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4.5 PROPERTIES OF THE DEFINITE INTEGRAL

This section contains some fundamental properties of the definite integral. Most of the proofs are difficult and have been placed in Appendix I.

Theorem 4.21

> If c is a real number, then
> $$\int_a^b c\,dx = c(b-a).$$

PROOF Let f be the constant function defined by $f(x) = c$ for every x in $[a, b]$. If P is a partition of $[a, b]$, then for every Riemann sum of f,

$$\sum_k f(w_k)\Delta x_k = \sum_k c\Delta x_k = c\sum_k \Delta x_k = c(b-a).$$

(The last equality is true because the sum $\sum_k \Delta x_k$ is the length of the interval $[a, b]$.) Consequently,

$$\left|\sum_k f(w_k)\Delta x_k - c(b-a)\right| = |c(b-a) - c(b-a)| = 0,$$

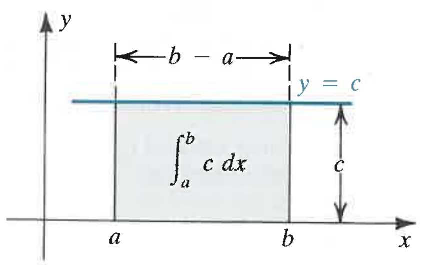
which is less than any positive number ϵ *regardless* of the size of $\|P\|$. Thus, by Definition (4.15), with $L = c(b-a)$,

$$\lim_{\|P\|\to 0}\sum_k f(w_k)\Delta x_k = \lim_{\|P\|\to 0}\sum_k c\Delta x_k = c(b-a).$$

By Definition (4.16), we thus obtain

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a). \quad ■$$

Figure 4.20



Note that if $c > 0$, then Theorem (4.21) agrees with Theorem (4.19): As illustrated in Figure 4.20, the graph of f is the horizontal line $y = c$, and the region under the graph from a to b is a rectangle with sides of lengths c and $b - a$. Hence the area $\int_a^b f(x)\,dx$ of the rectangle is $c(b-a)$.

EXAMPLE ■ 1 Evaluate $\int_{-2}^{3} 7\,dx$.

SOLUTION Using Theorem (4.21) yields

$$\int_{-2}^{3} 7\,dx = 7[3-(-2)] = 7(5) = 35.$$

If $c = 1$ in Theorem (4.21), we shall abbreviate the integrand as follows:

$$\int_a^b dx = b - a$$

If a function f is integrable on $[a, b]$ and c is a real number, then, by Theorem (4.11)(ii), a Riemann sum of the function cf may be written

$$\sum_k cf(w_k)\Delta x_k = c\sum_k f(w_k)\Delta x_k.$$

We can prove that the limit of the sums on the left of the last equation is equal to c times the limit of the sums on the right. This gives us the next theorem. A proof may be found in Appendix I.

Theorem 4.22

> If f is integrable on $[a, b]$ and c is any real number, then cf is integrable on $[a, b]$ and
> $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Theorem (4.22) is sometimes stated as follows: *A constant factor in the integrand may be taken outside the integral sign.* It is *not* permissible to take expressions involving variables outside the integral sign in this manner.

If two functions f and g are defined on $[a, b]$, then, by Theorem (4.11)(i), a Riemann sum of $f + g$ may be written

$$\sum_k [f(w_k) + g(w_k)]\Delta x_k = \sum_k f(w_k)\Delta x_k + \sum_k g(w_k)\Delta x_k$$

We can show that if f and g are integrable, then the limit of the sums on the left may be found by adding the limits of the two sums on the right. This fact is stated in integral form in (i) of the next theorem. A proof of (i) may be found in Appendix I. The analogous result for differences is stated in (ii) of the theorem.

Theorem 4.23

> If f and g are integrable on $[a, b]$, then $f + g$ and $f - g$ are integrable on $[a, b]$ and
>
> **(i)** $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
>
> **(ii)** $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Theorem (4.23)(i) may be extended to any finite number of functions. Thus, if $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$, then so is their sum and

$$\int_a^b [f_1(x) + f_2(x) + \cdots + f_n(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx.$$

EXAMPLE ■ 2 It will follow from the results in Section 4.6 that

$$\int_0^2 x^3\,dx = 4 \quad\text{and}\quad \int_0^2 x\,dx = 2.$$

Use these facts to evaluate $\int_0^2 (5x^3 - 3x + 6)\,dx$.

SOLUTION We may proceed as follows:

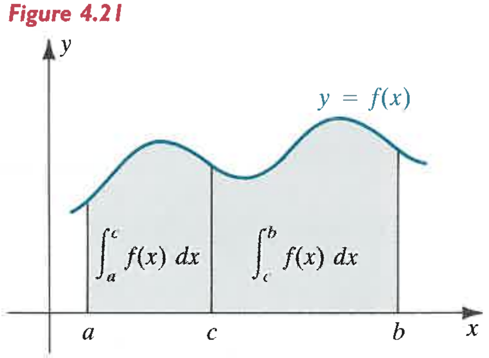
$$\begin{aligned}\int_0^2 (5x^3 - 3x + 6)\,dx &= \int_0^2 5x^3\,dx - \int_0^2 3x\,dx + \int_0^2 6\,dx\\ &= 5\int_0^2 x^3\,dx - 3\int_0^2 x\,dx + 6(2-0)\\ &= 5(4) - 3(2) + 12 = 26\end{aligned}$$

If f is continuous on $[a, b]$ and $f(x) \geq 0$ for every x in $[a, b]$, then, by Theorem (4.19), the integral $\int_a^b f(x)\,dx$ is the area under the graph of f from a to b. Similarly, if $a < c < b$, then the integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are the areas under the graph of f from a to c and from c to b, respectively, as illustrated in Figure 4.21. Since the area from a to b is the sum of the two smaller areas, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The next theorem shows that the preceding equality is true under a more general hypothesis. The proof is given in Appendix I.

Figure 4.21



Theorem 4.24

If $a < c < b$ and if f is integrable on both $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The following result is a generalization of Theorem (4.24) to the case where c is not necessarily between a and b.

Theorem 4.25

If f is integrable on a closed interval and if a, b, and c are any three numbers in the interval, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

PROOF If a, b, and c are all different, then there are six possible ways of arranging these three numbers. The theorem should be verified for each of these cases and also for the cases in which two or all three of the numbers are equal. We shall verify one case. Suppose that $c < a < b$. By Theorem (4.24),

$$\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx,$$

which, in turn, may be written

$$\int_a^b f(x)\,dx = -\int_c^a f(x)\,dx + \int_c^b f(x)\,dx.$$

The conclusion of the theorem now follows from the fact that interchanging the limits of integration changes the sign of the integral (see Definition 4.17). ■

EXAMPLE ■ 3 Express as one integral:

$$\int_2^7 f(x)\,dx - \int_5^7 f(x)\,dx$$

SOLUTION First we interchange the limits of the second integral using Definition (4.17) and then use Theorem (4.25) with $a = 2, b = 5,$ and $c = 7$:

$$\begin{aligned}\int_2^7 f(x)\,dx - \int_5^7 f(x)\,dx &= \int_2^7 f(x)\,dx + \int_7^5 f(x)\,dx\\ &= \int_2^5 f(x)\,dx\end{aligned}$$

As an alternative solution, by recognizing that

$$\int_2^7 f(x)\,dx = \int_2^5 f(x)\,dx + \int_5^7 f(x)\,dx,$$

the previous result immediately follows.

If f and g are continuous on $[a, b]$ and $f(x) \geq g(x) \geq 0$ for every x in $[a, b]$, then the area under the graph of f from a to b is greater than or equal to the area under the graph of g from a to b. The corollary to the next theorem is a generalization of this fact to arbitrary integrable functions. The proof of the theorem is given in Appendix I.

Theorem 4.26

If f is integrable on $[a, b]$ and $f(x) \geq 0$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

Corollary 4.27

If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for every x in $[a, b]$, then

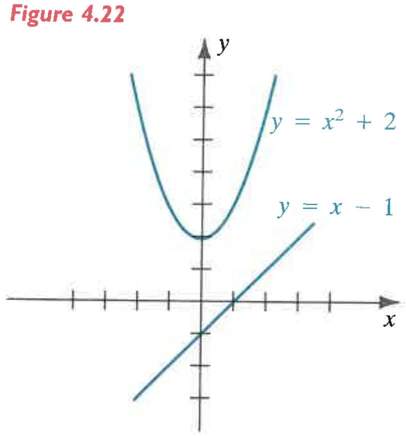
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

PROOF By Theorem (4.23), $f - g$ is integrable. Moreover, since $f(x) \geq g(x)$, $f(x) - g(x) \geq 0$ for every x in $[a, b]$. Hence, by Theorem (4.26),

$$\int_a^b [f(x) - g(x)]\,dx \geq 0.$$

Applying Theorem (4.23)(ii) leads to the desired conclusion. ■

Figure 4.22



EXAMPLE 4 Show that $\int_{-1}^2 (x^2 + 2)\,dx \geq \int_{-1}^2 (x - 1)\,dx$.

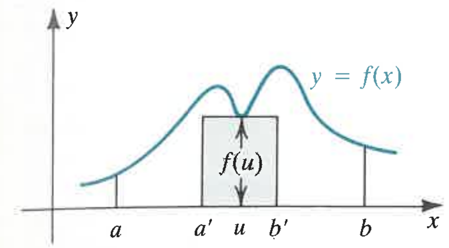
SOLUTION The graphs of $y = x^2 + 2$ and $y = x - 1$ are sketched in Figure 4.22. Since

$$x^2 + 2 \geq x - 1$$

for every x in $[-1, 2]$, the conclusion follows from Corollary (4.27).

Suppose, in Theorem (4.26), that f is continuous and that, in addition to the condition $f(x) \geq 0$, we have $f(c) > 0$ for some c in $[a, b]$. In this case, $\lim_{x\to c} f(x) > 0$, and, by Theorem (1.6), there is a subinterval $[a', b']$ of $[a, b]$ throughout which $f(x)$ is positive. If $f(u)$ is the minimum value of f on $[a', b']$ (see Figure 4.23), then the area under the graph of f from a to b is at least as large as the area $f(u)(b' - a')$ of the pictured rectangle. Consequently, $\int_a^b f(x)\,dx > 0$. It now follows, as in the proof of Corollary (4.27), that if f and g are continuous on $[a, b]$, if $f(x) \geq g(x)$ throughout $[a, b]$, and if $f(x) > g(x)$ for some x in $[a, b]$, then $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$. This fact will be used in the proof of the next theorem.

Figure 4.23



Mean Value Theorem for Definite Integrals 4.28

If f is continuous on a closed interval $[a, b]$, then there is a number z in the open interval (a, b) such that

$$\int_a^b f(x)\,dx = f(z)(b - a).$$

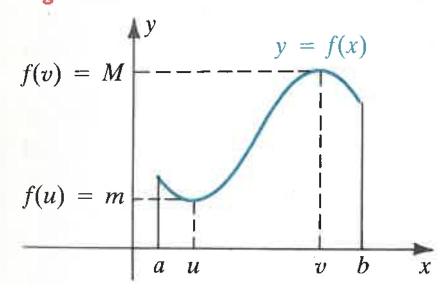
PROOF If f is a constant function, then $f(x) = c$ for some number c, and by Theorem (4.21),

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b - a) = f(z)(b - a)$$

for every number z in (a, b).

Next, assume that f is not a constant function and suppose that m and M are the minimum and maximum values of f, respectively, on $[a, b]$. Let $f(u) = m$ and $f(v) = M$ for some u and v in $[a, b]$, as illustrated in Figure 4.24 for the case in which $f(x)$ is positive throughout $[a, b]$. Since f is not a constant function, $m < f(x) < M$ for some x in $[a, b]$. Hence, by the remark immediately preceding this theorem,

Figure 4.24



$$\int_a^b m\,dx < \int_a^b f(x)\,dx < \int_a^b M\,dx.$$

Applying Theorem (4.21) yields

$$m(b - a) < \int_a^b f(x)\,dx < M(b - a).$$

Dividing by $b - a$ and recalling that $m = f(u)$ and $M = f(v)$ gives us

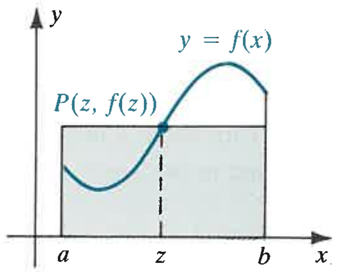
$$f(u) < \frac{1}{b - a}\int_a^b f(x)\,dx < f(v).$$

Since $[1/(b - a)]\int_a^b f(x)\,dx$ is a number between $f(u)$ and $f(v)$, it follows from the intermediate value theorem (1.26) that there is a number z, with $u < z < v$, such that

$$f(z) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Multiplying both sides by $b - a$ gives us the conclusion of the theorem. ■

Figure 4.25

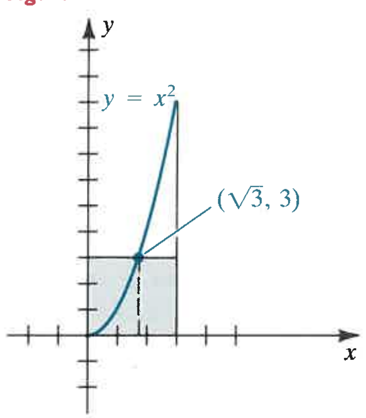


The number z of Theorem (4.28) is not necessarily unique; however, the theorem guarantees that *at least* one number z will produce the desired result.

