the lowlands of a new formation, into which its spurs reach out for many miles and in which the waters gushing from it begets new life."* Gauss saw the essence of analysis, the branch of mathematics including calculus, as the rigorous use of infinite processes. Newton, Leibniz, Euler, and Lagrange all manipulated infinite series masterfully but failed to prove that the results obtained were correct. Gauss's insistence on rigor fundamentally changed mathematics.

Gauss made profound discoveries in nearly all areas of pure and applied mathematics. His work established new directions in number theory, algebra, non-Euclidean geometry, statistics, differential geometry, analytical dynamics, potential theory, magnetism, and optics. His doctoral thesis, for example, gave the first proof of the fundamental theorem of algebra: Every polynomial with complex coefficients has at least one complex root. Not only did this work of a 22-year-old establish an important theorem, it also saw the introduction of a coherent account of complex numbers and their geometric representation, a subject of central importance in mathematics.

*Felix Klein, *Vorlesungen über die Entwicklung der Mathematik*, Teil I. Berlin: J. Springer, 1926, p. 62.

8.2 CONVERGENT OR DIVERGENT SERIES

We may use sequences to define expressions of the form

$$0.6 + 0.06 + 0.006 + 0.0006 + 0.00006 + \cdots,$$

where the three dots indicate that the sum continues indefinitely. In Definition (8.11), we call such an expression an *infinite series*. Since only finite sums may be added algebraically, we must *define* what is meant by this "infinite sum." As we shall see, the key to the definition is to consider the *sequence of partial sums* $\{S_n\}$, where S_k *is the sum of the first k numbers of the infinite series*. For the preceding illustration,

$$S_1 = 0.6$$
$$S_2 = 0.6 + 0.06 = 0.66$$
$$S_3 = 0.6 + 0.06 + 0.006 = 0.666$$
$$S_4 = 0.6 + 0.06 + 0.006 + 0.0006 = 0.6666$$

and so on. Thus, the sequence of partial sums $\{S_n\}$ may be written

$$0.6, 0.66, 0.666, 0.6666, 0.66666, \ldots.$$

It will follow from Theorem (8.15) that

$$S_n \to \tfrac{2}{3} \quad \text{as} \quad n \to \infty.$$

From an intuitive point of view, the more numbers of the infinite series that we add, the closer the sum gets to $\frac{2}{3}$. Thus, we write

$$\tfrac{2}{3} = 0.6 + 0.06 + 0.006 + 0.0006 + \cdots$$

and call $\frac{2}{3}$ the *sum* of the infinite series.

With this special case in mind, let us introduce terminology that will be used throughout the remainder of this chapter. In the following definition, we assume that $a_1, a_2, \ldots, a_n, \ldots$ are the terms of some sequence.

Definition 8.11

An **infinite series** (or simply a **series**) is an expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots,$$

or, in summation notation,

$$\sum_{n=1}^{\infty} a_n, \quad \text{or} \quad \sum a_n.$$

Each number a_k is a **term** of the series, and a_n is the ***n*****th term**.

Sometimes there is confusion between the concept of a series and that of a sequence. Remember that a series is an expression that represents an *infinite sum* of numbers. A sequence is a collection of numbers that are in one-to-one correspondence with the positive integers. The sequence of partial sums in the next definition is a special type of sequence that we obtain by using the terms of a series.

As in the special case introduced at the beginning of this section, we define the *sequence of partial sums* of a series as follows.

Definition 8.12

(i) The ***k*****th partial sum** S_k of the series $\sum a_n$ is

$$S_k = a_1 + a_2 + \cdots + a_k.$$

(ii) The **sequence of partial sums** of the series $\sum a_n$ is

$$S_1, S_2, S_3, \ldots, S_n, \ldots.$$

By Definition (8.12)(i),

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4. \end{aligned}$$

To calculate S_5, S_6, S_7, and so on, we add more terms of the series. Thus, S_{1000} is the sum of the first one thousand terms of $\sum a_n$. If the sequence $\{S_n\}$ has a limit S, we call S the *sum* of the series $\sum a_n$, as in the next definition.

Definition 8.13

A series $\sum a_n$ is **convergent** (or **converges**) if its sequence of partial sums $\{S_n\}$ converges—that is, if

$$\lim_{n\to\infty} S_n = S \quad \text{for some real number } S.$$

The limit S is the **sum** of the series $\sum a_n$, and we write

$$S = a_1 + a_2 + \cdots + a_n + \cdots.$$

The series $\sum a_n$ is **divergent** (or **diverges**) if $\{S_n\}$ diverges. A divergent series has no sum.

For most series, it is very difficult to find a formula for S_n. However, as we shall see in later sections, it may be possible to establish the convergence or divergence of a series using other methods. In the remainder of this section, we consider several important series for which we *can* find a formula for S_n.

EXAMPLE 1 Given the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots,$$

(a) find S_1, S_2, S_3, S_4, S_5, and S_6

(b) find S_n

(c) show that the series converges and find its sum

SOLUTION

(a) By Definition (8.12), the first six partial sums are as follows:

$$\begin{aligned} S_1 &= \frac{1}{1\cdot 2} = \frac{1}{2} \\ S_2 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3} \\ S_3 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4} \\ S_4 &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5} \\ S_5 &= S_4 + a_5 = \frac{4}{5} + \frac{1}{5\cdot 6} = \frac{5}{6} \\ S_6 &= S_5 + a_6 = \frac{5}{6} + \frac{1}{6\cdot 7} = \frac{6}{7} \end{aligned}$$

(b) To find S_n, we shall write the terms of the series in a different way. Using partial fractions, we can show that

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Consequently, the nth partial sum of the series may be written

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Regrouping, we see that all numbers except the first and last cancel, and hence

$$S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

(c) Using the formula for S_n obtained in part (b), we obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Thus, the series converges and has the sum 1. As in Definition (8.13), we may write

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots.$$

The series $\sum 1/[n(n+1)]$ of Example 1 is called a **telescoping series**, since writing S_n as shown in part (b) of the solution causes the terms to *telescope* to $1 - [1/(n+1)]$.

EXAMPLE ■ 2 Given the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 + (-1) + 1 + (-1) + \cdots + (-1)^{n-1} + \cdots,$$

(a) find S_1, S_2, S_3, S_4, S_5, and S_6
(b) find S_n
(c) show that the series diverges

SOLUTION

(a) By Definition (8.12),

$$S_1 = 1, \quad S_2 = 0, \quad S_3 = 1, \quad S_4 = 0, \quad S_5 = 1, \quad \text{and} \quad S_6 = 0.$$

(b) We can write S_n as follows:

$$S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(c) Since the sequence of partial sums $\{S_n\}$ oscillates between 1 and 0, it follows that $\lim_{n\to\infty} S_n$ does not exist. Hence, the series diverges.

EXAMPLE ■ 3 Prove that the following series is divergent:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

SOLUTION Let us group the terms of the series as follows:

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \cdots + \tfrac{1}{16}) + (\tfrac{1}{17} + \cdots + \tfrac{1}{32}) + \cdots$$

Note that each group contains twice the number of terms as the preceding group. Moreover, since increasing the denominator *decreases* the value of a fraction, we have the following:

$$\tfrac{1}{3} + \tfrac{1}{4} > \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$
$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{2}$$
$$\tfrac{1}{9} + \tfrac{1}{10} + \cdots + \tfrac{1}{16} > \tfrac{1}{16} + \tfrac{1}{16} + \cdots + \tfrac{1}{16} = \tfrac{1}{2}$$
$$\tfrac{1}{17} + \tfrac{1}{18} + \cdots + \tfrac{1}{32} > \tfrac{1}{32} + \tfrac{1}{32} + \cdots + \tfrac{1}{32} = \tfrac{1}{2}$$

Since the sum of the terms within each set of parentheses is greater than $\frac{1}{2}$, we obtain the following inequalities:

$$S_4 > 1 + \tfrac{1}{2} + \tfrac{1}{2} > 3(\tfrac{1}{2})$$
$$S_8 > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} > 4(\tfrac{1}{2})$$
$$S_{16} > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} > 5(\tfrac{1}{2})$$
$$S_{32} > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} > 6(\tfrac{1}{2})$$

It can be shown, by mathematical induction, that

$$S_{2^k} > (k+1)(\tfrac{1}{2}) \quad \text{for every positive integer } k.$$

It follows that S_n can be made as large as desired by taking n sufficiently large—that is, $\lim_{n\to\infty} S_n = \infty$. Since $\{S_n\}$ diverges, the given series diverges.

We can add numerical support to the inequalities in Example 3 by computing a few of the partial sums on a calculator or a computer. Using $k = 9, 10, 11,$ and 12 gives

$$S_{512} \approx 6.81652 \qquad S_{1024} \approx 7.50918$$
$$S_{2048} \approx 8.20208 \qquad S_{4096} \approx 8.89510.$$

Although a calculator or a computer will not provide a proof of convergence or divergence, we frequently have reason to compute partial sums.

The series in Example 3 will be useful in later developments. It is given the following special name.

Definition 8.14

The **harmonic series** is the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

In the next section, we shall give another proof of the divergence of the harmonic series.

Certain types of series occur frequently in solutions of applied problems. One of the most important is the **geometric series**

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots,$$

where a and r are real numbers, with $a \neq 0$.

Theorem 8.15

Let $a \neq 0$. The geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

(i) converges and has the sum $S = \dfrac{a}{1-r}$ if $|r| < 1$

(ii) diverges if $|r| \geq 1$

PROOF If $r = 1$, then $S_n = a + a + \cdots + a = na$ and the series diverges, since $\lim_{n\to\infty} S_n$ does not exist.

If $r = -1$, then $S_k = a$ if k is odd and $S_k = 0$ if k is even. Since the sequence of partial sums oscillates between a and 0, the series diverges.

If $r \neq 1$, then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Subtracting corresponding sides of these equations, we obtain

$$(1-r)S_n = a - ar^n.$$

Dividing both sides by $1 - r$ gives us

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right)\\ &= \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{ar^n}{1-r}\\ &= \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n.\end{aligned}$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, by Theorem (8.6)(i), and hence

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = S.$$

If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist, by Theorem (8.6)(ii), and hence $\lim_{n\to\infty} S_n$ does not exist. In this case, the series diverges. ■

EXAMPLE 4 Prove that the following series converges, and find its sum:

$$0.6 + 0.06 + 0.006 + \cdots + \frac{6}{10^n} + \cdots$$

SOLUTION This is the series considered at the beginning of this section. It is geometric with $a = 0.6$ and $r = 0.1$. Since $|r| < 1$, we conclude from Theorem (8.15)(i) that the series converges and has the sum

$$S = \frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{2}{3}.$$

Thus, $$\frac{2}{3} = 0.6 + 0.06 + 0.006 + \cdots + \frac{6}{10^n} + \cdots.$$

This justifies the nonterminating decimal notation $\frac{2}{3} = 0.66666\ldots$.

EXAMPLE 5 Prove that the following series converges, and find its sum:

$$2 + \frac{2}{3} + \frac{2}{3^2} + \cdots + \frac{2}{3^{n-1}} + \cdots$$

SOLUTION The series converges, since it is geometric with $r = \frac{1}{3} < 1$. By Theorem (8.15)(i), the sum is

$$S = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3.$$

Theorem 8.16

If a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

PROOF The nth term a_n of the series can be expressed as

$$a_n = S_n - S_{n-1}.$$

If S is the sum of the series $\sum a_n$, then we know $\lim_{n\to\infty} S_n = S$ and also $\lim_{n\to\infty} S_{n-1} = S$. Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0. \quad ■$$

CAUTION The preceding theorem states that *if* a series converges, *then* the limit of its nth term a_n as $n \to \infty$ is 0. The converse is false—that is, *if* $\lim_{n\to\infty} a_n = 0$, *it does not necessarily follow* that the series $\sum a_n$ is convergent. The harmonic series (8.14) is an illustration of a divergent series $\sum a_n$ for which $\lim_{n\to\infty} a_n = 0$. Consequently, to establish convergence of a series, *it is not enough* to prove that $\lim_{n\to\infty} a_n = 0$, since that may be true for divergent as well as for convergent series.

The next result is a corollary of Theorem (8.16) and the preceding remarks.

nth-Term Test 8.17

(i) If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

(ii) If $\lim_{n\to\infty} a_n = 0$, then further investigation is necessary to determine whether the series $\sum a_n$ is convergent or divergent.

The next illustration shows how to apply the nth-term test to a series.

ILLUSTRATION

Series	nth-term test	Conclusion
$\sum_{n=1}^{\infty} \frac{n}{2n+1}$	$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$	Diverges, by (8.17)(i)
$\sum_{n=1}^{\infty} \frac{1}{n^2}$	$\lim_{n\to\infty} \frac{1}{n^2} = 0$	Further investigation is necessary, by (8.17)(ii)
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$	Further investigation is necessary, by (8.17)(ii)
$\sum_{n=1}^{\infty} \frac{e^n}{n}$	$\lim_{n\to\infty} \frac{e^n}{n} = \infty$	Diverges, by (8.17)(i)

We shall see in the next section that the second series in the illustration converges and that the third series diverges.

The next theorem states that if corresponding terms of two series are identical after a certain term, then both series converge or both series diverge.

Theorem 8.18

If $\sum a_n$ and $\sum b_n$ are series such that $a_j = b_j$ for every $j > k$, where k is a positive integer, then both series converge or both series diverge.

PROOF By hypothesis, we may write the following:

$$\sum a_n = a_1 + a_2 + \cdots + a_k + a_{k+1} + \cdots + a_n + \cdots$$
$$\sum b_n = b_1 + b_2 + \cdots + b_k + a_{k+1} + \cdots + a_n + \cdots$$

Let S_n and T_n denote the nth partial sums of $\sum a_n$ and $\sum b_n$, respectively. It follows that if $n \geq k$, then

$$S_n - S_k = T_n - T_k,$$

or

$$S_n = T_n + (S_k - T_k).$$

Consequently, $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n + (S_k - T_k),$$

and hence either both of the limits exist or both do not exist. This gives us the desired conclusion. If both series converge, then their sums differ by $S_k - T_k$. ■

Theorem (8.18) implies that changing a finite number of terms of a series has no effect on its convergence or divergence (although it does change the sum of a convergent series). In particular, if we replace the first k terms of $\sum a_n$ by 0, convergence is unaffected. It follows that the series

$$a_{k+1} + a_{k+2} + \cdots + a_n + \cdots$$

converges or diverges if $\sum a_n$ converges or diverges, respectively. The series $a_{k+1} + a_{k+2} + \cdots$ is obtained from $\sum a_n$ by **deleting the first k terms.**

Let us state this result for reference as follows.

Theorem 8.19

For any positive integer k, the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots \quad \text{and} \quad \sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \cdots$$

either both converge or both diverge.

EXAMPLE 6 Show that the following series converges:

$$\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+2)(n+3)} + \cdots$$

SOLUTION The series can be obtained by deleting the first two terms of the convergent telescoping series of Example 1. Hence, by Theorem (8.19), the given series converges.

The proof of the next theorem follows directly from Definition (8.13).

Theorem 8.20

If $\sum a_n$ and $\sum b_n$ are convergent series with sums A and B, respectively, then

(i) $\sum (a_n + b_n)$ converges and has sum $A + B$

(ii) $\sum ca_n$ converges and has sum cA for every real number c

(iii) $\sum (a_n - b_n)$ converges and has sum $A - B$

It is also easy to show that if $\sum a_n$ diverges, then so does $\sum ca_n$ for every $c \neq 0$.

EXAMPLE ■ 7 Prove that the following series converges, and find its sum:

$$\sum_{n=1}^{\infty}\left[\frac{7}{n(n+1)}+\frac{2}{3^{n-1}}\right]$$

SOLUTION The telescoping series $\sum 1/[n(n+1)]$ was considered in Example 1, where we found that it converges and has the sum 1. Using Theorem (8.20)(ii) with $c=7$ and $a_n=1/[n(n+1)]$, we see that the series $\sum 7/[n(n+1)]$ converges and has the sum $7(1)=7$.

The geometric series $\sum 2/3^{n-1}$ converges and has the sum 3 (see Example 5). Hence, by Theorem (8.20)(i), the given series converges and has the sum $7+3=10$.

Theorem 8.21

If $\sum a_n$ is a convergent series and $\sum b_n$ is divergent, then the series $\sum(a_n+b_n)$ is divergent.

PROOF As in the statement of the theorem, let $\sum a_n$ be convergent and $\sum b_n$ be divergent. We shall give an indirect proof—that is, we shall assume that the conclusion of the theorem is *false* and arrive at a contradiction. Thus, *suppose* that $\sum(a_n+b_n)$ is convergent. Applying Theorem (8.20)(iii), we find that the series

$$\sum[(a_n+b_n)-a_n]=\sum b_n$$

is convergent. This result contradicts the fact that $\sum b_n$ is divergent, and hence our supposition is false—that is, $\sum(a_n+b_n)$ is divergent. ■

EXAMPLE ■ 8 Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{5^n}+\frac{1}{n}\right).$$

SOLUTION Since $\sum(1/5^n)$ is a convergent geometric series and $\sum(1/n)$ is the divergent harmonic series, then by Theorem (8.21), the given series diverges.

Infinite series often occur in applications in which we want to estimate the long-term behavior of a process that changes at regularly spaced intervals. The next example illustrates such a situation.

EXAMPLE ■ 9 A chemical plant produces pesticide that contains a molecule potentially harmful to people if the concentration is too high. The plant flushes out the tanks containing the pesticide once a week, and the discharge flows into a river that feeds the water reservoir of a nearby town.

The dangerous molecule breaks down gradually in water so that 90% of the amount remaining each week is dissipated by the end of the next week. Suppose that D units of the molecule are discharged each week.

(a) Find the number of units of the molecule in the river after n weeks.

(b) Estimate the amount of the molecule in the water supply after a very long time.

(c) If the toxic level of the molecule is T units, how large an amount of the molecule can the plant discharge each week?

SOLUTION

(a) Let A_n denote the amount of the molecule in the river immediately after the nth weekly discharge. The amount A_n is equal to the amount of the current discharge plus the amount remaining from previous discharges. Since 90% of the molecule that was in the river the week before is now gone, only 10% remains, so we have

$$A_n=D+\tfrac{1}{10}A_{n-1}.$$

Hence,

$$A_1=D,\qquad A_2=D+\tfrac{1}{10}A_1=D+\tfrac{1}{10}D=D\left(1+\tfrac{1}{10}\right)$$

and

$$A_3=D+\tfrac{1}{10}A_2=D+\tfrac{1}{10}\left[D\left(1+\tfrac{1}{10}\right)\right]=D\left[1+\left(\tfrac{1}{10}\right)+\left(\tfrac{1}{10}\right)^2\right].$$

Similarly, we obtain

$$A_4=D+\tfrac{1}{10}A_3=D\left[1+\left(\tfrac{1}{10}\right)+\left(\tfrac{1}{10}\right)^2+\left(\tfrac{1}{10}\right)^3\right].$$

We can show, by mathematical induction, that the amount of the molecule in the reservoir after n weeks is

$$A_n=D\left[1+\left(\tfrac{1}{10}\right)+\left(\tfrac{1}{10}\right)^2+\cdots+\left(\tfrac{1}{10}\right)^{n-1}\right].$$

(b) As n increases, the amount of the molecule in the water supply approaches

$$D\left[1+\left(\tfrac{1}{10}\right)+\left(\tfrac{1}{10}\right)^2+\cdots+\left(\tfrac{1}{10}\right)^{n-1}+\cdots\right],$$

which is a geometric series with $a=D$ and $r=\frac{1}{10}$. By Theorem (8.15)(i), the series converges to

$$S=\frac{D}{1-\frac{1}{10}}=\frac{10D}{9}.$$

Hence, in the long run, the water supply will contain about $\frac{10}{9}D$ units.

(c) To keep the long-term level below T units, we must have

$$\frac{10D}{9}<T$$

or

$$D<\tfrac{9}{10}T,$$

so the plant can discharge up to 90% of the toxic level each week.

We may apply infinite series to the $S \to I \to S$ epidemic discussed at the end of Section 8.1. Suppose that instead of I_n (the number ill on day n), we are interested in the *total number* S_n of individuals who have been ill at some time between the first and nth days. As in our earlier discussion, let us overestimate S_n by approximating the number F_{n+1} of new cases on day $n+1$ by aNI_n. Thus,

$$\begin{aligned} S_n &= I_1 + F_2 + F_3 + F_4 + \cdots + F_n \\ &= I_1 + aNI_1 + aNI_2 + aNI_3 + \cdots + aNI_{n-1}. \end{aligned}$$

Recalling that $I_n = r^{n-1}I_1$, with $r = 1 + aN - b$, we obtain

$$\begin{aligned} S_n &= I_1 + aNI_1 + aNrI_1 + aNr^2I_1 + \cdots + aNr^{n-2}I_1 \\ &= I_1 + aNI_1(1 + r + r^2 + \cdots + r^{n-2}). \end{aligned}$$

As in the proof of Theorem (8.15), this may be written

$$S_n = I_1 + aNI_1\left(\frac{1}{1-r} - \frac{r^{n-1}}{1-r}\right).$$

If $r < 1$, then

$$\begin{aligned} \lim_{n\to\infty} S_n &= I_1 + aNI_1\left(\frac{1}{1-r}\right) \\ &= I_1\left(1 + \frac{aN}{1-r}\right) \\ &= I_1\left(1 + \frac{aN}{b-aN}\right) \\ &= I_1\left(\frac{b}{b-aN}\right). \end{aligned}$$

If a and b are approximated from early data, this result enables health officials to determine an upper bound for the total number of individuals who will be ill at some stage of the epidemic.

EXERCISES 8.2

Exer. 1–6: Use the method of Example 1 to find (a) S_1, S_2, and S_3; (b) S_n; and (c) the sum of the series, if it converges.

1 $\displaystyle\sum_{n=1}^{\infty} \frac{-2}{(2n+5)(2n+3)}$

2 $\displaystyle\sum_{n=1}^{\infty} \frac{5}{(5n+2)(5n+7)}$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

4 $\displaystyle\sum_{n=1}^{\infty} \frac{-1}{9n^2+3n-2}$

5 $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ (*Hint:* Rationalize the denominator.)

Exer. 7–16: Use Theorem (8.15) to determine whether the geometric series converges or diverges; if it converges, find its sum.