The mean value theorem has an interesting geometric interpretation if $f(x) \geq 0$ on $[a, b]$. In this case, $\int_a^b f(x)\,dx$ is the area under the graph of f from a to b. If, as in Figure 4.25, a horizontal line is drawn through the point $P(z, f(z))$, then the area of the rectangular region bounded by this line, the x-axis, and the lines $x = a$ and $x = b$ is $f(z)(b - a)$, which, according to Theorem (4.28), is the same as the area under the graph of f from a to b.

EXAMPLE 5 It will follow from the results of Section 4.6 that $\int_0^3 x^2\,dx = 9$. Find a number z that satisfies the conclusion of the mean value theorem (4.28) for this definite integral.

Figure 4.26



SOLUTION The graph of $f(x) = x^2$ for $0 \leq x \leq 3$ is sketched in Figure 4.26. By the mean value theorem, there is a number z between 0 and 3 such that

$$\int_0^3 x^2\,dx = f(z)(3 - 0) = z^2(3).$$

This result implies that

$$9 = 3z^2, \quad \text{or} \quad z^2 = 3.$$

The solutions of the last equation are $z = \pm\sqrt{3}$; however, $-\sqrt{3}$ is not in $[0, 3]$. The number $z = \sqrt{3}$ satisfies the conclusion of the theorem.

If we consider the horizontal line through $P(\sqrt{3}, 3)$, then the area of the rectangle bounded by this line, the x-axis, and the lines $x = 0$ and $x = 3$ is equal to the area under the graph of f from $x = 0$ to $x = 3$ (see Figure 4.26).

In statistics, the term **arithmetic mean** is used for the **average** of a set of numbers. Therefore, the arithmetic mean of two numbers a and b is $(a + b)/2$, the arithmetic mean of three numbers $a, b,$ and c is $(a + b + c)/3$, and so on. To see the relationship between arithmetic means and the word *mean* used in *mean value theorem*, let us rewrite the conclusion of (4.28) as

$$f(z) = \frac{1}{b - a}\int_a^b f(x)\,dx$$

and express the definite integral as a limit of sums. If we specialize Definition (4.16) by using a regular partition P with n subintervals, then

$$f(z) = \frac{1}{b - a}\lim_{n\to\infty}\sum_{k=1}^{n} f(w_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[f(w_k)\frac{\Delta x}{b - a}\right]$$

for any number w_k in the kth subinterval of P and $\Delta x = (b - a)/n$. Since $\Delta x/(b - a) = 1/n$, we obtain

$$f(z) = \lim_{n\to\infty}\sum_{k=1}^{n}\left[f(w_k)\frac{\Delta x}{b - a}\right] = \lim_{n\to\infty}\sum_{k=1}^{n}\left[f(w_k)\frac{1}{n}\right],$$

or

$$f(z) = \lim_{n\to\infty}\left[\frac{f(w_1) + f(w_2) + \cdots + f(w_n)}{n}\right].$$

This result shows that we may regard the number $f(z)$ in the mean value theorem (4.28) as a limit of the arithmetic means (averages) of the function values $f(w_1), f(w_2), \ldots, f(w_n)$ as n increases without bound. This fact is the motivation for the next definition.

Definition 4.29

Let f be continuous on $[a, b]$. The **average value** f_{av} of f on $[a, b]$ is

$$f_{av} = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Note that, by the mean value theorem for definite integrals, if f is continuous on $[a, b]$, then

$$f_{av} = f(z) \quad \text{for some } z \text{ in } [a, b].$$

EXAMPLE 6 Given $\int_0^3 x^2\,dx = 9$, find the average value of f on $[0, 3]$.

SOLUTION By Definition (4.29), with $a = 0$, $b = 3$, and $f(x) = x^2$,

$$f_{av} = \frac{1}{3 - 0}\int_0^3 x^2\,dx = \frac{1}{3}\cdot 9 = 3.$$

In the interval $[0, 3]$, the function values $f(x) = x^2$ range from $f(0) = 0$ to $f(3) = 9$. Note that the function f *takes on* its average value 3 at the number $z = \sqrt{3}$.

EXERCISES 4.5

Exer. 1–6: Evaluate the integral.

1 $\int_{-2}^{4} 5\,dx$ 2 $\int_{1}^{10} \sqrt{2}\,dx$ 3 $\int_{6}^{2} 3\,dx$ 4 $\int_{4}^{-3} dx$ 5 $\int_{-1}^{1} dx$ 6 $\int_{2}^{2} 100\,dx$

Exer. 7–10: It will follow from the results in Section 4.6 that

$$\int_1^4 x^2\,dx = 21 \quad \text{and} \quad \int_1^4 x\,dx = \tfrac{15}{2}.$$

Use these facts to evaluate the integral.

7 $\int_1^4 (3x^2+5)\,dx$ **8** $\int_1^4 (6x-1)\,dx$

9 $\int_1^4 (2-9x-4x^2)\,dx$ **10** $\int_1^4 (3x+2)^2\,dx$

Exer. 11–16: Verify the inequality without evaluating the integrals.

11 $\int_1^2 (3x^2+4)\,dx \ge \int_1^2 (2x^2+5)\,dx$

12 $\int_1^4 (2x+2)\,dx \le \int_1^4 (3x+1)\,dx$

13 $\int_2^4 (x^2-6x+8)\,dx \le 0$ **14** $\int_2^4 (5x^2-x+1)\,dx \ge 0$

15 $\int_0^{2\pi} (1+\sin x)\,dx \ge 0$ **16** $\int_{-\pi/3}^{\pi/3} (\sec x - 2)\,dx \le 0$

Exer. 17–22: Express as one integral.

17 $\int_5^1 f(x)\,dx + \int_{-3}^5 f(x)\,dx$

18 $\int_4^1 f(x)\,dx + \int_6^4 f(x)\,dx$

19 $\int_c^d f(x)\,dx + \int_e^c f(x)\,dx$

20 $\int_{-2}^6 f(x)\,dx - \int_{-2}^2 f(x)\,dx$

21 $\int_c^{c+h} f(x)\,dx - \int_c^h f(x)\,dx$

22 $\int_c^m f(x)\,dx - \int_d^m f(x)\,dx$

Exer. 23–30: The given integral $\int_a^b f(x)\,dx$ may be verified using the results in Section 4.6. (a) Find a number z that satisfies the conclusion of the mean value theorem (4.28). (b) Find the average value of f on $[a, b]$.

23 $\int_0^3 3x^2\,dx = 27$ **24** $\int_{-4}^{-1} \frac{3}{x^2}\,dx = \frac{9}{4}$

25 $\int_{-2}^1 (x^2+1)\,dx = 6$

26 $\int_{-1}^3 (3x^2-2x+3)\,dx = 32$

27 $\int_{-1}^8 3\sqrt{x+1}\,dx = 54$ **28** $\int_{-2}^{-1} \frac{8}{x^3}\,dx = -3$

29 $\int_1^2 (4x^3-1)\,dx = 14$

30 $\int_1^4 (2+3\sqrt{x})\,dx = 20$

c **Exer. 31–32: The given integral may be verified using results in Section 4.6. Use Newton's method to approximate, to three decimal places, a number z that satisfies the conclusion of the mean value theorem (4.28).**

31 $\int_{-2}^3 (8x^3+3x-1)\,dx = 132.5$

32 $\int_{\pi/6}^{\pi/4} (1-\cos 4x)\,dx = \frac{\pi}{12} + \frac{\sqrt{3}}{8}$

33 Let f and g be integrable on $[a, b]$. If c and d are any real numbers, prove that

$$\int_a^b [cf(x)+dg(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx.$$

34 If f is continuous on $[a, b]$, prove that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

(*Hint:* $-|f(x)| \le f(x) \le |f(x)|$.)

4.6 THE FUNDAMENTAL THEOREM OF CALCULUS

This section contains one of the most important theorems in calculus. In addition to being useful in evaluating definite integrals, the theorem also exhibits the relationship between derivatives and definite integrals. This theorem, aptly called *the fundamental theorem of calculus*, was discovered independently by Sir Isaac Newton and by Gottfried Wilhelm Leibniz. It is primarily because of this discovery that both men are credited with the invention of calculus.

Figure 4.27

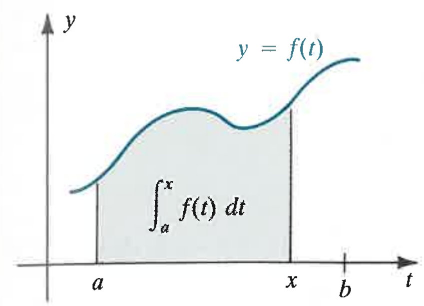
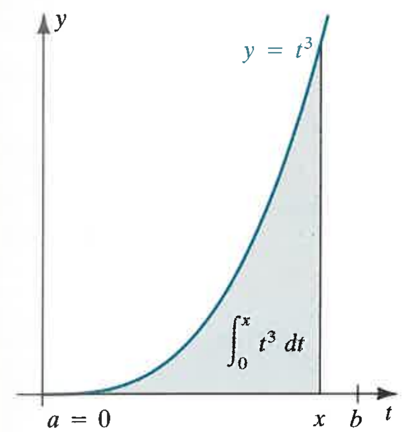


Figure 4.28



To avoid confusion in the following discussion, we shall use t as the independent variable and denote the definite integral of f from a to b by $\int_a^b f(t)\,dt$. If f is continuous on $[a, b]$ and $a \le x \le b$, then f is continuous on $[a, x]$; therefore, by Theorem (4.20), f is integrable on $[a, x]$. Consequently, the formula

$$G(x) = \int_a^x f(t)\,dt$$

determines a function G with domain $[a, b]$, since for each x in $[a, b]$, there corresponds a unique number $G(x)$.

To obtain a geometric interpretation of $G(x)$, suppose that $f(t) \ge 0$ for every t in $[a, b]$. In this case, we see from Theorem (4.19) that $G(x)$ is the area of the region under the graph of f from a to x (see Figure 4.27).

As a specific illustration, consider $f(t) = t^3$ with $a = 0$ and $b > 0$ (see Figure 4.28). In Example 8 of Section 4.3, we proved that the area under the graph of f from 0 to b is $\frac{1}{4}b^4$. Hence the area from 0 to x is

$$G(x) = \int_0^x t^3\,dt = \tfrac{1}{4}x^4.$$

This gives us an explicit form for the function G if $f(t) = t^3$. Note that in this illustration,

$$G'(x) = \frac{d}{dx}\left(\tfrac{1}{4}x^4\right) = x^3 = f(x).$$

Thus, by Definition (4.1), G is an antiderivative of f. This result is not an accident. Part I of the next theorem brings out the remarkable fact that if f is *any* continuous function and $G(x) = \int_a^x f(t)\,dt$, then G is an antiderivative of f. Part II of the theorem shows how *any* antiderivative may be used to find the value of $\int_a^b f(x)\,dx$.