7 $3 + \frac{3}{4} + \cdots + \frac{3}{4^{n-1}} + \cdots$

8 $3 + \frac{3}{(-4)} + \cdots + \frac{3}{(-4)^{n-1}} + \cdots$

9 $1 + \left(\frac{-1}{\sqrt{5}}\right) + \cdots + \left(\frac{-1}{\sqrt{5}}\right)^{n-1} + \cdots$

10 $1 + \left(\frac{e}{3}\right) + \cdots + \left(\frac{e}{3}\right)^{n-1} + \cdots$

11 $0.37 + 0.0037 + \cdots + \frac{37}{(100)^n} + \cdots$

12 $0.628 + 0.000628 + \cdots + \frac{628}{(1000)^n} + \cdots$

13 $\displaystyle\sum_{n=1}^{\infty} 2^{-n}3^{n-1}$

14 $\displaystyle\sum_{n=1}^{\infty} (-5)^{n-1}4^{-n}$

15 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}$

16 $\displaystyle\sum_{n=1}^{\infty} (\sqrt{2})^{n-1}$

Exer. 17–20: Use Theorem (8.15) to find all values of x for which the series converges, and find the sum of the series.

17 $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots$

18 $1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$

19 $\frac{1}{2} + \frac{(x-3)}{4} + \frac{(x-3)^2}{8} + \cdots + \frac{(x-3)^n}{2^{n+1}} + \cdots$

20 $3 + (x-1) + \frac{(x-1)^2}{3} + \cdots + \frac{(x-1)^n}{3^{n-1}} + \cdots$

Exer. 21–24: The overbar indicates that the digits underneath repeat indefinitely. Express the repeating decimal as a series, and find the rational number it represents.

21 $0.\overline{23}$

22 $5.\overline{146}$

23 $3.2\overline{394}$

24 $2.7\overline{1828}$

Exer. 25–32: Use Example 1 or 3 and Theorem (8.19) or (8.20) to determine whether the series converges or diverges.

25 $\frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots + \frac{1}{(n+3)(n+4)} + \cdots$

26 $\frac{1}{10\cdot 11} + \frac{1}{11\cdot 12} + \cdots + \frac{1}{(n+9)(n+10)} + \cdots$

27 $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \cdots + \frac{5}{n(n+1)} + \cdots$

28 $\frac{-1}{1\cdot 2} + \frac{-1}{2\cdot 3} + \cdots + \frac{-1}{n(n+1)} + \cdots$

29 $\frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+3} + \cdots$

30 $6^{-1} + 7^{-1} + \cdots + (n+5)^{-1} + \cdots$

31 $3 + \frac{3}{2} + \cdots + \frac{3}{n} + \cdots$

32 $-4 - 2 - \frac{4}{3} - \cdots - \frac{4}{n} - \cdots$

Exer. 33–40: Use the nth-term test (8.17) to determine whether the series diverges or needs further investigation.

33 $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{5n-1}$

34 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+(0.3)^n}$

35 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3}$

36 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n+1}$

37 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{e}}$

38 $\displaystyle\sum_{n=1}^{\infty} n\sin\frac{1}{n}$

39 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\ln(n+1)}$

40 $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{2n}{7n-5}\right)$

Exer. 41–48: Use known convergent or divergent series, together with Theorem (8.20) or (8.21), to determine whether the series is convergent or divergent; if it converges, find its sum.

41 $\displaystyle\sum_{n=3}^{\infty} \left[\left(\frac{1}{4}\right)^n + \left(\frac{3}{4}\right)^n\right]$

42 $\displaystyle\sum_{n=1}^{\infty} \left[\left(\frac{3}{2}\right)^n + \left(\frac{2}{3}\right)^n\right]$

43 $\displaystyle\sum_{n=1}^{\infty} (2^{-n} - 2^{-3n})$

44 $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3^n} - \frac{1}{4^n}\right)$

45 $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{8^n} + \frac{1}{n(n+1)}\right]$

46 $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{n(n+1)} - \frac{4}{n}\right]$

47 $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{n+2} - \frac{5}{n+3}\right)$

48 $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n}\right)$

c **Exer. 49–50:** For the given convergent series, (a) approximate S_1, S_2, and S_3 to five decimal places and (b) approximate the sum of the series to three decimal places.

49 $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{4^n}$

50 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{e^{(n^2)}}$

c **51** Let S_n be the nth partial sum of the harmonic series. If $M = 3$, use the method of Example 3 to find a positive integer m such that $S_m \geq M$, and approximate S_m to two decimal places.

c **52** Work Exercise 51 if $M = 8$.

C **Exer. 53–56: Compute partial sums S_n for the series, using $n = 4, 8, 12, 16$, and 20.**

53 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **54** $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

55 $\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$ **56** $\sum_{n=1}^{\infty} \frac{n}{e^n}$

57 Prove or disprove: If $\sum a_n$ and $\sum b_n$ both diverge, then $\sum (a_n + b_n)$ diverges.

58 What is wrong with the following "proof" that the divergent geometric series $\sum_{n=1}^{\infty} (-1)^{n+1}$ has the sum 0? (See Example 2.)

$$\sum_{n=1}^{\infty} (-1)^{n+1} = [1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \cdots$$
$$= 0 + 0 + 0 + \cdots = 0$$

59 A rubber ball is dropped from a height of 10 m. If it rebounds approximately one-half the distance after each fall, use a geometric series to approximate the total distance that the ball travels before coming to rest.

60 The bob of a pendulum swings through an arc 24 cm long on its first swing. If each successive swing is approximately five-sixths the length of the preceding swing, use a geometric series to approximate the total distance that the bob travels before coming to rest.

61 If a dosage of Q units of a certain drug is administered to an individual, then the amount remaining in the bloodstream at the end of t minutes is given by Qe^{-ct}, where $c > 0$. Suppose this same dosage is given at successive T-minute intervals.

(a) Show that the amount $A(k)$ of the drug in the bloodstream immediately after the kth dose is given by $A(k) = \sum_{n=0}^{k-1} Qe^{-ncT}$.

(b) Find an upper bound for the amount of the drug in the bloodstream after any number of doses.

(c) Find the smallest time between doses that will ensure that $A(k)$ does not exceed a certain level M for $M > Q$.

62 Suppose that each dollar introduced into the economy recirculates as follows: 85% of the original dollar is spent, then 85% of that \$0.85 is spent, and so on. Find the economic impact (the total amount spent) if \$1,000,000 is introduced into the economy.

63 In a pest eradication program, N sterilized male flies are released into the general population each day, and 90% of these flies will survive a given day.

(a) Show that the number of sterilized flies in the population after n days is

$$N + (0.9)N + \cdots + (0.9)^{n-1}N.$$

(b) If the *long-range* goal of the program is to keep 20,000 sterilized males in the population, how many such flies should be released each day?

64 A certain drug has a half-life in the bloodstream of about 2 hr. Doses of K milligrams will be administered every 4 hr, with K still to be determined.

(a) Show that the number of milligrams of drug in the bloodstream after the nth dose has been administered is

$$K + \tfrac{1}{4}K + \cdots + (\tfrac{1}{4})^{n-1}K,$$

and that this sum is approximately $\frac{4}{3}K$ for large values of n.

(b) If more than 500 mg of the drug in the bloodstream is considered to be a dangerous level, find the largest possible dose that can be given repeatedly over a long period of time.

(c) Refer to Exercise 61. If the dose K is 50 mg, how frequently can the drug be safely administered?

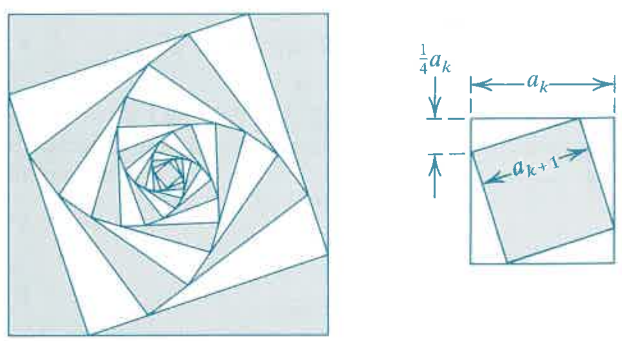
65 The first figure shows some terms of a sequence of squares $S_1, S_2, \ldots, S_k, \ldots$. Let a_k, A_k, and P_k denote the side, area, and perimeter, respectively, of the square S_k. The square S_{k+1} is constructed from S_k by connecting four points on S_k, with each point a distance of $\frac{1}{4}a_k$ from a vertex, as shown in the second figure.

(a) Find a relationship between a_{k+1} and a_k.

(b) Find a_n, A_n, and P_n.

(c) Calculate $\sum_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} A_n$.

Exercise 65

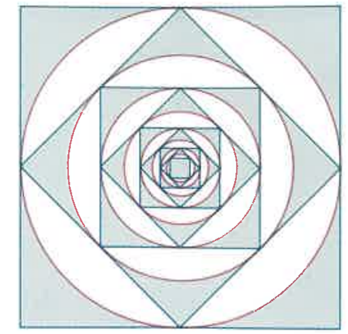


66 The figure shows several terms of a sequence consisting of alternating circles and squares. Each circle is inscribed in a square, and each square (excluding the largest) is inscribed in a circle. Let S_n denote the area of the nth square and C_n the area of the nth circle.

(a) Find relationships between S_n and C_n and between C_n and S_{n+1}.

(b) What portion of the largest square is shaded in the figure?

Exercise 66



8.3 POSITIVE-TERM SERIES

In the preceding section, we established the convergence or divergence of several series by finding a formula for the nth partial sum S_n and then determining whether or not $\lim_{n\to\infty} S_n$ exists. Unfortunately, except in special cases such as a geometric series or a telescoping series, it is often impossible to find an explicit formula for S_n. However, we can develop tests for convergence or divergence of a series $\sum a_n$ that use the nth term a_n. *These tests will not give us the sum S of the series*, but instead will tell us only whether the sum *exists*. This result is sufficient in most applications, because knowing that the sum exists, we can usually approximate it to any degree of accuracy by adding a sufficient number of terms of the series.

In this section, we consider only **positive-term series**—that is, series $\sum a_n$ such that $a_n > 0$ for every n. Although this approach may appear to be very specialized, positive-term series are the foundation for all of our future work with series. As we shall see later, the convergence or divergence of an *arbitrary* series can often be determined from that of a related positive-term series.

The next theorem shows that to establish convergence or divergence of a positive-term series, it is sufficient to determine whether the sequence of partial sums $\{S_n\}$ is bounded.

***Theorem* 8.22**

If $\sum a_n$ is a positive-term series and if there exists a number M such that

$$S_n = a_1 + a_2 + \cdots + a_n < M$$

for every n, then the series converges and has a sum $S \le M$. If no such M exists, the series diverges.