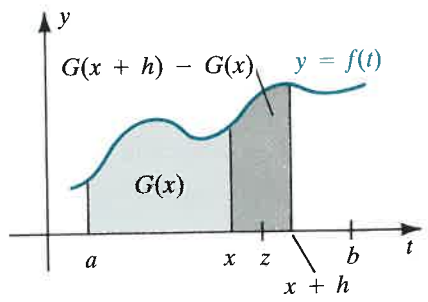
Fundamental Theorem of Calculus 4.30

> Suppose f is continuous on a closed interval $[a, b]$.
>
> **Part I** If the function G is defined by
>
> $$G(x) = \int_a^x f(t)\,dt$$
>
> for every x in $[a, b]$, then G is an antiderivative of f on $[a, b]$.
>
> **Part II** If F is any antiderivative of f on $[a, b]$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF To establish Part I, we must show that if x is in $[a, b]$, then $G'(x) = f(x)$—that is,

$$\lim_{h\to 0} \frac{G(x+h)-G(x)}{h} = f(x).$$

Figure 4.29



Before giving a formal proof, let us consider some geometric aspects of this limit. If $f(x) \geq 0$ throughout $[a, b]$, then $G(x)$ is the area under the graph of f from a to x, as illustrated in Figure 4.29. If $h > 0$, then the difference $G(x+h) - G(x)$ is the area under the graph of f from x to $x+h$, and the number h is the length of the interval $[x, x+h]$. We shall show that

$$\frac{G(x+h) - G(x)}{h} = f(z)$$

for some value z between x and $x+h$. Apparently, if $h \to 0$, then $z \to x$ and $f(z) \to f(x)$, which is what we wish to prove.

Let us now give a rigorous proof that $G'(x) = f(x)$. If x and $x+h$ are in $[a, b]$, then using the definition of G together with Definition (4.17) and Theorem (4.24) yields

$$\begin{aligned} G(x+h) - G(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt. \end{aligned}$$

Consequently, if $h \neq 0$, then

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

If $h > 0$, then, by the mean value theorem (4.28), there is a number z in the open interval $(x, x+h)$ such that

$$\int_x^{x+h} f(t)\,dt = f(z)h$$

and, therefore,

$$\frac{G(x+h) - G(x)}{h} = f(z).$$

Since $x < z < x+h$, it follows from the continuity of f that

$$\lim_{h\to 0^+} f(z) = \lim_{z\to x^+} f(z) = f(x)$$

and hence

$$\lim_{h\to 0^+} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0^+} f(z) = f(x).$$

If $h < 0$, then we may prove in similar fashion that

$$\lim_{h\to 0^-} \frac{G(x+h) - G(x)}{h} = f(x).$$

The two preceding one-sided limits imply that

$$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = f(x).$$

This completes the proof of Part I.

To prove Part II, let F be any antiderivative of f and let G be the special antiderivative defined in Part I. From Theorem (4.2), we know that there is a constant C such that

$$G(x) = F(x) + C$$

for every x in $[a, b]$. Hence, from the definition of G,

$$\int_a^x f(t)\,dt = F(x) + C$$

for every x in $[a, b]$. If we let $x = a$ and use the fact that $\int_a^a f(t)\,dt = 0$, we obtain $0 = F(a) + C$, or $C = -F(a)$. Consequently,

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

This is an identity for every x in $[a, b]$, so we may substitute b for x, obtaining

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Replacing the variable t by x gives us the conclusion of Part II. ■

We often denote the difference $F(b) - F(a)$ either by $F(x)]_a^b$ or by $[F(x)]_a^b$. Part II of the fundamental theorem may then be expressed as follows.

Corollary 4.31

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a).$$

The formula in Corollary (4.31) is also valid if $a \geq b$. If $a > b$, then, by Definition (4.17),

$$\begin{aligned} \int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx \\ &= -[F(a) - F(b)] \\ &= F(b) - F(a). \end{aligned}$$

If $a = b$, then by Definition (4.18),

$$\int_a^a f(x)\,dx = 0 = F(a) - F(a).$$

Corollary (4.31) allows us to evaluate a definite integral very easily if we can find an antiderivative of the integrand. For example, since an antiderivative of x^3 is $\frac{1}{4}x^4$, we have

$$\int_0^b x^3\,dx = \frac{1}{4}x^4\Big]_0^b = \frac{1}{4}b^4 - \frac{1}{4}(0)^4 = \frac{1}{4}b^4.$$

Those who doubt the importance of the fundamental theorem should compare this simple computation with the limit of sums calculation discussed in Example 8 of Section 4.3.

EXAMPLE ■ 1 Evaluate $\int_{-2}^{3}(6x^2-5)\,dx$.

SOLUTION An antiderivative of $6x^2-5$ is $F(x)=2x^3-5x$. Applying Corollary (4.31), we get

$$\begin{aligned}\int_{-2}^{3}(6x^2-5)\,dx &= \left[2x^3-5x\right]_{-2}^{3}\\ &= [2(3)^3-5(3)]-[2(-2)^3-5(-2)]\\ &= [54-15]-[-16+10]=45.\end{aligned}$$

Note that if $F(x)+C$ is used in place of $F(x)$ in Corollary (4.31), the same result is obtained, since

$$\begin{aligned}\left[F(x)+C\right]_a^b &= [F(b)+C]-[F(a)+C]\\ &= F(b)-F(a)\\ &= \left[F(x)\right]_a^b.\end{aligned}$$

In particular, since

$$\int f(x)\,dx = F(x)+C,$$

where $F'(x)=f(x)$, we obtain the following theorem.

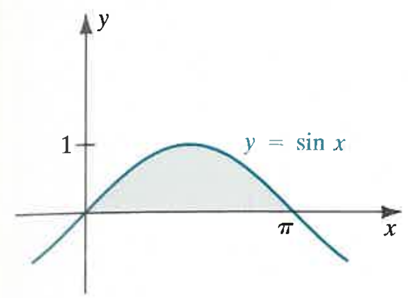
Theorem 4.32

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

Theorem (4.32) states that *a definite integral can be evaluated by evaluating the corresponding indefinite integral.* As with previous cases, when using Theorem (4.32), it is unnecessary to include the constant of integration C for the indefinite integral.

EXAMPLE ■ 2 Find the area A of the region between the graph of $y=\sin x$ and the x-axis from $x=0$ to $x=\pi$.

Figure 4.30



SOLUTION The region is sketched in Figure 4.30. Applying Theorems (4.19) and (4.32) gives us the following:

$$\begin{aligned}A=\int_0^{\pi}\sin x\,dx &= \left[\int \sin x\,dx\right]_0^{\pi}\\ &= \left[-\cos x\right]_0^{\pi}\\ &= -\cos\pi-(-\cos 0)\\ &= -(-1)+1=2\end{aligned}$$

By Theorem (4.32), we can use any formula for indefinite integration to obtain a formula for definite integrals. To illustrate, using Table (4.4), we obtain

$$\int_a^b x^r\,dx = \left[\frac{x^{r+1}}{r+1}\right]_a^b \quad \text{if } r\neq -1$$

$$\int_a^b \sin x\,dx = \left[-\cos x\right]_a^b$$

$$\int_a^b \sec^2 x\,dx = \left[\tan x\right]_a^b.$$

EXAMPLE ■ 3 Evaluate $\int_{-1}^{2}(x^3+1)^2\,dx$.

SOLUTION We first square the integrand and then apply the power rule to each term as follows:

$$\begin{aligned}\int_{-1}^{2}(x^3+1)^2\,dx &= \int_{-1}^{2}(x^6+2x^3+1)\,dx\\ &= \left[\frac{x^7}{7}+2\cdot\frac{x^4}{4}+x\right]_{-1}^{2}\\ &= \left[\frac{2^7}{7}+2\cdot\frac{2^4}{4}+2\right]-\left[\frac{(-1)^7}{7}+2\frac{(-1)^4}{4}+(-1)\right]\\ &= \frac{405}{14}\end{aligned}$$

EXAMPLE ■ 4 Evaluate $\int_1^4\left(5x-2\sqrt{x}+\frac{32}{x^3}\right)dx$.

SOLUTION We begin by changing the form of the integrand so that the power rule may be applied to each term. Thus,

$$\begin{aligned}\int_1^4 (5x - 2x^{1/2} + 32x^{-3})\,dx &= \left[5\left(\frac{x^2}{2}\right) - 2\left(\frac{x^{3/2}}{3/2}\right) + 32\left(\frac{x^{-2}}{-2}\right)\right]_1^4\\ &= \left[\frac{5}{2}x^2 - \frac{4}{3}x^{3/2} - \frac{16}{x^2}\right]_1^4\\ &= \left[\frac{5}{2}(4)^2 - \frac{4}{3}(4)^{3/2} - \frac{16}{4^2}\right] - \left[\frac{5}{2} - \frac{4}{3} - 16\right]\\ &= \frac{259}{6}.\end{aligned}$$

CAUTION

A common *misuse* of the fundamental theorem of calculus is to make the false interpretation that Corollary (4.31) asserts that if F is a function such that $F'(x) = f(x)$ for some function f, then $\int_a^b f(x)\,dx = F(b) - F(a)$. Guided by this false interpretation, we might make the following fallacious argument:

$$\text{"If } F(x) = \frac{-1}{x}, \quad \text{then} \quad F'(x) = \frac{1}{x^2}$$

$$\text{and so} \quad \int_{-1}^{1} \frac{1}{x^2}\,dx = F(1) - F(-1) = -2."$$

This reasoning is incorrect because it tries to make use of the *conclusion* of Corollary (4.31) in a situation in which the *hypothesis* is not true. Theorems in mathematics are of the form: If a certain set of conditions holds (the hypothesis), then certain conclusions must be true. In Corollary (4.31), the hypothesis is that the function f is continuous on the interval $[a, b]$. In this instance, the function f is not continuous on the interval $[-1, 1]$: Not only is f undefined at $x = 0$, but $\lim_{x\to 0} f(x)$ does not even exist.

Before we apply any theorem in a particular situation, we must check that all of the conditions in the hypothesis are true.

The method of substitution developed for indefinite integrals may also be used to evaluate a definite integral. We could use (4.7) to find an indefinite integral (that is, an antiderivative) and then apply the fundamental theorem of calculus. Another method, which is sometimes shorter, is to change the limits of integration. Using (4.7) together with the fundamental theorem gives us the following formula, with $F' = f$:

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big]_a^b$$

The number on the right may be written

$$F(g(b)) - F(g(a)) = F(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,du.$$

This result gives us the following theorem, provided f and g' are integrable.

Theorem 4.33

If $u = g(x)$, then $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$

Theorem (4.33) states that after making the substitution $u = g(x)$ and $du = g'(x)\,dx$, we may use the values of g that correspond to $x = a$ and $x = b$, respectively, as the limits of the integral involving u. It is then unnecessary to return to the variable x after integrating. This technique is illustrated in the next example.

EXAMPLE 5 Evaluate $\displaystyle\int_2^{10} \frac{3}{\sqrt{5x-1}}\,dx.$

SOLUTION Let us begin by writing the integral as

$$3\int_2^{10} \frac{1}{\sqrt{5x-1}}\,dx.$$

The expression $\sqrt{5x-1}$ in the integrand suggests the following substitution:

$$u = 5x - 1, \qquad du = 5\,dx$$

The form of du indicates that we should introduce the factor 5 into the integrand and then compensate by multiplying the integral by $\frac{1}{5}$, as follows:

$$3\int_2^{10} \frac{1}{\sqrt{5x-1}}\,dx = \frac{3}{5}\int_2^{10} \frac{1}{\sqrt{5x-1}}\,5\,dx$$

We next calculate the values of $u = 5x - 1$ that correspond to the limits of integration $x = 2$ and $x = 10$:

(i) If $x = 2$, then $u = 5(2) - 1 = 9$.

(ii) If $x = 10$, then $u = 5(10) - 1 = 49$.

Substituting in the integrand and changing the limits of integration as in Theorem (4.33) gives us

$$\begin{aligned}3\int_2^{10} \frac{1}{\sqrt{5x-1}}\,dx &= \frac{3}{5}\int_2^{10} \frac{1}{\sqrt{5x-1}}\,5\,dx\\ &= \frac{3}{5}\int_9^{49} \frac{1}{\sqrt{u}}\,du = \frac{3}{5}\int_9^{49} u^{-1/2}\,du\\ &= \left[\left(\frac{3}{5}\right)\frac{u^{1/2}}{1/2}\right]_9^{49} = \frac{6}{5}[49^{1/2} - 9^{1/2}] = \frac{24}{5}.\end{aligned}$$

EXAMPLE 6 Evaluate $\int_0^{\pi/4} (1+\sin 2x)^3 \cos 2x\,dx$.