PROOF If $\{S_n\}$ is the sequence of partial sums of the positive-term series $\sum a_n$, then

$$S_1 < S_2 < \cdots < S_n < \cdots$$

and therefore $\{S_n\}$ is monotonic. If there exists a number M such that $S_n < M$ for every n, then $\{S_n\}$ is bounded monotonic. As in the proof of Theorem (8.9),

$$\lim_{n\to\infty} S_n = S \le M$$

for some S, and hence the series converges. If no such M exists, then $\lim_{n\to\infty} S_n = \infty$ and the series diverges. ■

We may use the nth term a_n of a series $\sum a_n$ to define a function f such that $f(n) = a_n$ for every positive integer n. In some cases, if we replace n with x, we obtain a function that is defined for every *real* number $x \ge 1$. For example,

$$\text{given} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{let} \quad f(n) = \frac{1}{n^2}.$$

Replacing n with x, we obtain $f(x) = 1/x^2$, which gives us the desired function f. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} f(n) = f(1) + f(2) + \cdots + f(n) + \cdots.$$

The next result shows that if a function f obtained in this way satisfies certain conditions, then we may use the improper integral $\int_1^\infty f(x)\,dx$ to test the series $\sum_{n=1}^{\infty} f(n)$ for convergence or divergence.

***Integral Test* 8.23**

If $\sum a_n$ is a series, let $f(n) = a_n$ and let f be the function obtained by replacing n with x. If f is positive-valued, continuous, and decreasing for every real number $x \ge 1$, then the series $\sum a_n$

(i) converges if $\int_1^\infty f(x)\,dx$ converges

(ii) diverges if $\int_1^\infty f(x)\,dx$ diverges

PROOF As in the hypotheses, we let $f(n) = a_n$ and consider $f(x)$ for every real number $x \ge 1$. A typical graph of this positive-valued, continuous, decreasing function is sketched in Figure 8.8. If n is a positive integer greater than 1, the area of the inscribed rectangular polygon illustrated in Figure 8.8 is

$$\sum_{k=2}^{n} f(k) = f(2) + f(3) + \cdots + f(n).$$

Figure 8.8

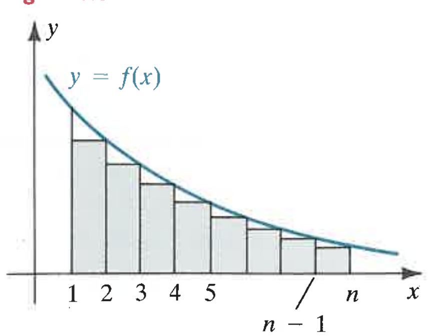
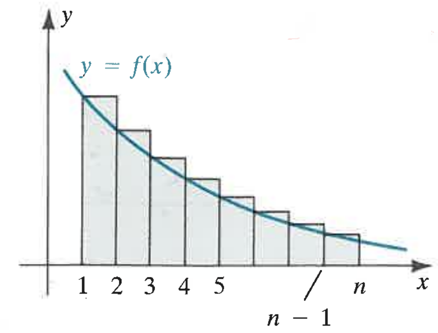


Figure 8.9



Similarly, the area of the circumscribed rectangular polygon illustrated in Figure 8.9 is

$$\sum_{k=1}^{n-1} f(k) = f(1) + f(2) + \cdots + f(n-1).$$

Since $\int_1^n f(x)\,dx$ is the area under the graph of f from 1 to n,

$$\sum_{k=2}^{n} f(k) \le \int_1^n f(x)\,dx \le \sum_{k=1}^{n-1} f(k).$$

Let S_n be the nth partial sum of the series $f(1) + f(2) + \cdots + f(n) + \cdots$, then this inequality may be written

$$S_n - f(1) \le \int_1^n f(x)\,dx \le S_{n-1}.$$

The preceding inequality implies that if the integral $\int_1^\infty f(x)\,dx$ converges and equals $K > 0$, then

$$S_n - f(1) \le K, \quad \text{or} \quad S_n \le K + f(1)$$

for every positive integer n. Hence, by Theorem (8.22), the series $\sum f(n)$ converges.

If the improper integral diverges, then

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty,$$

and since $\int_1^n f(x)\,dx \le S_{n-1}$, we also have $\lim_{n\to\infty} S_{n-1} = \infty$—that is, the series $\sum f(n)$ diverges. ■

In using the integral test (8.23), it is necessary to consider

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx.$$

Thus, we must integrate $f(x)$ and then take a limit. If $f(x)$ is not readily integrable, a different test for convergence or divergence should be used.

EXAMPLE ■ 1 Use the integral test (8.23) to prove that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

diverges (see Example 3 of Section 8.2).

SOLUTION Since $a_n = 1/n$, we let $f(n) = 1/n$. Replacing n by x gives us $f(x) = 1/x$. Because f is positive-valued, continuous, and decreasing for $x \geq 1$, we can apply the integral test (8.23):

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} [\ln x]_1^t$$
$$= \lim_{t\to\infty} [\ln t - \ln 1] = \infty$$

The series diverges, by (8.23)(ii).

EXAMPLE 2 Determine whether the infinite series $\sum ne^{-n^2}$ converges or diverges.

SOLUTION Since $a_n = ne^{-n^2}$, we let $f(n) = ne^{-n^2}$ and consider $f(x) = xe^{-x^2}$. If $x \geq 1$, then f is positive-valued and continuous. The first derivative may be used to determine whether f is decreasing. Since

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) < 0,$$

f is decreasing on $[1, \infty)$. We may therefore apply the integral test as follows:

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx = \lim_{t\to\infty} \left[\left(-\tfrac{1}{2}\right)e^{-x^2}\right]_1^t$$
$$= \left(-\frac{1}{2}\right) \lim_{t\to\infty} \left[\frac{1}{e^{(t^2)}} - \frac{1}{e}\right] = \frac{1}{2e}$$

Hence the series converges, by (8.23)(i).

In Example 2, we proved that the series $\sum ne^{-n^2}$ converges and therefore has a sum S. However, *we have not found the numerical value of* S. The number $1/(2e)$ in the solution is the value of an improper integral, not the sum of the series. If desired, we could *approximate* S by using a partial sum S_n, with n sufficiently large. (See Exercise 59.)

An integral test may also be used if the function f satisfies the conditions of (8.23) for every $x \geq k$ for some positive integer k. In this case, we merely replace the integral in (8.23) by $\int_k^{\infty} f(x)\,dx$. This corresponds to deleting the first $k - 1$ terms of the series.

The following series, which is a generalization of the harmonic series (8.14), will be useful when we apply comparision tests later in this section.

Definition 8.24

A ***p*-series**, or a **hyperharmonic series**, is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots,$$

where p is a positive real number.

Note that if $p = 1$ in (8.24), we obtain the harmonic series. The following theorem provides information about convergence or divergence of p-series.

Theorem 8.25

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

(i) converges if $p > 1$

(ii) diverges if $p \leq 1$

PROOF The special case $p = 1$ is the divergent harmonic series. Suppose that p is a positive real number and $p \neq 1$. We use the integral test (8.23), letting $f(n) = 1/n^p$ and considering $f(x) = 1/x^p = x^{-p}$. The function f is positive-valued and continuous for $x \geq 1$. Moveover, for these values of x we see that $f'(x) = -px^{-p-1} < 0$, and hence f is decreasing. Thus, f satisfies the conditions stated in the integral test (8.23), and we consider

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^t$$
$$= \frac{1}{1-p} \lim_{t\to\infty} (t^{1-p} - 1).$$

If $p > 1$, then $p - 1 > 0$ and the last expression may be written

$$\frac{1}{1-p} \lim_{t\to\infty} \left(\frac{1}{t^{p-1}} - 1\right) = \frac{1}{1-p}(0 - 1) = \frac{1}{p-1}.$$

Thus, by (8.23)(i), the p-series converges if $p > 1$.

If $0 < p < 1$, then $1 - p > 0$ and

$$\frac{1}{1-p} \lim_{t\to\infty} (t^{1-p} - 1) = \infty.$$

Hence, by (8.23)(ii), the p-series diverges.

If $p \leq 0$, then $\lim_{n\to\infty}(1/n^p) \neq 0$ and, by the nth-term test (8.17)(i), the series diverges. ■

The following illustration contains some specific p-series.

ILLUSTRATION

p-Series	Value of p	Conclusion
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$	$p = 2$	Converges, by (8.25)(i), since $2 > 1$
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$	$p = \frac{1}{2}$	Diverges, by (8.25)(ii), since $\frac{1}{2} < 1$
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots$	$p = \frac{3}{2}$	Converges, by (8.25)(i), since $\frac{3}{2} > 1$
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots$	$p = \frac{1}{3}$	Diverges, by (8.25)(ii), since $\frac{1}{3} < 1$

The next theorem allows us to use known convergent (divergent) series to establish the convergence (divergence) of other series.

Basic Comparision Tests 8.26

> Let $\sum a_n$ and $\sum b_n$ be positive-term series.
>
> **(i)** If $\sum b_n$ converges and $a_n \le b_n$ for every positive integer n, then $\sum a_n$ converges.
>
> **(ii)** If $\sum b_n$ diverges and $a_n \ge b_n$ for every positive integer n, then $\sum a_n$ diverges.

PROOF Let S_n and T_n denote the nth partial sums of $\sum a_n$ and $\sum b_n$, respectively. Suppose $\sum b_n$ converges and has the sum T. If $a_n \le b_n$ for every n, then $S_n \le T_n < T$ and hence, by Theorem (8.22), $\sum a_n$ converges. This proves part (i).

To prove (ii), suppose $\sum b_n$ diverges and $a_n \ge b_n$ for every n. Then $S_n \ge T_n$, and since T_n increases without bound as n becomes infinite, so does S_n. Consequently, $\sum a_n$ diverges. ■

The convergence or divergence of a series is not affected by deleting a finite number of terms, so the condition $a_n \le b_n$ or $a_n \ge b_n$ of (8.26) is only required from the kth term on, for some positive integer k.

A series $\sum d_n$ is said to **dominate** a series $\sum c_n$ if $d_n \ge c_n$ for every positive integer n. In this terminology, (8.26)(i) states that *a positive-term series that is dominated by a convergent series is also convergent.* Part (ii) states that *a series that dominates a divergent positive-term series is also divergent.*

EXAMPLE ■ 3 Determine whether the series converges or diverges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ **(b)** $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\sqrt{n}-1}$

SOLUTION

(a) For every $n \ge 1$,

$$\frac{1}{2+5^n} < \frac{1}{5^n} = \left(\frac{1}{5}\right)^n.$$

Since $\sum (1/5)^n$ is a convergent geometric series, the given series converges, by the basic comparison test (8.26)(i).

(b) The p-series $\sum 1/\sqrt{n}$ diverges, and hence so does the series obtained by disregarding the first term $1/\sqrt{1}$. If $n \ge 2$, then

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \quad \text{and hence} \quad \frac{3}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}.$$

It follows from the basic comparison test (8.26)(ii) that the given series diverges.

When we use a basic comparison test, we must first decide on a suitable series $\sum b_n$ and then prove that either $a_n \le b_n$ or $a_n \ge b_n$ for every n greater than some positive integer k. This proof can be very difficult if a_n is a complicated expression. The following comparison test is often easier to apply, because after deciding on $\sum b_n$, we need only take a limit of the quotient a_n/b_n as $n \to \infty$.

Limit Comparison Test 8.27

> Let $\sum a_n$ and $\sum b_n$ be positive-term series. If there is a positive real number c such that
>
> $$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$
>
> then either both series converge or both series diverge.

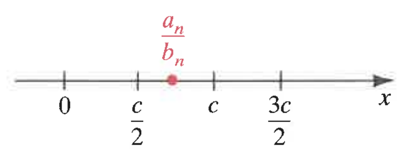
PROOF If $\lim_{n\to\infty} (a_n/b_n) = c > 0$, then a_n/b_n is close to c if n is large. Hence, there exists a number N such that

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \quad \text{whenever} \quad n > N$$

(see Figure 8.10). This is equivalent to

$$\frac{c}{2} b_n < a_n < \frac{3c}{2} b_n \quad \text{whenever} \quad n > N.$$

Figure 8.10



If the series $\sum a_n$ converges, then $\sum (c/2) b_n$ also converges, because it is dominated by $\sum a_n$. Applying (8.20)(ii), we find that the series

$$\sum b_n = \sum \left(\frac{2}{c}\right)\left(\frac{c}{2}\right) b_n$$

converges.

Conversely, if $\sum b_n$ converges, then so does $\sum a_n$, since it is dominated by the convergent series $\sum (3c/2) b_n$. We have proved that $\sum a_n$ converges if and only if $\sum b_n$ converges. Consequently, $\sum a_n$ diverges if and only if $\sum b_n$ diverges. ■

If, in (8.27), the limit equals 0 or ∞, it may be possible to determine whether the series $\sum a_n$ converges or diverges by using the comparison test stated in Exercise 51 or 52, respectively.

To find a suitable series $\sum b_n$ to use in the limit comparison test (8.27) when a_n is a quotient, a good procedure is to *delete all terms in the numerator and the denominator of a_n except those that have the greatest effect on the magnitude.* We may also replace any constant factor c by 1, since $\sum b_n$ and $\sum c b_n$ either both converge or both diverge (see Theorem 8.20). The next illustration demonstrates this procedure for several series $\sum a_n$.

ILLUSTRATION

a_n	Deleting terms of least magnitude	Choice of b_n in (8.27)
$\dfrac{3n+1}{4n^3+n^2-2}$	$\dfrac{3n}{4n^3}=\dfrac{3}{4n^2}$	$\dfrac{1}{n^2}$
$\dfrac{5}{\sqrt{n^2+2n+7}}$	$\dfrac{5}{\sqrt{n^2}}=\dfrac{5}{n}$	$\dfrac{1}{n}$
$\dfrac{\sqrt[3]{n^2+4}}{6n^2-n-1}$	$\dfrac{\sqrt[3]{n^2}}{6n^2}=\dfrac{n^{2/3}}{6n^2}=\dfrac{1}{6n^{4/3}}$	$\dfrac{1}{n^{4/3}}$

EXAMPLE ■ 4 Determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2+1}}$ **(b)** $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+5n}{2^n(n^2+1)}$

SOLUTION

(a) The nth term of the series is

$$a_n=\frac{1}{\sqrt[3]{n^2+1}}.$$

If we delete the number 1 in the radicand, we obtain $b_n=1/\sqrt[3]{n^2}$, which is the nth term of a divergent p-series, with $p=\frac{2}{3}$. Applying the limit comparison test (8.27) gives us the following:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2+1}}=\lim_{n\to\infty}\sqrt[3]{\frac{n^2}{n^2+1}}=1>0$$

Since $\sum b_n$ diverges, so does $\sum a_n$.

It is important to note that we cannot use $b_n=1/\sqrt[3]{n^2}$ with the basic comparison test (8.26), because $a_n<b_n$ instead of $a_n\geq b_n$.

(b) The nth term of the series is

$$a_n=\frac{3n^2+5n}{2^n n^2+2^n}.$$

Deleting the terms of least magnitude in the numerator and the denominator, we obtain

$$\frac{3n^2}{2^n n^2}=\frac{3}{2^n},$$

and hence we choose $b_n=1/2^n$. Applying the limit comparison test (8.27) gives us

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{x\to\infty}\frac{3n^2+5n}{2^n(n^2+1)}\cdot\frac{2^n}{1}=\lim_{x\to\infty}\frac{3n^2+5n}{n^2+1}=3>0.$$

Since, by Theorem (8.15)(i), $\sum b_n$ is a convergent geometric series (with $r=\frac{1}{2}<1$), the series $\sum a_n$ is also convergent.

EXAMPLE ■ 5 Let $a_n=\dfrac{8n+\sqrt{n}}{5+n^2+n^{7/2}}$. Determine whether $\sum a_n$ converges or diverges.

SOLUTION To find a suitable comparison series $\sum b_n$, we delete all but the highest powers of n in the numerator and the denominator, obtaining

$$\frac{8n}{n^{7/2}}=\frac{8}{n^{5/2}}.$$

Applying the limit comparison test (8.27), with $b_n=1/n^{5/2}$, we find

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{8n+n^{1/2}}{5+n^2+n^{7/2}}\cdot\frac{n^{5/2}}{1}$$

$$=\lim_{n\to\infty}\frac{8n^{7/2}+n^3}{5+n^2+n^{7/2}}=8>0.$$

Since $\sum b_n$ is a convergent p-series with $p=\frac{5}{2}>1$, it follows from (8.27) that $\sum a_n$ is also convergent.

With a programmable calculator or a computer, it is relatively easy to find the nth partial sum S_n for a given infinite series. If the infinite series converges to the sum S, then given an $\epsilon>0$, we can find an N such that S_N is within ϵ of S. In some cases, we can determine a value for N without explicitly knowing S. Whenever this is possible, we can obtain good approximations to the sum of the infinite series. One such case occurs when the terms of $\{a_n\}$ form a positive, decreasing sequence.

As in the integral test (8.23), if $\sum a_n$ is a series, let $f(n)=a_n$ and let f be the function obtained by replacing n with x. If f is continuous and decreasing for $x>N$ for some integer N, then it can be shown that the error in approximating the sum of the given series by $\sum_{n=1}^{N}a_n$ is less than $\int_N^\infty f(x)\,dx$ (see Exercise 53).

EXAMPLE ■ 6 Approximate the sum of the series $\sum_{n=1}^{\infty}(1/n^3)$ with an error smaller than 10^{-5}.

SOLUTION The given series converges by Theorem (8.25) with $p=3$. The function $f(x)=1/x^3$ is positive, continuous, and decreasing for all $x>0$—the last condition is true since $f'(x)=-3/x^4$ is negative. We also have

$$\int_N^\infty f(x)\,dx=\int_N^\infty\frac{1}{x^3}\,dx=\lim_{t\to\infty}\int_N^t\frac{1}{x^3}\,dx$$

$$=\lim_{t\to\infty}\left[\frac{-1}{2x^2}\right]_N^t=\lim_{t\to\infty}\left[\frac{-1}{2t^2}+\frac{1}{2N^2}\right]=\frac{1}{2N^2}.$$

From the result of Exercise 53, we see that $\sum_{n=1}^{N}(1/n^3)$ is within $1/(2N^2)$ of the sum of the infinite series $\sum_{n=1}^{\infty}(1/n^3)$. If we wish the error in our approximation to be less than 10^{-5}, then we must choose N such that

$$\frac{1}{2N^2} < 10^{-5},$$

which is equivalent to $N^2 > 10^5/2 = 50{,}000$, or $N > \sqrt{50{,}000} \approx 223.6$.

Thus, by summing the first 224 terms, we can approximate $\sum_{n=1}^{\infty}(1/n^3)$ with an error smaller than 10^{-5}. A simple computer program yields the result $\sum_{n=1}^{224}(1/n^3) \approx 1.20204698$, which is our approximation to $\sum_{n=1}^{\infty}(1/n^3)$. By our error estimate, we have

$$1.20204698 < \sum_{n=1}^{\infty}\frac{1}{n^3} < 1.20204698 + 0.000005.$$

Thus, the true value of $\sum_{n=1}^{\infty}(1/n^3)$ lies in the interval

$$[1.20204698,\ 1.20205198].$$

We conclude this section with several general remarks about positive-term series. Suppose that $\sum a_n$ is a positive-term series and the terms are grouped in some manner, such as

$$(a_1 + a_2) + a_3 + (a_4 + a_5 + a_6 + a_7) + \cdots.$$

If we denote the last series by $\sum b_n$, so that

$$b_1 = a_1 + a_2, \quad b_2 = a_3, \quad b_3 = a_4 + a_5 + a_6 + a_7, \quad \ldots,$$

then any partial sum of the series $\sum b_n$ is also a partial sum of $\sum a_n$. It follows that if $\sum a_n$ converges, then $\sum b_n$ converges and has the same sum. A similar argument may be used for any grouping of the terms of $\sum a_n$. Thus, *if a positive-term series converges, then the series obtained by grouping the terms in any manner also converges and has the same sum.* We cannot make a similar statement about arbitrary divergent series. For example, the terms of the divergent series $\sum(-1)^n$ may be grouped to produce a convergent series (see Exercise 58 of Section 8.2).

Next, suppose that a convergent positive-term series $\sum a_n$ has the sum S and that a new series $\sum b_n$ is formed by rearranging the terms in some way. For example, $\sum b_n$ could be the series

$$a_2 + a_8 + a_1 + a_5 + a_7 + a_3 + \cdots.$$

If T_n is the nth partial sum of $\sum b_n$, then it is a sum of terms of $\sum a_n$. If m is the largest of the subscripts associated with the terms a_k in T_n, then $T_n \le S_m < S$. Consequently, $T_n < S$ for every n. Applying Theorem (8.22), we find that $\sum b_n$ converges and has a sum $T \le S$. The preceding proof is independent of the particular rearrangement of terms. We may also regard the series $\sum a_n$ as having been obtained by rearranging the terms of $\sum b_n$ and hence, by the same argument, $S \le T$. We have proved that *if the terms of a convergent positive-term series $\sum a_n$ are rearranged in any manner, then the resulting series converges and has the same sum.*

EXERCISES 8.3

Exer. 1–12: (a) Show that the function f determined by the nth term of the series satisfies the hypotheses of the integral test. (b) Use the integral test to determine whether the series converges or diverges.

1 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3+2n)^2}$

2 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(4+n)^{3/2}}$

3 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n+7}$

4 $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

5 $\displaystyle\sum_{n=1}^{\infty}n^2e^{-n^3}$

6 $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(2n-5)}$

7 $\displaystyle\sum_{n=3}^{\infty}\frac{\ln n}{n}$

8 $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

9 $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$

10 $\displaystyle\sum_{n=4}^{\infty}\left(\frac{1}{n-3}-\frac{1}{n}\right)$

11 $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan n}{1+n^2}$

12 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+16n^2}$

Exer. 13–20: Use a basic comparison test to determine whether the series converges or diverges.

13 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4+n^2+1}$

14 $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

15 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n3^n}$

16 $\displaystyle\sum_{n=1}^{\infty}\frac{2+\cos n}{n^2}$

17 $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan n}{n}$

18 $\displaystyle\sum_{n=1}^{\infty}\frac{\operatorname{arcsec} n}{(0.5)^n}$

19 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^n}$

20 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$

Exer. 21–28: Use the limit comparison test to determine whether the series converges or diverges.

21 $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+4}$

22 $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3+\sqrt{n}}$

23 $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{4n^3-5n}}$

24 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)(n+2)}}$

25 $\displaystyle\sum_{n=1}^{\infty}\frac{8n^2-7}{e^n(n+1)^2}$

26 $\displaystyle\sum_{n=1}^{\infty}\frac{3n+5}{n2^n}$

27 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+9}$

28 $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$

Exer. 29–46: Determine whether the series converges or diverges.