SOLUTION The integrand suggests the power rule $\int_a^b u^3\,du = \left[\frac{1}{4}u^4\right]_a^b$. Thus, we let

$$u = 1 + \sin 2x, \qquad du = 2\cos 2x\,dx.$$

The form of du indicates that we should introduce the factor 2 into the integrand and multiply the integral by $\frac{1}{2}$, as follows:

$$\int_0^{\pi/4} (1+\sin 2x)^3 \cos 2x\,dx = \tfrac{1}{2}\int_0^{\pi/4} (1+\sin 2x)^3\, 2\cos 2x\,dx.$$

We next calculate the values of $u = 1 + \sin 2x$ that correspond to the limits of integration $x = 0$ and $x = \pi/4$:

(i) If $x = 0$, then $u = 1 + \sin 0 = 1 + 0 = 1$.

(ii) If $x = \dfrac{\pi}{4}$, then $u = 1 + \sin\dfrac{\pi}{2} = 1 + 1 = 2$.

Substituting in the integrand and changing the limits of integration gives us

$$\int_0^{\pi/4} (1+\sin 2x)^3 \cos 2x\,dx = \tfrac{1}{2}\int_1^2 u^3\,du$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{8}[16-1] = \frac{15}{8}.$$

The following theorem illustrates a useful technique for evaluating certain definite integrals.

Theorem 4.34

Let f be continuous on $[-a, a]$.

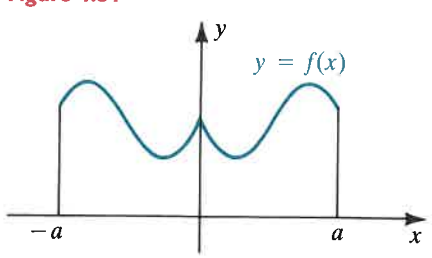
(i) If f is an even function,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

(ii) If f is an odd function,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Figure 4.31



PROOF We shall prove (i). If f is an even function, then the graph of f is symmetric with respect to the y-axis. As a special case, if $f(x) \geq 0$ for every x in $[0, a]$, we have a situation similar to that in Figure 4.31, and hence the area under the graph of f from $x = -a$ to $x = a$ is twice that from $x = 0$ to $x = a$. This gives us the formula in (i).

To show that the formula is true if $f(x) < 0$ for some x, we may proceed as follows. Using, successively, Theorem (4.24), Definition (4.17), and Theorem (4.22), we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^{-a} f(x)(-dx) + \int_0^a f(x)\,dx.$$

Since f is even, $f(-x) = f(x)$, and the last equality may be written

$$\int_{-a}^{a} f(x)\,dx = \int_0^{-a} f(-x)(-dx) + \int_0^a f(x)\,dx.$$

If, in the first integral on the right, we substitute $u = -x$, $du = -dx$ and observe that $u = a$ when $x = -a$, we obtain

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(u)\,du + \int_0^a f(x)\,dx.$$

The last two integrals on the right are equal, since the variables are *dummy variables*, and, therefore,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx. \quad ■$$

EXAMPLE 7 Evaluate

(a) $\int_{-1}^{1} (x^4 + 3x^2 + 1)\,dx$

(b) $\int_{-1}^{1} (x^5 + 3x^3 + x)\,dx$

(c) $\int_{-5}^{5} (2x^3 + 3x^2 + 7x)\,dx$

SOLUTION

(a) Since the integrand determines an even function, we may apply Theorem (4.34)(i):

$$\int_{-1}^{1} (x^4 + 3x^2 + 1)\,dx = 2\int_0^1 (x^4 + 3x^2 + 1)\,dx$$

$$= 2\left[\frac{x^5}{5} + x^3 + x\right]_0^1 = \frac{22}{5}$$

(b) The integrand is odd, so we apply Theorem (4.34)(ii):

$$\int_{-1}^{1} (x^5 + 3x^3 + x)\,dx = 0$$

(c) The function given by $2x^3 + 7x$ is odd but the function given by $3x^2$ is even, so we apply Theorem (4.34)(ii) *and* (i):

$$\int_{-5}^{5} (2x^3 + 3x^2 + 7x)\,dx = \int_{-5}^{5} (2x^3 + 7x)\,dx + \int_{-5}^{5} 3x^2\,dx$$
$$= 0 + 2\int_{0}^{5} 3x^2\,dx$$
$$= 2\left[x^3\right]_0^5 = 250$$

The technique of defining a function by means of a definite integral, as in Part I of the fundamental theorem of calculus (4.30), will have a very important application in Chapter 6, when we consider logarithmic functions. Recall, from (4.30), that if f is continuous on $[a, b]$ and $G(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$, then G is an antiderivative of f—that is, $(d/dx)(G(x)) = f(x)$. This result may be stated in integral form as follows:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

The preceding formula is generalized in the next theorem.

Theorem 4.35

Let f be continuous on $[a, b]$. If $a \le c \le b$, then for every x in $[a, b]$,

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x).$$

PROOF If F is an antiderivative of f, then

$$\frac{d}{dx}\int_c^x f(t)\,dt = \frac{d}{dx}(F(x) - F(c))$$
$$= \frac{d}{dx}(F(x)) - \frac{d}{dx}(F(c))$$
$$= f(x) - 0 = f(x).$$

EXAMPLE 8 If $G(x) = \int_1^x \frac{1}{t}\,dt$ and $x > 0$, find $G'(x)$.

SOLUTION We apply Theorem (4.35) with $c = 1$ and $f(x) = 1/x$. If we choose a and b such that $0 < a \le 1 \le b$, then f is continuous on $[a, b]$. Hence, by Theorem (4.35), for every x in $[a, b]$,

$$G'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

In (4.29), we defined the *average value* f_{av} of a function f on $[a, b]$ as follows:

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The next example indicates why this terminology is appropriate in applications.

EXAMPLE 9 Suppose that a point P moving on a coordinate line has a continuous velocity function v. Show that the average value of v on $[a, b]$ equals the average velocity during the time interval $[a, b]$.

SOLUTION By Definition (4.29) with $f = v$,

$$v_{av} = \frac{1}{b-a}\int_a^b v(t)\,dt.$$

If s is the position function of P, then $s'(t) = v(t)$—that is, $s(t)$ is an antiderivative of $v(t)$. Hence, by the fundamental theorem of calculus,

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(t)\Big]_a^b = s(b) - s(a).$$

Substituting in the formula for v_{av} give us

$$v_{av} = \frac{s(b) - s(a)}{b-a},$$

which is the average velocity of P on $[a, b]$ (see Definition 2.2).

Results similar to that in Example 9 occur in discussions of average acceleration, average marginal cost, average marginal revenue, and many other applications of the derivative (see Exercises 49–54).

EXERCISES 4.6

Exer. 1–36: Evaluate the integral.

1 $\int_1^4 (x^2 - 4x - 3)\,dx$

2 $\int_{-2}^3 (5 + x - 6x^2)\,dx$

3 $\int_{-2}^3 (8z^3 + 3z - 1)\,dz$

4 $\int_0^2 (z^4 - 2z^3)\,dz$

5 $\int_7^{12} dx$

6 $\int_{-6}^{-1} 8\,dx$

7 $\int_1^2 \frac{5}{x^6}\,dx$

8 $\int_1^4 \sqrt{16x^5}\,dx$

9 $\int_4^9 \frac{t-3}{\sqrt{t}}\,dt$

10 $\int_{-1}^{-2} \frac{2t-7}{t^3}\,dt$

11 $\int_{-8}^8 (\sqrt[3]{s^2} + 2)\,ds$

12 $\int_1^0 s^2(\sqrt[3]{s} - \sqrt{s})\,ds$

13 $\int_{-1}^0 (2x+3)^2\,dx$

14 $\int_1^2 (4x^{-5} - 5x^4)\,dx$

15 $\int_3^2 \frac{x^2-1}{x-1}\,dx$

16 $\int_0^{-1} \frac{x^3+8}{x+2}\,dx$

17 $\int_1^1 (4x^2-5)^{100}\,dx$

18 $\int_5^5 \sqrt[3]{x^2+\sqrt{x^5+1}}\,dx$

19 $\int_1^3 \frac{2x^3-4x^2+5}{x^2}\,dx$

20 $\int_{-2}^{-1}\left(x-\frac{1}{x}\right)^2 dx$

21 $\int_{-3}^{6} |x-4|\,dx$

22 $\int_{-1}^{5} |2x-3|\,dx$

23 $\int_1^4 \sqrt{5-x}\,dx$

24 $\int_1^5 \sqrt[3]{2x-1}\,dx$

25 $\int_{-1}^{1} (v^2-1)^3 v\,dv$

26 $\int_{-2}^{0} \frac{v^2}{(v^3-2)^2}\,dv$

27 $\int_0^1 \frac{1}{(3-2x)^2}\,dx$

28 $\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$

29 $\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^3}\,dx$

30 $\int_0^1 (3-x^4)^3x^3\,dx$

31 $\int_{\pi/2}^{\pi}\cos(\tfrac{1}{3}x)\,dx$

32 $\int_0^{\pi/2} 3\sin(\tfrac{1}{2}x)\,dx$

33 $\int_{\pi/4}^{\pi/3}(4\sin 2\theta+6\cos 3\theta)\,d\theta$

34 $\int_{\pi/6}^{\pi/4}(1-\cos 4\theta)\,d\theta$

35 $\int_{-\pi/6}^{\pi/6}(x+\sin 5x)\,dx$

36 $\int_0^{\pi/3}\frac{\sin x}{\cos^2 x}\,dx$

Exer. 37–40: Is the calculation or argument valid? Explain.

37 $\int_0^{\pi}\sec^2 x\,dx = \left[\tan x\right]_0^{\pi} = \tan\pi - \tan 0 = 0 - 0 = 0$

38 $\int_0^{\pi}\cos^2 x\,dx = \left[\frac{x}{2}+\frac{\sin 2x}{4}\right]_0^{\pi} = \left(\frac{\pi}{2}+0\right)-(0+0) = \frac{\pi}{2}$

39 If $f(x)=x^3$, then since $f(-x)=-f(x)$, we have $\int_{-1}^{0} f(x)\,dx = -\int_0^1 f(x)\,dx$ and hence $\int_{-1}^{1} f(x)\,dx = 0$.

40 If $f(x)=1/x^3$, then since $f(-x)=-f(x)$, we have $\int_{-1}^{0} f(x)\,dx = -\int_0^1 f(x)\,dx$ and hence $\int_{-1}^{1} f(x)\,dx = 0$.

Exer. 41–44: (a) Find a number z that satisfies the conclusion of the mean value theorem (4.28) for the given integral $\int_a^b f(x)\,dx$. (b) Find the average value of f on $[a, b]$.

41 $\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$

42 $\int_{-2}^{0}\sqrt[3]{x+1}\,dx$

43 $\int_0^5 \sqrt{x+4}\,dx$

44 $\int_{-3}^{2}\sqrt{6-x}\,dx$

Exer. 45–48: Find the derivative without integrating.

45 $\frac{d}{dx}\int_0^3 \sqrt{x^2+16}\,dx$

46 $\frac{d}{dx}\int_0^1 x\sqrt{x^2+4}\,dx$

47 $\frac{d}{dx}\int_0^x \frac{1}{t+1}\,dt$

48 $\frac{d}{dx}\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt, \quad |x|<1$

49 A point P is moving on a coordinate line with a continuous acceleration function a. If v is the velocity function, then the *average acceleration* on a time interval $[t_1, t_2]$ is

$$\frac{v(t_2)-v(t_1)}{t_2-t_1}.$$

Show that the average acceleration is equal to the average value of a on $[t_1, t_2]$.

50 If a function f has a continuous derivative on $[a, b]$, show that the average rate of change of $f(x)$ with respect to x on $[a, b]$ (see Definition 2.4) is equal to the average value of f' on $[a, b]$.

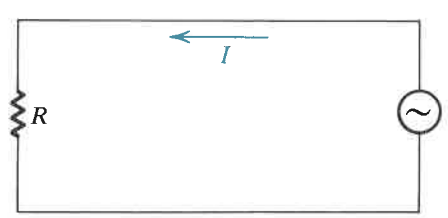
51 The vertical distribution of velocity of the water in a river may be approximated by $v = c(d-y)^{1/6}$, where v is the velocity (in meters per second) at a depth of y meters below the water surface, d is the depth of the river, and c is a positive constant.

(a) Find a formula for the average velocity v_{av} in terms of d and c.

(b) If v_0 is the velocity at the surface, show that $v_{av} = \frac{6}{7}v_0$.