29 $\displaystyle\sum_{n=1}^{\infty}\frac{2n+n^2}{n^3+1}$

30 $\displaystyle\sum_{n=1}^{\infty}\frac{n^5+4n^3+1}{2n^8+n^4+2}$

31 $\displaystyle\sum_{n=1}^{\infty}\frac{1+2^n}{1+3^n}$

32 $\displaystyle\sum_{n=4}^{\infty}\frac{3n}{2n^2-7}$

33 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{5n^2+1}}$

34 $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^4}$

35 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{2n+1}}$

36 $\displaystyle\sum_{n=1}^{\infty}\frac{n+\ln n}{n^3+n+1}$

37 $\displaystyle\sum_{n=1}^{\infty}ne^{-n}$

38 $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$

39 $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$

40 $\displaystyle\sum_{n=1}^{\infty}\tan\frac{1}{n}$

41 $\displaystyle\sum_{n=1}^{\infty}\frac{(2n+1)^3}{(n^3+1)^2}$

42 $\displaystyle\sum_{n=1}^{\infty}\frac{n+\ln n}{n^2+1}$

43 $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+2^n}{n+3^n}$

44 $\displaystyle\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{2^n}\right)$

45 $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$

46 $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n+2^n}{n+5^n}$

Exer. 47–48: Find every real number k for which the series converges.

47 $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^k\ln n}$

48 $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^k}$

49 (a) Use the proof of the integral test (8.23) to show that, for every positive integer $n > 1$,

$$\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n.$$

(b) Estimate the number of terms of the harmonic series that should be added so that $S_n > 100$.

50 Consider the hypothetical problem illustrated in the figure on the following page: Starting with a ball of radius 1 ft, a person stacks balls vertically such that if r_k is the radius of the kth ball, then $r_{n+1} = r_n\sqrt{n/(n+1)}$ for each positive integer n.

(a) Show that the height of the stack can be made arbitrarily large.

(b) If the balls are made of a material that weighs 1 lb/ft^3, show that the total weight of the stack is always less than 4π pounds.

Exercise 50

51 Suppose that $\sum a_n$ and $\sum b_n$ are positive-term series. Prove that if $\lim_{n\to\infty}(a_n/b_n) = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges. (This is not necessarily true for series that contain negative terms.)

52 Prove that if $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

53 Let $\sum a_n$ be a convergent, positive-term series. Let $f(n) = a_n$, and suppose f is continuous and decreasing for $x \ge N$ for some integer N. Prove that the error in approximating the sum of the given series by $\sum_{n=1}^{N} a_n$ is less than $\int_N^\infty f(x)\,dx$.

Exer. 54–56: Use Exercise 53 to estimate the smallest number of terms that can be added to approximate the sum of the series with an error less than E.

54 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$; $E = 0.001$ **55** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$; $E = 0.01$

56 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$; $E = 0.05$

57 Prove that if a positive-term series $\sum a_n$ converges, then $\sum(1/a_n)$ diverges.

58 Prove that if a positive-term series $\sum a_n$ converges, then $\sum\sqrt{a_n a_{n+1}}$ converges. (*Hint:* First show that the following is true: $\sqrt{a_n a_{n+1}} \le (a_n + a_{n+1})/2$.)

c **Exer. 59–64: Approximate the sum of the given series to three decimal places. (Use Exercise 53 to justify the accuracy of your answer.)**

59 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ **60** $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

61 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ **62** $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

63 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ **64** $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

c **65** Graph, on the same coordinate axes, $y = x$ and $y = \ln(x^k)$ for $k = 1, 2, 3$ and $1 \le x \le 20$. Then use the graphs to predict whether the series $\sum_{n=2}^{\infty}(1/\ln(n^k))$ converges or diverges for $k = 1, 2,$ and 3.

c **66** Graph, on the same coordinate axes, $y = x$ and $y = (\ln x)^k$ for $k = 1, 2, 3$ and $1 \le x \le 200$. Then use the graphs to predict whether the series $\sum_{n=2}^{\infty}(1/(\ln n)^k)$ converges or diverges for $k = 1, 2,$ and 3.

8.4 THE RATIO AND ROOT TESTS

For the integral test to be applied to a positive-term series $\sum a_n$ with $a_n = f(n)$, the terms must be decreasing and we must be able to integrate $f(x)$. These conditions often rule out series that involve factorials and other complicated expressions. In this section, we examine two tests that can be used to help determine convergence or divergence when other tests are not applicable. Unfortunately, as indicated by part (iii) of both tests, they are inconclusive for certain series.

Ratio Test 8.28

Let $\sum a_n$ be a positive-term series, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

(i) If $L < 1$, the series is convergent.

(ii) If $L > 1$ or $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \infty$, the series is divergent.

(iii) If $L = 1$, apply a different test; the series may be convergent or divergent.

PROOF

(i) Suppose that $\lim_{n\to\infty}(a_{n+1}/a_n) = L < 1$. Let r be any number such that $0 \le L < r < 1$. Since a_{n+1}/a_n is close to L if n is large, there exists an integer N such that whenever $n \ge N$,

$$\frac{a_{n+1}}{a_n} < r \quad \text{or} \quad a_{n+1} < a_n r.$$

Substituting $N, N+1, N+2, \ldots$ for n, we obtain

$$a_{N+1} < a_N r$$
$$a_{N+2} < a_{N+1} r < a_N r^2$$
$$a_{N+3} < a_{N+2} r < a_N r^3$$

and, in general,

$$a_{N+m} < a_N r^m \quad \text{whenever} \quad m > 0.$$

It follows from the basic comparison test (8.26)(i) that the series

$$a_{N+1} + a_{N+2} + \cdots + a_{N+m} + \cdots$$

converges, since its terms are less than the corresponding terms of the convergent geometric series

$$a_N r + a_N r^2 + \cdots + a_N r^n + \cdots.$$

Since convergence or divergence is unaffected by discarding a finite number of terms (see Theorem (8.19)), the series $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) Suppose that $\lim_{n\to\infty}(a_{n+1}/a_n) = L > 1$. If r is a real number such that $L > r > 1$, then there exists an integer N such that

$$\frac{a_{n+1}}{a_n} > r > 1 \quad \text{whenever} \quad n \ge N.$$

Consequently, $a_{n+1} > a_n$ if $n \ge N$. Thus, $\lim_{n\to\infty} a_n \ne 0$ and, by the nth-term test (8.17)(i), the series $\sum a_n$ diverges.

The proof for $\lim_{n\to\infty}(a_{n+1}/a_n) = \infty$ is similar and is left as an exercise.

(iii) The ratio test is inconclusive if

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

for it is easy to verify that the limit is 1 for both the convergent series $\sum(1/n^2)$ and the divergent series $\sum(1/n)$. Consequently, *if the limit is* 1, *then a different test must be used.* ■

EXAMPLE ■ 1 Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ **(b)** $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

SOLUTION

(a) Applying the ratio test (8.28), we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(a_{n+1} \cdot \frac{1}{a_n}\right) \\ &= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \\ &= \lim_{n\to\infty} \frac{3}{n+1} = 0.\end{aligned}$$

Since $0 < 1$, the series is convergent.

(b) Applying the ratio test (8.28), we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^2} \cdot \frac{n^2}{3^n} \\ &= \lim_{n\to\infty} \frac{3n^2}{n^2+2n+1} = 3.\end{aligned}$$

Since $3 > 1$, the series diverges, by (8.28)(ii).

EXAMPLE ■ 2 Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Applying the ratio test gives us

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\ &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)} \cdot \frac{1}{n^n} \\ &= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \\ &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.\end{aligned}$$

The last equality is a consequence of Theorem (6.32)(ii). Since $e > 1$, the series diverges.

If $\sum a_n$ is a series such that $\lim_{n\to\infty}(a_{n+1}/a_n) = 1$, we must use a different test (see (iii) of (8.28)). The next illustration contains several series of this type and suggestions on how to show convergence or divergence.

ILLUSTRATION

Series $\sum a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$	Suggestion
$\sum_{n=1}^{\infty} \frac{2n^2+3n+4}{5n^5-7n^3+n}$	1	Show convergence by using the limit comparison test (8.27) with $b_n = 1/n^3$.
$\sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^3+5n+3}}$	1	Show divergence by using the limit comparison test (8.27) with $b_n = 1/\sqrt{n}$.
$\sum_{n=1}^{\infty} \frac{\ln n}{n}$	1	Show divergence by using the integral test (8.23).

The following test is often useful if a_n contains powers of n.

Root Test 8.29

Let $\sum a_n$ be a positive-term series, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = L.$$

(i) If $L < 1$, the series is convergent.
(ii) If $L > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, the series is divergent.
(iii) If $L = 1$, apply a different test; the series may be convergent or divergent.

PROOF If $L < 1$, consider any number r such that $0 \le L < r < 1$. By the definition of limit, there exists a positive integer N such that if $n \ge N$, then

$$\sqrt[n]{a_n} < r \quad \text{or} \quad a_n < r^n.$$

Since $0 < r < 1$, $\sum_{n=N}^{\infty} r^n$ is a convergent geometric series, and hence, by the basic comparison test (8.26), $\sum_{n=N}^{\infty} a_n$ converges. Consequently, $\sum_{n=1}^{\infty} a_n$ converges. This proves (i). The remainder of the proof is similar to that used for the ratio test. ■

EXAMPLE ■ 3 Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}.$$

SOLUTION Applying the root test (8.29) yields

$$\lim_{n\to\infty}\sqrt[n]{\frac{2^{3n+1}}{n^n}} = \lim_{n\to\infty}\left(\frac{2^{3n+1}}{n^n}\right)^{1/n}$$
$$= \lim_{n\to\infty}\frac{2^{3+(1/n)}}{n} = 0.$$

Since $0 < 1$, the series converges. We could have applied the ratio test (8.28); however, the process of evaluating the limit would have been more complicated.

EXERCISES 8.4

Exer. 1–10: Find $\lim_{n\to\infty}(a_{n+1}/a_n)$, and use the ratio test (8.28) to determine if the series converges or diverges or if the test is inconclusive.

1 $\sum_{n=1}^{\infty}\frac{3n+1}{2^n}$ **2** $\sum_{n=1}^{\infty}\frac{3^n}{n^2+4}$

3 $\sum_{n=1}^{\infty}\frac{5^n}{n(3^{n+1})}$ **4** $\sum_{n=1}^{\infty}\frac{2^{n-1}}{5^n(n+1)}$

5 $\sum_{n=1}^{\infty}\frac{100^n}{n!}$ **6** $\sum_{n=1}^{\infty}\frac{n^{10}+10}{n!}$

7 $\sum_{n=1}^{\infty}\frac{n+3}{n^2+2n+5}$ **8** $\sum_{n=1}^{\infty}\frac{3n}{\sqrt{n^3+1}}$

9 $\sum_{n=1}^{\infty}\frac{n!}{e^n}$ **10** $\sum_{n=1}^{\infty}\frac{n!}{(n+1)^5}$

Exer. 11–18: Find $\lim_{n\to\infty}\sqrt[n]{a_n}$, and use the root test (8.29) to determine if the series converges or diverges or if the test is inconclusive.