52 In the electrical circuit shown in the figure, the alternating current I is given by $I = I_M \sin\omega t$, where t is the time and I_M is the maximum current. The rate P at which heat is being produced in the resistor of R ohms is given by $P = I^2R$. Compute the *average rate* of production of heat over one complete cycle (from $t=0$ to $t = 2\pi/\omega$). (*Hint:* Use the half-angle formula for the sine.)

Exercise 52



53 If a ball is dropped from a height of s_0 feet above the ground and air resistance is negligible, then the distance that it falls in t seconds is $16t^2$ feet. Use Definition (4.29) to show that the average velocity for the ball's journey to the ground is $4\sqrt{s_0}$ ft/sec.

54 A meteorologist determines that the temperature T (in °F) on a cold winter day is given by

$$T = \tfrac{1}{20}t(t-12)(t-24),$$

where t is time (in hours) and $t=0$ corresponds to midnight. Find the average temperature between 6 A.M. and 12 noon.

55 If g is differentiable and f is continuous for every x, prove that

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x).$$

56 Extend the formula in Exercise 55 to

$$\frac{d}{dx}\int_{k(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x) - f(k(x))k'(x).$$

Exer. 57–60: Use Exercises 55 and 56 to find the derivative.

57 $\frac{d}{dx}\int_2^{x^4}\frac{t}{\sqrt{t^3+2}}\,dt$

58 $\frac{d}{dx}\int_0^{x^2}\sqrt[3]{t^4+1}\,dt$

59 $\frac{d}{dx}\int_{3x}^{x^3}(t^3+1)^{10}\,dt$

60 $\frac{d}{dx}\int_{1/x}^{\sqrt{x}}\sqrt{t^4+t^2+4}\,dt$

4.7 NUMERICAL INTEGRATION

In this section, we will study several techniques of numerical integration that help us approximate definite integrals to any desired degree of accuracy. Evaluating a definite integral $\int_a^b f(x)\,dx$ by the fundamental theorem of calculus requires having an antiderivative for f. If we cannot obtain an antiderivative, we may use these numerical methods to obtain very accurate approximations. To emphasize their geometric nature, we illustrate these methods for functions with $f(x) \ge 0$ on $[a, b]$.

RECTANGLE RULES

Recalling Definition (4.16) and assuming that the definite integral $\int_a^b f(x)\,dx$ exists, we approximate its value, as a sum of areas of rectangles, using any Riemann sum of f. In particular, if we use a regular partition with $\Delta x = (b-a)/n$, then $x_k = a + k\Delta x$ for $k = 0, 1, 2, \ldots, n$, and

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(w_k)\Delta x,$$

where w_k is any number in the kth subinterval $[x_{k-1}, x_k]$ of the partition. (Refer to Figure 4.12 on page 379.) Each term $f(w_k)\Delta x$ in the sum is the area of a rectangle of width Δx and height $f(w_k)$. The accuracy of such an approximation to $\int_a^b f(x)\,dx$ by rectangles is affected by both the location of w_k within each subinterval and the width Δx of the rectangles.

As we saw in Section 4.4, by locating each w_k at a left-hand endpoint x_{k-1}, we obtain a left endpoint approximation. Alternatively, by locating each w_k at a right-hand endpoint x_k, we obtain a right endpoint approximation. A third possibility is to let w_k be the midpoint of each subinterval;

then $w_k = (x_{k-1} + x_k)/2$. This choice of location for w_k gives a midpoint approximation. Using the notation $x_{k-1/2}$ to indicate this midpoint, $(x_{k-1} + x_k)/2$, we formalize the three choices for the location of w_k in the following rules.

Rectangle Rules 4.36

For a regular partition of an interval $[a, b]$ with n subintervals, each of width $\Delta x = (b - a)/n$, the definite integral $\int_a^b f(x)\,dx$ is approximated by

(i) the **left rectangle rule:**

$$L_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x$$

(ii) the **right rectangle rule:**

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x$$

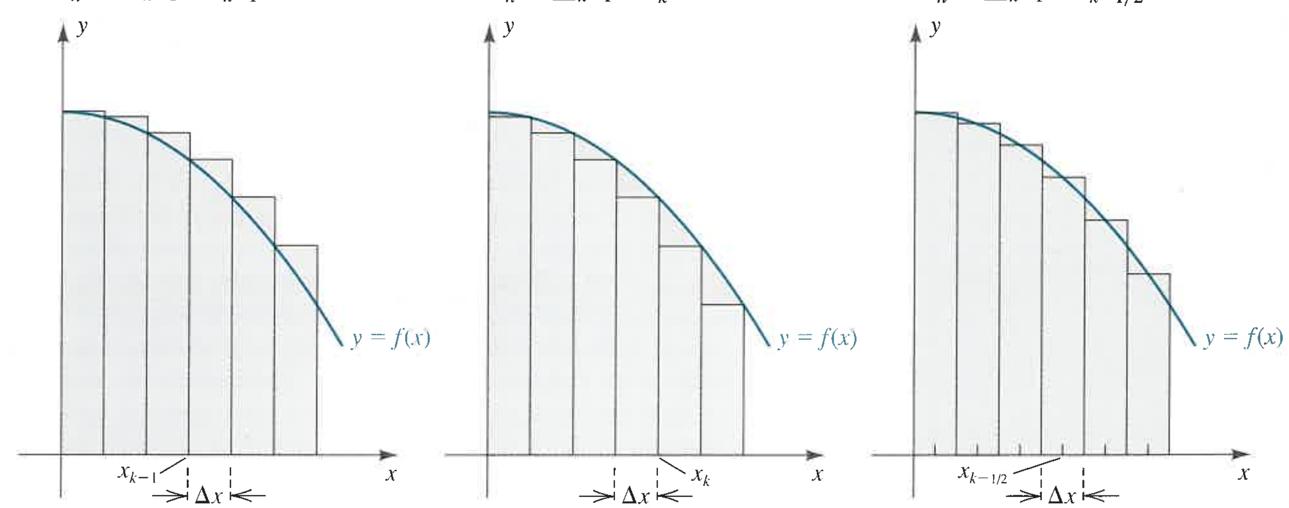
(iii) the **midpoint rule:**

$$M_n = \sum_{k=1}^{n} f(x_{k-1/2})\Delta x$$

If a function is strictly increasing or strictly decreasing over the interval, then the endpoint rules in (4.36) give the areas of the inscribed and circumscribed rectangles. Figure 4.32 shows a function f that is decreasing over the interval $[a, b]$. In Figure 4.32(a), the left rectangle rule L_n gives the sum of the areas of the circumscribed rectangles; it overestimates the definite integral. The gray-shaded area represents the error resulting from the left rectangle rule. That is, the gray-shaded regions are contained within the circumscribed rectangles but are not under the graph of f.

Figure 4.32

(a) $L_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x$ (b) $R_n = \sum_{k=1}^{n} f(x_k)\Delta x$ (c) $M_n = \sum_{k=1}^{n} f(x_{k-1/2})\Delta x$



Similarly, for a decreasing function, as shown in Figure 4.32(b), the right rectangle rule gives the sum of the areas of the inscribed rectangles, and it underestimates the definite integral. The gray-shaded area shows the resulting error, which is made up of the regions under the graph that are not included within the inscribed rectangles. Finally, we see in Figure 4.32(c) that the midpoint rule appears to give a better approximation of the definite integral. As indicated by the gray-shaded area, the resulting error includes regions under the graph that are not within the rectangles as well as portions of the rectangles that are not under the graph. These areas of error may partially offset each other and yield a more accurate estimate of the original definite integral. Thus, the midpoint rule M_n often gives a number that lies between the left rectangle rule L_n and the right rectangle rule R_n.

In the next example, we apply the left rectangle, the right rectangle, and the midpoint rules to determine approximations for the definite integral of a specific function on a prescribed interval.

EXAMPLE ■ 1 Approximate $\int_1^2 1/x\,dx$ using a regular partition with $n = 4$, using

(a) the midpoint rule M_n **(b)** the left rectangle rule L_n

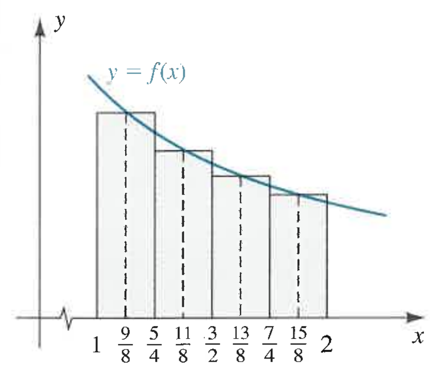
(c) the right rectangle rule R_n

SOLUTION With $n = 4$, we have $\Delta x = (b - a)/n = (2 - 1)/4 = 1/4$, and the function f is given by $f(x) = 1/x$. The endpoints of the subintervals are $x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{3}{2}, x_3 = \frac{7}{4}$, and $x_4 = 2$.

(a) The midpoints are $x_{1/2} = \frac{9}{8}, x_{3/2} = \frac{11}{8}, x_{5/2} = \frac{13}{8}$, and $x_{7/2} = \frac{15}{8}$ (see Figure 4.33). By (4.36)(iii), we obtain

Figure 4.33

$f(x) = 1/x$



$$\int_a^b f(x)\,dx = \int_1^2 \frac{1}{x}\,dx \approx M_4 = \sum_{k=1}^{4} f(x_{k-1/2})\Delta x$$

$$= \sum_{k=1}^{4} \left(\frac{1}{x_{k-1/2}}\right)\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\sum_{k=1}^{4}\left(\frac{1}{x_{k-1/2}}\right)$$

$$= \tfrac{1}{4}\left(\tfrac{8}{9} + \tfrac{8}{11} + \tfrac{8}{13} + \tfrac{8}{15}\right) = \tfrac{4448}{6435}$$

$$\approx 0.6912198912.$$

(b) The left-hand endpoints are $1, \frac{5}{4}, \frac{3}{2}$, and $\frac{7}{4}$. By (4.36)(i),

$$\int_1^2 \frac{1}{x}\,dx \approx L_4 = \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) = \frac{319}{420}$$

$$\approx 0.7595238095.$$

(c) The right-hand endpoints are $\frac{5}{4}, \frac{3}{2}, \frac{7}{4}$, and 2. By (4.36)(ii),

$$\int_1^2 \frac{1}{x}\,dx \approx R_4 = \frac{1}{4}\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}+\frac{1}{2}\right) = \tfrac{533}{840} \approx 0.6345238095.$$

In Chapter 6, we will see that the correct value to ten decimal places for $\int_1^2 1/x\,dx$ is 0.6931471806. We note that the midpoint rule, which yields a number between those given by the left and the right rectangle rules, gives a better approximation than either endpoint rule.

Note that in each of the rectangle rules (4.36), Δx is a constant factor so that we can also write these rules as

$$L_n = \Delta x \sum_{k=1}^{n} f(x_{k-1}), \quad R_n = \Delta x \sum_{k=1}^{n} f(x_k), \quad M_n = \Delta x \sum_{k=1}^{n} f(x_{k-1/2}).$$

Once we compute the left endpoint or the right endpoint approximation, it is easy to determine the other endpoint approximation since the right-hand endpoint of one subinterval is the left-hand endpoint of the next interval:

$$L_n = \Delta x\left[f(x_0) + \sum_{k=1}^{n-1} f(x_k)\right] \quad \text{and} \quad R_n = \Delta x\left[\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right]$$

If we let $C = \sum_{k=1}^{n-1} f(x_k)$ and note that $x_0 = a$ and $x_n = b$, we have

$$R_n - L_n = \Delta x[C + f(b)] - \Delta x[f(a) + C] = \Delta x[f(b) - f(a)]$$

or, equivalently,

$$L_n = R_n + \Delta x[f(a) - f(b)].$$

Thus, for the case of Example 1, we can find the left endpoint approximation from the right endpoint approximation, as follows:

$$\begin{aligned} L_4 &= R_4 + (\tfrac{1}{4})[f(1) - f(2)] \\ &\approx 0.6345238095 + (\tfrac{1}{4})(1 - \tfrac{1}{2}) \\ &= 0.7595238095 \end{aligned}$$

TRAPEZOIDAL RULES

Since the left and right rectangle rules often yield under- or overestimates, it is natural to consider a numerical integration rule based on their average, T_n:

$$\begin{aligned} T_n = \tfrac{1}{2}(L_n + R_n) &= (\tfrac{1}{2})\left(\sum_{k=1}^{n} f(x_{k-1})\Delta x + \sum_{k=1}^{n} f(x_k)\Delta x\right) \\ &= \sum_{k=1}^{n} (\tfrac{1}{2})\left[f(x_{k-1}) + f(x_k)\right]\Delta x. \end{aligned}$$

Trapezoidal Rule 4.37

For a regular partition of an interval $[a, b]$ with n subintervals, each of width $\Delta x = (b-a)/n$, the definite integral $\int_a^b f(x)\,dx$ is approximated by the **trapezoidal rule**:

$$\begin{aligned} T_n &= \tfrac{1}{2}(L_n + R_n) = \sum_{k=1}^{n} \tfrac{1}{2}\left[f(x_{k-1}) + f(x_k)\right]\Delta x \\ &= \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \end{aligned}$$

Note that since each term in the sum for T_n has a constant factor $(\frac{1}{2})\Delta x = (b-a)/2n$, we can also write the sum as

$$T_n = \frac{b-a}{2n}\sum_{k=1}^{n}\left[f(x_{k-1}) + f(x_k)\right].$$

The last equality in (4.37) follows from the relation between the left rectangle and right rectangle rules,

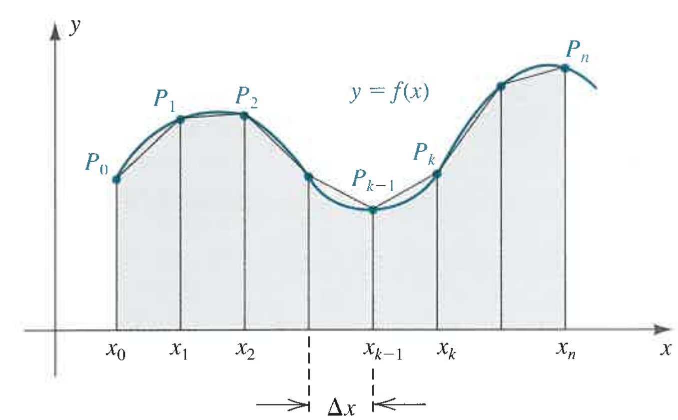
$$L_n = R_n + \Delta x[f(a) - f(b)] = R_n + \Delta x[f(x_0) - f(x_n)],$$

and the definition of the trapezoidal rule as the average of L_n and R_n:

$$\begin{aligned} T_n &= \tfrac{1}{2}(L_n + R_n) = \tfrac{1}{2}(R_n + \Delta x[f(x_0) - f(x_n)] + R_n) \\ &= \tfrac{1}{2}\left(2R_n + \Delta x[f(x_0) - f(x_n)]\right) \\ &= \tfrac{1}{2}\left(2\Delta x \sum_{k=1}^{n} f(x_k) + \Delta x\left[f(x_0) - f(x_n)\right]\right) \\ &= \frac{\Delta x}{2}\left(2\left[f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)\right] + f(x_0) - f(x_n)\right) \\ &= \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \end{aligned}$$

Figure 4.34 provides a graphical interpretation of the trapezoidal rule. Each term $\frac{1}{2}[f(x_{k-1}) + f(x_k)]\Delta x$ in the sum is the area of a trapezoid

Figure 4.34



formed by the secant line joining the endpoints of the graph over the kth subinterval $[x_{k-1}, x_k]$, the interval itself, and the vertical segments above x_{k-1} and x_k. The gray-shaded regions in the figure show the error when we approximate the area under the graph of f by the area of the trapezoid.

EXAMPLE 2 Approximate $\int_1^2 1/x\,dx$ using a regular partition with $n = 4$, using the trapezoidal rule T_n.

SOLUTION With the results of Example 1, we have

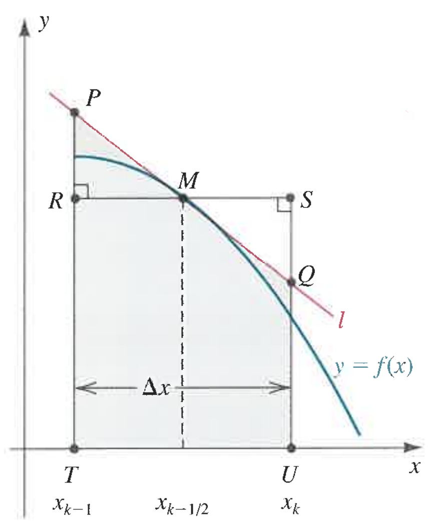
$$\begin{aligned} T_4 &= \tfrac{1}{2}(L_4 + R_4) \\ &= \tfrac{1}{2}(0.7595238095 + 0.6345238095) \\ &= 0.6970238095, \end{aligned}$$

which is closer to the correct value (to ten decimal places) of 0.6931471806 than either L_4 or R_4.

Alternatively, we can compute T_4 directly from the last form of the trapezoidal rule in (4.37):

$$\begin{aligned} T_n &= \frac{2-1}{2(4)}\left[f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2)\right] \\ &= \tfrac{1}{8}\left[1 + 2(\tfrac{4}{5}) + 2(\tfrac{2}{3}) + 2(\tfrac{4}{7}) + (\tfrac{1}{2})\right] \\ &= \tfrac{1}{8}(1 + \tfrac{8}{5} + \tfrac{4}{3} + \tfrac{8}{7} + \tfrac{1}{2}) \\ &= \tfrac{1}{8}(\tfrac{1171}{210}) = \tfrac{1171}{1680} \approx 0.6970238095 \end{aligned}$$

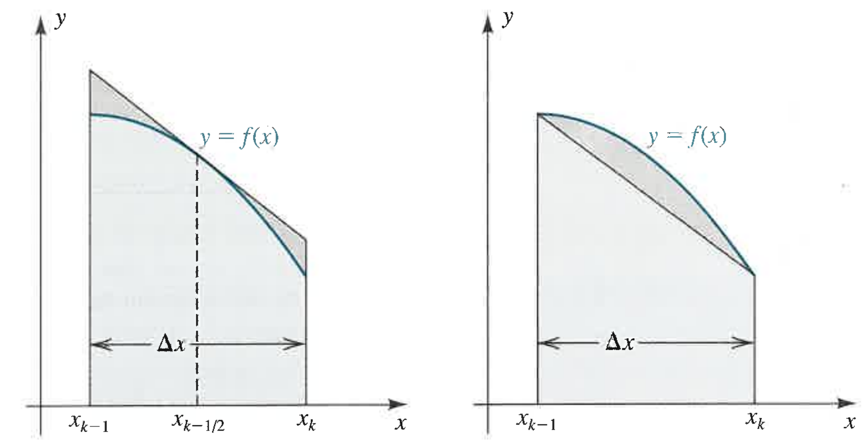
Figure 4.35



We can obtain other trapezoidal approximations for the area under the graph of f over a subinterval. For example, as illustrated in Figure 4.35, we can construct a nonvertical line l through the point M, which lies on the graph over the midpoint of the interval. Extending this line until it meets the vertical lines over x_{k-1} and x_k at points P and Q, respectively, forms a trapezoid $TPQU$. Adding a horizontal line through M forms a rectangle $TRSU$, whose area is one of the terms of the midpoint rule. Using elementary geometry, it can be shown that the area of the trapezoid $TPQU$ is equal to the area of the rectangle $TRSU$.

Note that this result is independent of the shape of the graph of f. If we take any nonvertical line through M, *the resulting trapezoid has the same area as the midpoint rectangle*. Thus, in addition to having a second trapezoidal approximation, we also have an alternative geometric way of viewing the midpoint rule. By an appropriate choice of the line l, we may be able to see if the midpoint rule gives an underestimate or an overestimate of the definite integral. We may be able to choose the line l, as in Figure 4.36, so that the entire area under the curve is below the line; thus the midpoint rule will overestimate the definite integral on this subinterval.

Figure 4.36



If we examine the difference between the midpoint rule and the trapezoidal rule on one subinterval (Figure 4.36), we see that the midpoint rule is an overestimate and the trapezoidal rule is an underestimate. The midpoint rule has about half the error of the trapezoidal rule.

SIMPSON'S RULE

Recall that the trapezoidal rule, which averages the results of the left rectangle rule and the right rectangle rule, is an improvement over each of them. We may do even better by combining the midpoint rule and the trapezoidal rule. The British mathematician Thomas Simpson (1710–1761) suggested a combination, using a "weighted average," where M_n is counted twice as heavily as T_n.

Simpson's Rule 4.38

For a regular partition of an interval $[a, b]$ with n subintervals, each of width $\Delta x = (b-a)/n$, the definite integral $\int_a^b f(x)\,dx$ is approximated by **Simpson's rule:**

$$\begin{aligned} S_n &= \tfrac{1}{3}(2M_n + T_n) \\ &= \frac{b-a}{6n}[f(x_0) + 4f(x_{1/2}) + 2f(x_1) + 4f(x_{3/2}) \\ &\qquad + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(x_{n-1/2}) + f(x_n)] \end{aligned}$$

The last equality in (4.38) follows from the fact that

$$M_n = [f(x_{1/2}) + f(x_{3/2}) + f(x_{5/2}) + \cdots + f(x_{n-1/2})]\Delta x$$

can be combined with the final expression for the trapezoidal rule in (4.37).

The next example shows the result of applying Simpson's rule to the definite integral given in Examples 1 and 2.

EXAMPLE 3 Approximate $\int_1^2 1/x\,dx$ using a regular partition with $n = 4$, using Simpson's rule S_n.

SOLUTION Using the results of Examples 1 and 2, we have

$$S_4 = \tfrac{1}{3}(2M_4 + T_4)$$
$$\approx \tfrac{1}{3}[2(0.6912198912) + 0.6970238095] \approx 0.6931545306.$$

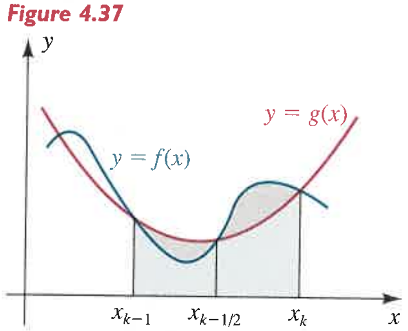
Comparing this result to the correct value (to ten decimal places) of 0.6931471806, we see that Simpson's rule gives the best approximation, followed by the midpoint rule and then the trapezoidal rule. Alternatively, we can compute S_4 directly from the last form of Simpson's rule in (4.38):

$$S_4 = \frac{2-1}{6(4)}\left[f(1) + 4f\left(\frac{9}{8}\right) + 2f\left(\frac{5}{4}\right) + 4f\left(\frac{11}{8}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{13}{8}\right) + 2f\left(\frac{7}{4}\right) + 4f\left(\frac{15}{8}\right) + f(2)\right]$$
$$= \tfrac{1}{24}\left(1 + \tfrac{32}{9} + \tfrac{8}{5} + \tfrac{32}{11} + \tfrac{4}{3} + \tfrac{32}{13} + \tfrac{8}{7} + \tfrac{32}{15} + \tfrac{1}{2}\right)$$
$$= \tfrac{1}{24}\left(\tfrac{35{,}969{,}064}{2{,}162{,}160}\right) = \left(\tfrac{1{,}498{,}711}{2{,}162{,}160}\right) \approx 0.6931545307$$

The numerical integration techniques we have considered up to now approximate the *region* under the graph lying over a small subinterval by a simpler region (a rectangle or a trapezoid) whose area is found by simple geometric formulas. Another conceptual approach to numerical integration also leads to Simpson's rule: We replace the *function* f by a simpler function g whose graph closely approximates the graph of f on each subinterval. We then integrate the simpler function by finding its antiderivative and approximate $\int_{x_{k-1}}^{x_k} f(x)\,dx$ by $\int_{x_{k-1}}^{x_k} g(x)\,dx$. In this perspective, note that a rectangle rule replaces f by a constant function. The trapezoidal rule replaces f by a linear function that matches the values of f at the endpoints of the subinterval.

For Simpson's rule, on each subinterval, we replace the function f by a *quadratic function* g that matches the value of f at the endpoints and the midpoint—that is, on each subinterval, the graph of g is a parabola with three points in common with the graph of f, as shown in Figure 4.37.

Figure 4.37



A quadratic function can be written in the form $g(x) = c + bx + ax^2$ for constants c, b, and a. It will be easier to use an equivalent form

$$g(x) = c_0 + c_1(x - x_{k-1/2}) + c_2(x - x_{k-1/2})^2$$

on the subinterval $[x_{k-1}, x_k]$.