11 $\sum_{n=1}^{\infty}\frac{1}{n^n}$ **12** $\sum_{n=1}^{\infty}\frac{(\ln n)^n}{n^{n/2}}$ **13** $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$

14 $\sum_{n=2}^{\infty}\frac{5^{n+1}}{(\ln n)^n}$ **15** $\sum_{n=1}^{\infty}\frac{n}{3^n}$ **16** $\sum_{n=1}^{\infty}\frac{n^{10}}{10^n}$

17 $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$ **18** $\sum_{n=2}^{\infty}\left(\frac{n}{\ln n}\right)^n$

Exer. 19–40: Determine whether the series converges or diverges.

19 $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$ **20** $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{3n+4}$

21 $\sum_{n=1}^{\infty}\frac{99^n(n^5+2)}{n^2 10^{2n}}$ **22** $\sum_{n=1}^{\infty}\frac{n3^{2n}}{5^{n-1}}$

23 $\sum_{n=1}^{\infty}\frac{2}{n^3+e^n}$ **24** $\sum_{n=1}^{\infty}\frac{n+1}{n^3+1}$

25 $\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^n n!$ **26** $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ **27** $\sum_{n=1}^{\infty}\frac{n^n}{10^{n+1}}$

28 $\sum_{n=1}^{\infty}\frac{10+2^n}{n!}$ **29** $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ **30** $\sum_{n=1}^{\infty}\frac{(2n)!}{2^n}$

31 $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ **32** $\sum_{n=1}^{\infty}\frac{(2n)^n}{(5n+3n^{-1})^n}$

33 $\sum_{n=1}^{\infty}\frac{\ln n}{(1.01)^n}$ **34** $\sum_{n=1}^{\infty}3^{1/n}$ **35** $\sum_{n=1}^{\infty}n\tan\frac{1}{n}$

36 $\sum_{n=1}^{\infty}\frac{\arctan n}{n^2}$ **37** $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ **38** $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^n}$

39 $1+\frac{1\cdot 3}{2!}+\frac{1\cdot 3\cdot 5}{3!}+\cdots + \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}+\cdots$

40 $\frac{1}{2}+\frac{1\cdot 4}{2\cdot 4}+\frac{1\cdot 4\cdot 7}{2\cdot 4\cdot 6}+\cdots + \frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3n-2)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}+\cdots$

8.5 ALTERNATING SERIES AND ABSOLUTE CONVERGENCE

The tests for convergence that we have discussed thus far can be applied only to positive-term series. We now consider infinite series that contain both positive and negative terms. One of the simplest, and most useful, series of this type is an **alternating series**, in which the terms are alternately positive and negative. It is customary to express an alternating series in one of the forms

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots$$

or $$-a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots$$

with $a_k > 0$ for every k. The next theorem provides the main test for convergence of these series. For convenience, we consider $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$. A similar proof holds for $\sum_{n=1}^{\infty}(-1)^n a_n$.

Alternating Series Test 8.30

The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots$$

is convergent if the following two conditions are satisfied:

(i) $a_k \ge a_{k+1} > 0$ for every k

(ii) $\lim_{n\to\infty} a_n = 0$

PROOF By condition (i), we may write

$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ge \cdots \ge a_k \ge a_{k+1} \ge \cdots.$$

Let us consider the partial sums

$$S_2, S_4, S_6, \ldots, S_{2n}, \ldots,$$

which contain an even number of terms of the series. Since

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n})$$

and $a_k - a_{k+1} \ge 0$ for every k, we see that

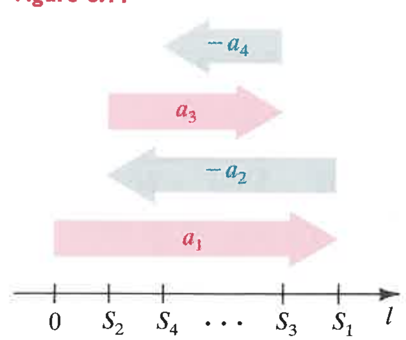
$$0 \le S_2 \le S_4 \le \cdots \le S_{2n} \le \cdots;$$

that is, $\{S_{2n}\}$ is a monotonic sequence. This fact is also evident from Figure 8.11, where we have used a coordinate line l to represent the following four partial sums of the series:

$$S_1 = a_1, \quad S_2 = a_1 - a_2, \quad S_3 = a_1 - a_2 + a_3, \quad S_4 = a_1 - a_2 + a_3 - a_4$$

You may find it instructive to locate the points on l that correspond to S_5 and S_6.

Figure 8.11



Referring to Figure 8.11, we see that $S_{2n} \le a_1$ for every positive integer n. This may also be proved algebraically by observing that

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \le a_1.$$

Thus, $\{S_{2n}\}$ is a *bounded* monotonic sequence. As in the proof of Theorem (8.9),

$$\lim_{n\to\infty} S_{2n} = S \le a_1$$

for some number S.

If we next consider a partial sum S_{2n+1} having an *odd* number of terms of the series, then $S_{2n+1} = S_{2n} + a_{2n+1}$ and, since $\lim_{n\to\infty} a_{2n+1} = 0$,

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} = S.$$

Because both the sequence of even partial sums and the sequence of odd partial sums have the same limit S, it follows that

$$\lim_{n\to\infty} S_n = S \le a_1.$$

That is, the series converges. ■

EXAMPLE 1 Determine whether the alternating series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$ **(b)** $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n-3}$

SOLUTION

(a) Let $$a_n = f(n) = \frac{2n}{4n^2-3}.$$

To apply the alternating series test (8.30), we must show that

(i) $a_k \ge a_{k+1}$ for every positive integer k

(ii) $\lim_{n\to\infty} a_n = 0$

There are several ways to prove (i). One method is to show that $f(x) = 2x/(4x^2-3)$ is decreasing for $x \ge 1$. By the quotient rule,

$$f'(x) = \frac{(4x^2-3)(2)-(2x)(8x)}{(4x^2-3)^2} = \frac{-8x^2-6}{(4x^2-3)^2} < 0.$$

By Theorem (3.15), $f(x)$ is decreasing and, therefore, $f(k) \ge f(k+1)$; that is, $a_k \ge a_{k+1}$ for every positive integer k.

We can also prove (i) directly, by proving that $a_k - a_{k+1} \ge 0$. Thus, if $a_n = 2n/(4n^2-3)$, then for every positive integer k,

$$a_k - a_{k+1} = \frac{2k}{4k^2-3} - \frac{2(k+1)}{4(k+1)^2-3} = \frac{8k^2+8k+6}{(4k^2-3)(4k^2+8k+1)} \ge 0.$$

Still another technique for proving that $a_k \ge a_{k+1}$ is to show that $a_{k+1}/a_k \le 1$.

To prove (ii), we see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{4n^2-3} = 0.$$

Thus, the alternating series converges.

(b) We can show that $a_k \ge a_{k+1}$ for every k; however,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{4n-3} = \frac{1}{2} \ne 0,$$

and hence the series diverges, by the nth-term test (8.17)(i).

The alternating series test (8.30) may be used if condition (i) holds for $k > m$ for some positive integer m, because this corresponds to deleting the first m terms of the series.

If a series converges, then the nth partial sum S_n can be used to approximate the sum S of the series. In many cases, it is difficult to determine the accuracy of the approximation. However, for an *alternating series*, the next theorem provides a simple way of estimating the error that is involved.

Theorem 8.31

Let $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ be an alternating series that satisfies conditions (i) and (ii) of the alternating series test. If S is the sum of the series and S_n is a partial sum, then

$$|S - S_n| \le a_{n+1};$$

that is, the error involved in approximating S by S_n is less than or equal to a_{n+1}.

PROOF The series obtained by deleting the first n terms of $\sum(-1)^{n-1}a_n$, namely,

$$(-1)^n a_{n+1} + (-1)^{n+1}a_{n+2} + (-1)^{n+2}a_{n+3} + \cdots,$$

also satisfies the conditions of (8.30) and therefore has a sum R_n. Thus,

$$S - S_n = R_n = (-1)^n(a_{n+1} - a_{n+2} + a_{n+3} - \cdots)$$

and $$|R_n| = a_{n+1} - a_{n+2} + a_{n+3} - \cdots.$$

Employing the same argument used in the proof of the alternating series test, we see that $|R_n| \le a_{n+1}$. Consequently,

$$E = |S - S_n| = |R_n| \le a_{n+1},$$

which is what we wished to prove. ■

In the next example, we use Theorem (8.31) to approximate the sum of an alternating series. In order to discuss the accuracy of an approximation, we must first agree on what is meant by one-decimal-place accuracy, two-decimal-place accuracy, and so on. Let us adopt the following convention. If E is the error in an approximation, then the approximation will be considered accurate to k decimal places if $|E| < 0.5 \times 10^{-k}$. For example, we have

$$\text{1-decimal-place accuracy if } |E| < 0.5 \times 10^{-1} = 0.05$$
$$\text{2-decimal-place accuracy if } |E| < 0.5 \times 10^{-2} = 0.005$$
$$\text{3-decimal-place accuracy if } |E| < 0.5 \times 10^{-3} = 0.0005.$$

EXAMPLE 2 Prove that the series

$$1 - \frac{1}{3!} + \frac{1}{5!} - \cdots + (-1)^{n-1}\frac{1}{(2n-1)!} + \cdots$$

is convergent, and approximate its sum S to five decimal places.

SOLUTION The nth term $a_n = 1/(2n-1)!$ has limit 0 as $n \to \infty$, and $a_k > a_{k+1}$ for every positive integer k. Hence the series converges, by the alternating series test. If we use S_n to approximate S, then, by Theorem (8.31), the error involved is less than or equal to $a_{n+1} = 1/(2n+1)!$. Calculating several values of a_{n+1}, we find that for $n = 4$,

$$a_5 = \frac{1}{9!} \approx 0.0000028 < 0.000005.$$

Hence, the partial sum S_4 approximates S to five decimal places. Since

$$S_4 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$$
$$= 1 - \tfrac{1}{6} + \tfrac{1}{120} - \tfrac{1}{5040} \approx 0.841468,$$

we have $S \approx 0.84147$.

It will follow from (8.48)(a) that the sum of the series is $\sin 1$, and hence $\sin 1 \approx 0.84147$.

The following concept is useful in investigating a series that contains both positive and negative terms but is not alternating. It allows us to use tests for positive-term series to establish convergence for other types of series (see Theorem 8.34).

Definition 8.32

A series $\sum a_n$ is **absolutely convergent** if the series

$$\sum |a_n| = |a_1| + |a_2| + \cdots + |a_n| + \cdots$$

is convergent.

Note that if $\sum a_n$ is a positive-term series, then $|a_n| = a_n$, and in this case, absolute convergence is the same as convergence.

EXAMPLE 3 Prove that the following alternating series is absolutely convergent:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots + (-1)^n\frac{1}{n^2} + \cdots$$

SOLUTION Taking the absolute value of each term gives us

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots,$$

which is a convergent p-series. Hence, by Definition (8.32), the alternating series is absolutely convergent.

EXAMPLE 4 The **alternating harmonic series** is

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots.$$

Show that this series is

(a) convergent **(b)** not absolutely convergent

SOLUTION

(a) Conditions (i) and (ii) of the alternating series test (8.30) are satisfied, because

$$\frac{1}{k} > \frac{1}{k+1} \quad \text{for every } k \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence, the alternating harmonic series is convergent.

(b) To examine the series for absolute convergence, we apply Definition (8.32) and consider

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{1}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

This series is the divergent harmonic series (see Example 3 of Section 8.2). Hence, by Definition (8.32), the alternating harmonic series is not absolutely convergent.

Series that are convergent but not absolutely convergent, such as the alternating harmonic series in Example 4, are given a special name, as indicated in the next definition.

Definition 8.33

A series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ is convergent and $\sum |a_n|$ is divergent.

The following theorem tells us that absolute convergence implies convergence.

Theorem 8.34

If a series $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

PROOF If we let $b_n = a_n + |a_n|$ and we make use of the property that $-|a_n| \le a_n \le |a_n|$, then

$$0 \le a_n + |a_n| \le 2|a_n|, \quad \text{or} \quad 0 \le b_n \le 2|a_n|.$$

If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent and hence, by Theorem (8.20)(ii), $\sum 2|a_n|$ is convergent. If we apply the basic comparison test (8.26), it follows that $\sum b_n$ is convergent. By (8.20)(iii), $\sum(b_n - |a_n|)$ is convergent. Since $b_n - |a_n| = a_n$, the proof is complete. ■

EXAMPLE 5 Let $\sum a_n$ be the series

$$\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7} - \frac{1}{2^8} + \cdots,$$

where the signs of the terms vary in pairs as indicated and where $|a_n| = 1/2^n$. Determine whether $\sum a_n$ converges or diverges.

SOLUTION The series is neither alternating nor geometric nor positive-term, so none of the earlier tests can be applied. Let us consider the series of absolute values:

$$\sum |a_n| = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} + \cdots$$

This series is geometric, with $r = \frac{1}{2}$, and since $\frac{1}{2} < 1$, it is convergent, by Theorem (8.15)(i). Thus the given series is absolutely convergent and hence, by Theorem (8.34), it is convergent.

EXAMPLE 6 Determine whether the following series is convergent or divergent:

$$\sin 1 + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \cdots + \frac{\sin n}{n^2} + \cdots$$

SOLUTION The series contains both positive and negative terms, but it is not an alternating series, because, for example, the first three terms are positive and the next three are negative. The series of absolute values is

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}.$$

Since $$\frac{|\sin n|}{n^2} \le \frac{1}{n^2},$$

the series of absolute values $\sum |(\sin n)/n^2|$ is dominated by the convergent p-series $\sum(1/n^2)$ and hence is convergent. Thus, the given series is absolutely convergent and therefore is convergent, by Theorem (8.34).

We see from the preceding discussion that an arbitrary series may be classified in exactly *one* of the following ways:

(i) absolutely convergent **(ii)** conditionally convergent **(iii)** divergent

Of course, for positive-term series, we need only determine convergence or divergence.

The following form of the ratio test may be used to investigate absolute convergence.

Ratio Test for Absolute Convergence 8.35

Let $\sum a_n$ be a series of nonzero terms, and suppose

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

(i) If $L < 1$, the series is absolutely convergent.

(ii) If $L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the series is divergent.

(iii) If $L = 1$, apply a different test; the series may be absolutely convergent, conditionally convergent, or divergent.

The proof is similar to that of (8.28). Note that for positive-term series the two ratio tests are identical.

We can also state a root test for absolute convergence. The statement is the same as that of (8.29), except that we replace $\sqrt[n]{a_n}$ with $\sqrt[n]{|a_n|}$.

EXAMPLE 7 Determine whether the following series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+4}{2^n}$$

SOLUTION Using the ratio test (8.35), we obtain

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2+4}{2^{n+1}} \cdot \frac{2^n}{n^2+4}\right|$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(\frac{n^2+2n+5}{n^2+4}\right) = \frac{1}{2}(1) = \frac{1}{2} < 1.$$

Hence, by (8.35)(i), the series is absolutely convergent.

It can be proved that if a series $\sum a_n$ is absolutely convergent and if the terms are rearranged in any manner, then the resulting series converges and has the same sum as the given series, which is not true for conditionally convergent series. If $\sum a_n$ is conditionally convergent, then by suitably rearranging terms, we can obtain either a divergent series or a series that converges and has any desired sum S. (See an advanced calculus text for details.)

We now have a variety of tests that can be used to investigate a series for convergence or divergence. Considerable skill is needed to determine which test is best suited for a particular series. This skill can be obtained by working many exercises involving different types of series. The following summary may be helpful in deciding which test to apply; however, some series cannot be investigated by any of these tests. In those cases, it may be necessary to use results from advanced mathematics courses.

Summary of Convergence and Divergence Tests for Series

Test	Series	Convergence or divergence	Comments
nth-term	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim_{n\to\infty} a_n = 0$
Geometric series	$\sum_{n=1}^{\infty} ar^{n-1}$	(i) Converges with sum $S = \dfrac{a}{1-r}$ if $\lvert r\rvert < 1$ (ii) Diverges if $\lvert r\rvert \geq 1$	Useful for comparison tests if the nth term a_n of a series is *similar* to ar^{n-1}
p-series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	(i) Converges if $p > 1$ (ii) Diverges if $p \leq 1$	Useful for comparison tests if the nth term a_n of a series is *similar* to $1/n^p$
Integral	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$	(i) Converges if $\int_1^{\infty} f(x)\,dx$ converges (ii) Diverges if $\int_1^{\infty} f(x)\,dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing, and readily integrable.
Comparison	$\sum a_n, \sum b_n$ $a_n > 0, b_n > 0$	(i) If $\sum b_n$ converges and $a_n \leq b_n$ for every n, then $\sum a_n$ converges. (ii) If $\sum b_n$ diverges and $a_n \geq b_n$ for every n, then $\sum a_n$ diverges. (iii) If $\lim_{n\to\infty}(a_n/b_n) = c$ for some positive real number c, then both series converge or both diverge.	The comparison series $\sum b_n$ is often a geometric series or a p-series. To find b_n in (iii), consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$	If $\lim_{n\to\infty}\left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert = L$ (or ∞), the series (i) converges (absolutely) if $L < 1$ (ii) diverges if $L > 1$ (or ∞)	Inconclusive if $L = 1$ Useful if a_n involves factorials or nth powers If $a_n > 0$ for every n, the absolute value sign may be disregarded.
Root	$\sum a_n$	If $\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert} = L$ (or ∞), the series (i) converges (absolutely) if $L < 1$ (ii) diverges if $L > 1$ (or ∞)	Inconclusive if $L = 1$ Useful if a_n involves nth powers If $a_n > 0$ for every n, the absolute value sign may be disregarded.
Alternating series	$\sum(-1)^n a_n$ $a_n > 0$	Converges if $a_k \geq a_{k+1}$ for every k and $\lim_{n\to\infty} a_n = 0$	Applicable only to an alternating series
$\sum\lvert a_n\rvert$	$\sum a_n$	If $\sum\lvert a_n\rvert$ converges, then $\sum a_n$ converges.	Useful for series that contain both positive and negative terms

EXERCISES 8.5

Exer. 1–4: Determine whether the series (a) satisfies conditions (i) and (ii) of the alternating series test (8.30) and (b) converges or diverges.

1 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n^2+7}$

2 $\sum_{n=1}^{\infty}(-1)^{n-1}n5^{-n}$

3 $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$

4 $\sum_{n=1}^{\infty}(-1)^n\dfrac{e^{2n}+1}{e^{2n}-1}$

Exer. 5–32: Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

5 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{\sqrt{2n+1}}$

6 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n^{2/3}}$

7 $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{\ln(n+1)}$

8 $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n^2+4}$

9 $\sum_{n=2}^{\infty}(-1)^n\dfrac{n}{\ln n}$

10 $\sum_{n=1}^{\infty}(-1)^n\dfrac{\ln n}{n}$

11 $\sum_{n=1}^{\infty}(-1)^n\dfrac{5}{n^3+1}$

12 $\sum_{n=1}^{\infty}(-1)^n e^{-n}$

13 $\sum_{n=1}^{\infty}\dfrac{(-10)^n}{n!}$

14 $\sum_{n=1}^{\infty}\dfrac{n!}{(-5)^n}$

15 $\sum_{n=1}^{\infty}(-1)^n\dfrac{n^2+3}{(2n-5)^2}$

16 $\sum_{n=1}^{\infty}\dfrac{\sin\sqrt{n}}{\sqrt{n^3+4}}$

17 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{\sqrt[3]{n}}{n+1}$

18 $\sum_{n=1}^{\infty}(-1)^n\dfrac{(n+1)^2}{n^5+1}$

19 $\sum_{n=1}^{\infty}\dfrac{\cos\frac{1}{6}\pi n}{n^2}$

20 $\sum_{n=1}^{\infty}(-1)^n\dfrac{\ln n}{(1.5)^n}$

21 $\sum_{n=1}^{\infty}(-1)^n n\sin\dfrac{1}{n}$

22 $\sum_{n=1}^{\infty}(-1)^n\dfrac{\arctan n}{n^2}$

23 $\sum_{n=2}^{\infty}(-1)^n\dfrac{1}{n\sqrt{\ln n}}$

24 $\sum_{n=1}^{\infty}(-1)^n\dfrac{2^{1/n}}{n!}$

25 $\sum_{n=1}^{\infty}\dfrac{n^n}{(-5)^n}$

26 $\sum_{n=1}^{\infty}\dfrac{(n^2+1)^n}{(-n)^n}$

27 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1+4^n}{1+3^n}$

28 $\sum_{n=1}^{\infty}(-1)^n\dfrac{n^4}{e^n}$

29 $\sum_{n=1}^{\infty}(-1)^n\dfrac{\cos\pi n}{n}$

30 $\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{(2n-1)\pi}{2}$

31 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{(n-4)^2+5}$

32 $\sum_{n=1}^{\infty}(-1)^n\dfrac{\ln n}{\sqrt[3]{n}}$

c **Exer. 33–38: Approximate the sum of each series to three decimal places.**

33 $\sum_{n=0}^{\infty}(-1)^n\dfrac{1}{n!}$

34 $\sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{1}{(2n)!}$

35 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n^3}$

36 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{1}{n^5}$

37 $\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{n+1}{5^n}$

38 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n}\left(\dfrac{1}{2}\right)^n$

c **Exer. 39–42: Use Theorem (8.31) to find a positive integer n such that S_n approximates the sum of the series to four decimal places.**

39 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n^2}$

40 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{\sqrt{n}}$

41 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n^n}$

42 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{n^3+1}$

Exer. 43–44: Show that the alternating series converges for every positive integer k.

43 $\sum_{n=1}^{\infty}(-1)^n\dfrac{(\ln n)^k}{n}$

44 $\sum_{n=1}^{\infty}(-1)^n\dfrac{1}{\sqrt[k]{n}}$

45 If $\sum a_n$ and $\sum b_n$ are both convergent series, is $\sum a_n b_n$ convergent? Explain.

46 If $\sum a_n$ and $\sum b_n$ are both divergent series, is $\sum a_n b_n$ divergent? Explain.

8.6 POWER SERIES

The most important reason for developing the theory in the previous sections is to represent functions as *power series*—that is, as series whose terms contain powers of a variable x. To illustrate, if we use the formula