We must first determine the values of the coefficients c_0, c_1, and c_2 so that the values of f and g are equal at the endpoints and at the midpoint of the subinterval. To do so, we need to satisfy the conditions

$$g(x_{k-1}) = f(x_{k-1}), \quad g(x_{k-1/2}) = f(x_{k-1/2}), \quad \text{and} \quad g(x_k) = f(x_k),$$

and then use these equations to determine c_0, c_1, and c_2.

1. At the midpoint $x = x_{k-1/2}$, we have

$$g(x_{k-1/2}) = c_0 + c_1(0) + c_2(0)^2 = c_0.$$

For agreement at the midpoint, we need $g(x_{k-1/2}) = f(x_{k-1/2})$, so we have $c_0 = f(x_{k-1/2})$.

2. At the left endpoint $x = x_{k-1}$, we have

$$x - x_{k-1/2} = x_{k-1} - x_{k-1/2} = -\tfrac{1}{2}\Delta x,$$

so $$g(x_{k-1}) = f(x_{k-1/2}) - \tfrac{1}{2}c_1\Delta x + \tfrac{1}{4}c_2(\Delta x)^2.$$

To obtain agreement with f at the left endpoint, we need $g(x_{k-1}) = f(x_{k-1})$—that is,

(I) $$f(x_{k-1/2}) - \tfrac{1}{2}c_1\Delta x + \tfrac{1}{4}c_2(\Delta x)^2 = f(x_{k-1}).$$

3. At the right endpoint $x = x_k$, we have $x - x_{k-1/2} = \frac{1}{2}\Delta x$, so

$$g(x_k) = f(x_{k-1/2}) + \tfrac{1}{2}c_1\Delta x + \tfrac{1}{4}c_2(\Delta x)^2.$$

To achieve agreement with f at the right endpoint, we must have $g(x_k) = f(x_k)$ or, equivalently,

(II) $$f(x_{k-1/2}) + \tfrac{1}{2}c_1\Delta x + \tfrac{1}{4}c_2(\Delta x)^2 = f(x_k).$$

Thus, in order for g and f to agree at the three points, we must solve the equations (I) and (II) for c_1 and c_2. Doing so yields

$$c_1 = \left\{\frac{f(x_k) - f(x_{k-1})}{\Delta x}\right\}, \quad c_2 = \left\{\frac{2[f(x_k) - 2f(x_{k-1/2}) + f(x_{k-1})]}{(\Delta x)^2}\right\}.$$

Thus, we have determined the coefficients (c_0, c_1, and c_2) of the quadratic function g.

Once we have explicitly found the function g, we approximate $\int_{x_{k-1}}^{x_k} f(x)\,dx$ by $\int_{x_{k-1}}^{x_k} g(x)\,dx$. The integral of the quadratic function is

$$\int_{x_{k-1}}^{x_k}\left[c_0 + c_1(x - x_{k-1/2}) + c_2(x - x_{k-1/2})^2\right]dx$$
$$= \left[c_0x + \tfrac{1}{2}c_1(x - x_{k-1/2})^2 + \tfrac{1}{3}c_2(x - x_{k-1/2})^3\right]_{x_{k-1}}^{x_k}$$
$$= \left[c_0x_k + \tfrac{1}{2}c_1(x_k - x_{k-1/2})^2 + \tfrac{1}{3}c_2(x_k - x_{k-1/2})^3\right]$$
$$- \left[c_0x_{k-1} + \tfrac{1}{2}c_1(x_{k-1} - x_{k-1/2})^2 + \tfrac{1}{3}c_2(x_{k-1} - x_{k-1/2})^3\right]$$
$$= c_0(x_k - x_{k-1}) + \tfrac{1}{2}c_1(\tfrac{1}{2}\Delta x)^2 + \tfrac{1}{3}c_2(\tfrac{1}{2}\Delta x)^3$$
$$-\tfrac{1}{2}c_1(-\tfrac{1}{2}\Delta x)^2 - \tfrac{1}{3}c_2(-\tfrac{1}{2}\Delta x)^3$$
$$= c_0(\Delta x) + \tfrac{1}{12}c_2(\Delta x)^3,$$

which becomes, after substituting the values for c_0 and c_2 we found above,

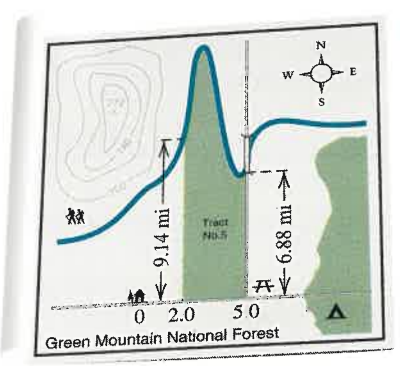
$$\tfrac{1}{6}\left[f(x_{k-1}) + 4f(x_{k-1/2}) + f(x_k)\right]\Delta x,$$

which are exactly the terms we have for Simpson's rule. This result gives an alternative justification for Simpson's rule: We can derive Simpson's rule either by beginning with a weighted average of the midpoint and trapezoidal rules or by approximating the graph of f with parabolas.

In some treatments of numerical integration, a different form of Simpson's rule is used. Instead of using quadratics whose values match those of the function f at the endpoints and the midpoint of each subinterval $[x_{k-1}, x_k]$, this other form uses an *even* value for n. It then divides the interval $[a, b]$ into $n/2$ subintervals and uses a quadratic for each of these $n/2$ subintervals. In this approach, the first quadratic matches f at x_0, x_1, x_2, the next quadratic matches f at x_2, x_3, x_4, and so forth. Note that Simpson's rule (4.38) can be used for an odd or an even value of n. If you use a software package on a computer or a built-in function on a calculator for numerical integration, consult the reference manual to determine which form of Simpson's rule is being used.

In the next example of numerical integration, we approximate the definite integral of a function known only by a table of function values with equally spaced x-coordinates. In applications, results obtained from an experiment frequently provide only function *values*, rather than a formula for the function.

Figure 4.38



EXAMPLE 4 Aerial surveys of a tract of the Green Mountain National Forest shown in Figure 4.38 measured the width of the forest (in miles) at regularly spaced intervals, $\frac{3}{10}$ mi apart. The gathered data are shown in the following table.

x	2.0	2.3	2.6	2.9	3.2	3.5	3.8	4.1	4.4	4.7	5.0
y	9.14	11.82	13.41	13.72	12.87	11.27	9.42	7.81	6.78	6.49	6.88

The Forest Service estimates that, on average, there are 125 mature trees per acre. Approximate the total number of mature trees in this tract of the forest using the rules of numerical integration.

SOLUTION We must first obtain estimates for the forest's land area in square miles. To do so, we use the data in the table and consider the forest area as the definite integral of the function $y = f(x)$ over the interval $[2, 5]$.

For the left and right rectangle rules and for the trapezoidal rule, we can choose $n = 10$ and $\Delta x = (5.0 - 2.0)/10 = 0.3$. Using our earlier observation that $R_n = \Delta x \left[\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right]$, we have

$$\begin{aligned} R_{10} &= (0.3)\left[\sum_{k=1}^{9} f(x_k) + f(x_{10})\right] \\ &= (0.3)[(11.82 + 13.41 + 13.72 + 12.87 + 11.27 \\ &\qquad + 9.42 + 7.81 + 6.78 + 6.49) + 6.88] \\ &= (0.3)(93.59 + 6.88) = 30.141 \end{aligned}$$

$$\text{and} \qquad \begin{aligned} L_{10} &= R_{10} + \Delta x[f(a) - f(b)] \\ &= 30.141 + (0.3)(9.14 - 6.88) = 30.819, \end{aligned}$$

$$\text{so that} \qquad T_{10} = \tfrac{1}{2}(L_{10} + R_{10}) = 30.48.$$

For the midpoint rule and Simpson's rule, we consider every other x-value as a midpoint so that $n = 5$ and $\Delta x = 0.6$. Then we compute

$$\begin{aligned} L_5 &= (9.14 + 13.41 + 12.87 + 9.42 + 6.78)(0.6) = 30.972, \\ R_5 &= (13.41 + 12.87 + 9.42 + 6.78 + 6.88)(0.6) = 29.616, \\ M_5 &= (11.82 + 13.72 + 11.27 + 7.81 + 6.49)(0.6) = 30.666, \\ T_5 &= (L_5 + R_5)/2 = 30.294, \\ \text{and} \quad S_5 &= (2M_5 + T_5)/3 = 30.542. \end{aligned}$$

The computed results are summarized in the following table, where we have rounded figures to two decimal places because the given data on the width of the forest can be assumed accurate to only two places.

n	L_n	R_n	M_n	T_n	S_n
5	30.97	29.62	30.67	30.29	30.54
10	30.82	30.14	—	30.48	—

From these figures, we estimate the tract of forest to be about 30.5 mi^2. Since there are 640 acres in a square mile, the forest is about 19,520 acres in extent. With an average of 125 trees per acre, the forest contains approximately (19,520)(125) = 2,440,000 mature trees.

DEPENDENCE ON Δx

We noted earlier in this section that both the location for w_k and the size Δx affect the accuracy of $\sum_{k=1}^{n} f(w_k)\Delta x$ as an approximation for $\int_a^b f(x)\,dx$. We have considered several different choices for locating w_k and now examine the size of Δx. Since the width $\Delta x = (b - a)/n$ depends on the number n of rectangles, the discussion will focus on n. Increasing n may increase the accuracy but it introduces more terms in the sum to calculate. We can find the approximations using a calculator or a computer program for different choices for n to see how increasing n improves the accuracy. The next example shows the numerical results for a particular definite integral using a program on a calculator that displays 12 significant digits and works internally with 14 digits.

EXAMPLE 5 Use the numerical integration rules to approximate the definite integral $\int_0^1 [4/(1 + x^2)]\,dx$ for $n = 2, 6, 18, 54$, and 162.

SOLUTION The following table displays the results of running the computer program:

n	L_n	R_n	M_n	T_n	S_n
2	3.6	2.6	3.16235294118	3.1	3.14156862745
6	3.30362973314	2.9702963998	3.14390742722	3.13696306647	3.14159264031
18	3.19663380591	3.08552269480	3.14184985518	3.14107825036	3.14159265357
54	3.16005401619	3.12301697915	3.14162123155	3.14153549767	3.14159265359
162	3.14775914244	3.13541346343	3.14159582892	3.14158630293	3.14159265359

We can make several observations on the basis of an examination of the table.

As we increase n, the values given by the midpoint, the trapezoidal, and Simpson's rules all seem to approach a number whose first six significant digits are 3.14159. For the left rectangle rule, increases in n produce decreases in the values of L_n. These values also get closer to 3.14159. For the right rectangle rule, increases in n produce increases in the values of R_n, which also get closer to 3.14159.

In going from one value of n to the next value, we see that the change in the approximated values is greater for the rectangle rules than for either the trapezoidal or Simpson's rule. For example, when n increases from 6 to 18, the value for L_n changes by 0.10699592723, whereas the value for S_n changes by only -0.00000001326.

To gain a better understanding of the effect on the approximations of increases in n, we can compare the results in the table of Example 5 with the exact value of the definite integral. The next example discusses such a comparison.

EXAMPLE ■ 6 The definite integral $\int_0^1 [4/(1+x^2)]\,dx$ has a value of π. (We will prove this fact in Chapter 6.)

(a) Using the results of Example 5, compute the errors for each approximation by finding the difference between π and the approximation.

(b) Investigate the ratios of error for each successive pair of values for n.

SOLUTION

(a) Given the known value π for the result, we use a calculator to compute each error, that is, the correct value π minus the approximated value. The following table lists the results.

n	L_n	R_n	M_n	T_n	S_n
2	−4.584 E −1	5.416 E −1	−2.076 E −2	4.159 E−2	2.403 E −5
6	−1.620 E −1	1.713 E −1	−2.315 E −3	4.630 E −3	1.328 E −8
18	−5.504 E −2	5.607 E −2	−2.572 E −4	5.144 E −4	1.82 E−11
54	−1.846 E −2	1.858 E −2	−2.858 E −5	5.716 E −5	3 E −13
162	−6.166 E −3	6.179 E −3	−3.175 E −6	6.351 E −6	1 E −13

The data in the table indicate that the error decreases as n increases. We note too that when Simpson's rule is used, the error is extremely small even when $n = 2$.

(b) The next table shows the ratio of a column entry and the entry below it.

$n, n+1$	L_n/L_{n+1}	R_n/R_{n+1}	M_n/M_{n+1}	T_n/T_{n+1}	S_n/S_{n+1}
2, 6	2.83	3.16	8.97	8.98	1809
6, 18	2.94	3.06	9.00	9.00	729.7
18, 54	2.98	3.02	9.00	9.00	60.7
54, 162	2.99	3.01	9.00	9.00	3

Note that each successive value of n is 3 times the preceding value. The errors for the left and right rectangle rules were approximately divided by 3 for each tripling of n, and the errors for the midpoint and trapezoidal rules were approximately divided by $9 = 3^2$. We see no pattern in the errors for Simpson's rule, which may be due to round-off errors that occur because one very small number is being divided by another.

The patterns we observed in the table of Example 6(b) are not coincidental. They follow from more general results about error estimates.

ERROR ESTIMATES

For the five numerical integration rules that we have considered in this section, we can find *error estimates*, or *bounds*, on the size of the error even if we do not know the exact value of the definite integral. If I is the actual value of the definite integral $\int_a^b f(x)\,dx$ and A_n is an approximated value using n rectangles, then the size of the error is $|I - A_n|$. By a **bound** on the size of the error, we mean a number B such that $|I - A_n| \le B$. We can obtain bounds that depend on the number n of subintervals and the maximum value of derivatives of the function f. We state without proof the theorem describing these error estimates.

Theorem 4.39

Let $I = \int_a^b f(x)\,dx$ be the definite integral being approximated. If f' is continuous and if K_1 is a positive number such that $|f'(x)| \le K_1$ for every x in $[a, b]$, then the **error estimates for the left rectangle rule** L_n and the **right rectangle rule** R_n are given by

$$|I - L_n| \le K_1 \frac{(b-a)^2}{2n} \quad \text{and} \quad |I - R_n| \le K_1 \frac{(b-a)^2}{2n}.$$

If f'' is continuous and if K_2 is a positive real number such that $|f''(x)| \le K_2$ for every x in $[a, b]$, then the **error estimates for the midpoint rule** M_n and the **trapezoidal rule** T_n are given by

$$|I - M_n| \le K_2 \frac{(b-a)^3}{24n^2} \quad \text{and} \quad |I - T_n| \le K_2 \frac{(b-a)^3}{12n^2}.$$

If $f^{(4)}$ is continuous and if K_4 is a positive real number such that $|f^{(4)}(x)| \le K_4$ for every x in $[a, b]$, then the **error estimate for Simpson's rule** S_n is given by

$$|I - S_n| \le K_4 \frac{(b-a)^5}{2880n^4}.$$

The next example illustrates how the error estimates in (4.39) can be used. If we can find values for K_1, K_2, or K_4, then we may use the estimates in (4.39) to determine how large n should be in order to ensure that a particular approximation is within a given margin of error.

EXAMPLE 7 Determine how large n must be in order to use the trapezoidal rule to approximate $I = \int_1^3 1/x\,dx$ with an error less than 10^{-3}.

SOLUTION From Theorem (4.39), we have

$$|I - T_n| \le K_2 \frac{(3-1)^3}{12n^2} = \frac{2K_2}{3n^2},$$

where K_2 is a bound on the absolute value of the second derivative of $f(x) = 1/x$ on the interval $[1, 3]$. Since $f''(x) = 2/x^3$ is positive and decreasing on $[1, 3]$, its maximum value is $f''(1) = 2$. Therefore, we have $|I - T_n| \le 4/(3n^2)$. To ensure that the error is less than 10^{-3}, we must choose n so that

$$\frac{4}{3n^2} < 10^{-3},$$

which is equivalent to

$$n^2 > \tfrac{4000}{3}, \quad \text{or} \quad n > \sqrt{\tfrac{4000}{3}} \approx 36.5.$$

Hence we should choose n to be at least 37 in order to guarantee an error less than 10^{-3}.

Before the availability of electronic computing devices, great efforts were made to estimate the constants K_1, K_2, and K_4. Once these numbers were known, the inequalities in (4.39) could be solved for n, as in Example 7, to determine how many subintervals n to use in order to obtain an approximation within a prescribed error. Today with inexpensive computing power (including hand-held programmable calculators), there is an alternative approach. We can obtain an approximation that is within a given margin of error by repeatedly computing a numerical integration rule for increasing values of n and observing the convergence of the estimates as n grows larger. We will illustrate this approach in Example 8.

This alternative approach is based on the fact that the error estimates (4.39) give the expected decrease in the error when we *change* n by a certain multiple. To illustrate, if we compare the error estimates for the trapezoidal rule with n and $5n$ subintervals, respectively, on the same definite integral, we have, by (4.39),

$$E_n = |I - T_n| \le K_2 \frac{(b-a)^3}{12n^2}$$

and

$$E_{5n} = |I - T_{5n}| \le K_2 \frac{(b-a)^3}{12(5n)^2}.$$

The ratio of these two error estimates is

$$\frac{E_n}{E_{5n}} = \frac{K_2[(b-a)^3/12n^2]}{K_2[(b-a)^3/12(5n)^2]} = 5^2 = 25.$$

Since $E_{5n} = \frac{1}{25}E_n$, we expect the error to decrease by a factor of 25 when we increase the number of subintervals from n to $5n$. We obtain similar expected decreases for the other numerical integration rules by examining the power of n in the denominator of the error estimates. For example, if we multiply n by 3, then, by (4.39), we expect the error in the left and right rectangle rules to be divided by 3, the error in the midpoint and trapezoidal rules to be divided by $3^2 = 9$, and the error in Simpson's rule to be divided by $3^4 = 81$.

EXAMPLE 8

(a) Use Simpson's rule to approximate the definite integral $\int_1^{12} 1/x\,dx$ for $n = 5, 10, 20, 40,$ and 80.

(b) Discuss the expected accuracy of the final result.

SOLUTION

(a) We use Simpson's rule (4.38) for the integrand $f(x) = 1/x$, and display the results in a table:

n	S_n
5	2.50179046384
10	2.48685897261
20	2.48507069664
40	2.48491806727
80	2.48490738622

(b) Each time we double n, the error estimation in (4.39) predicts that the error in Simpson's rule will be divided by $2^4 = 16$, which means that we will add at least one correct decimal digit each time we double n. Thus, the estimate

$$\int_1^{12} \frac{1}{x}\,dx \approx 2.4849$$

is correct to at least four decimal places.

For the function $f(x) = 1/x$, we can easily find $f^{(4)}(x) = 24x^{-5}$. The largest value for this positive decreasing function on the interval [1, 12] is $f^{(4)}(1) = 24$. Using the formal error estimate in (4.39) yields

$$|I - S_{80}| \le 24\frac{(12-1)^5}{2880(80)^4} \approx 3.28 \times 10^{-5}.$$

CAUTION In some instances, increasing the size of n does not necessarily lead to a more accurate approximation to the value of a definite integral. When a calculator or a computer is used to implement one of the numerical integration rules, round-off errors can occur when the size of the numbers becomes so small that they cannot be stored precisely in the machine. When n is very large, the numerical integration rules add a very large number of terms. If there are sufficiently many terms (a large value of n), the sum of the round-off errors can be large enough to produce a less accurate estimate for the value of the definite integral than does a smaller value of n. Courses in numerical analysis explore such issues in greater depth.

EXERCISES 4.7

c **Exer. 1–4: Use all five numerical integration rules with an appropriate n to approximate the definite integral of the function $y = f(x)$ over the interval [2, 5] when the function values are as given in the table.**

1

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	3.2	2.7	4.1	3.8	3.5	4.6	5.2

2

x	2.0	2.75	3.5	4.25	5.0
y	15.2	17.1	18.6	19.2	20.4

3

x	y
2.00	4.12
2.375	3.76
2.75	3.21
3.125	3,58
3.50	3.94
3.875	4.15
4.25	4.69
4.625	5.44
5.00	7.52

4

x	y
2.0	12.1
2.3	10.4
2.6	8.4
2.9	6.2
3.2	5.8
3.5	5.3
3.8	5.9
4.1	6.4
4.4	7.6
4.7	9.0
5.0	12.1

c **Exer. 5–6: (a) Use Riemann sums with both left-hand endpoints and right-hand endpoints to approximate the definite integral of the function $y = f(x)$ over the interval [4, 6] when the unequally spaced function values are as given in the table. (b) Find a trapezoidal rule estimate for the given partition of [4, 6] by averaging the two estimates from part (a).**

5

x	y
4.00	0.386
4.15	0.423
4.35	0.470
4.50	0.504
4.75	0.558
5.10	0.629
5.30	0.668
5.65	0.732
6.00	0.792

6

x	y
4.000	3.812
4.587	1.392
4.954	−2.250
5.223	−3.128
5.434	−2.435
5.608	−1.225
5.756	−0.029
5.886	0.950
6.000	1.637

Exer. 7–14: (a) Approximate the definite integral using the indicated rule for the given values of n. (b) Evaluate the definite integral exactly, and compute the errors for each approximation. (c) Determine how the error changes for successive computations.

7 $\int_1^{1.6} (2x-1)\,dx$; left rectangle rule for $n = 3, 6,$ and 12

8 $\int_1^{3} (x^2+1)\,dx$; right rectangle rule for $n = 4, 8,$ and 16

9 $\int_1^{5} x^3\,dx$; midpoint rule for $n = 2, 4,$ and 8

10 $\int_{-1}^{1} (x^2+5x+1)\,dx$; midpoint rule for $n = 1, 4,$ and 16

11 $\int_1^{5} x^3\,dx$; trapezoidal rule for $n = 2, 4,$ and 8

12 $\int_{-1}^{1} (x^2+5x+1)\,dx$; trapezoidal rule for $n = 1, 4,$ and 16

13 $\int_1^{5} x^3\,dx$; Simpson's rule for $n = 2, 4,$ and 8

14 $\int_{-1}^{1} (x^2+5x+1)\,dx$; Simpson's rule for $n = 1, 4,$ and 16

c **Exer. 15–18: (a) Approximate the definite integral using the indicated rule for the given values of n. (b) On the basis of the pattern of values, determine the expected accuracy for the approximation corresponding to the largest n.**

15 $\int_1^{3} \sqrt{1+x^3}\,dx$; trapezoidal rule for $n = 5, 10, 20,$ and 40

16 $\int_0^{5} 2^{-x^2}\,dx$; trapezoidal rule for $n = 2, 8, 32,$ and 128

17 $\int_0^{\pi} \cos(\sin x)\,dx$; Simpson's rule for $n = 2, 6, 18,$ and 54

18 $\int_0^{4} \frac{1}{1+x^3}\,dx$; Simpson's rule for $n = 8, 16, 32,$ and 64

Exer. 19–22: Use Theorem (4.39) to estimate the maximum error in approximating the definite integral for the given value of n, using (a) the trapezoidal rule and (b) Simpson's rule.

19 $\int_{-2}^{3} (\frac{1}{360}x^6 + \frac{1}{60}x^5)\,dx$; $n = 25$

20 $\int_{0}^{3} (-\frac{1}{12}x^4 + \frac{2}{3}x^3)\,dx$; $n = 24$

21 $\int_{1}^{5} \frac{1}{x^2}\,dx$; $n = 16$

22 $\int_{1}^{4} \frac{1}{35}x^{7/2}\,dx$; $n = 15$

Exer. 23–26: Using Theorem (4.39), find the least integer n such that the error estimate in approximating the definite integral is less than the given E when using (a) the left rectangle rule, (b) the midpoint rule, and (c) Simpson's rule.

23 $\int_{1}^{8} 81x^{8/3}\,dx$; $E = 0.05$

24 $\int_{1}^{2} \frac{1}{120x^2}\,dx$; $E = 0.1$

25 $\int_{1/2}^{1} \frac{1}{x}\,dx$; $E = 0.02$

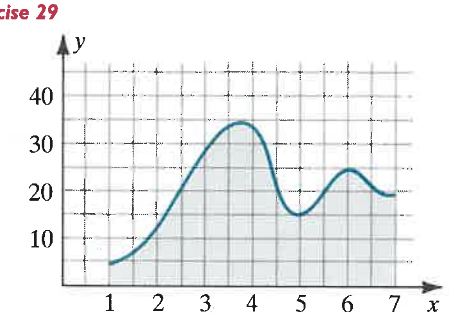
26 $\int_{0}^{3} \frac{1}{x+1}\,dx$; $E = 0.005$

27 If $f(x)$ is a polynomial of degree less than 4, prove that Simpson's rule gives the exact value of $\int_a^b f(x)\,dx$.

28 Suppose that f is continuous and that both f and f'' are nonnegative throughout $[a, b]$. Prove that $\int_a^b f(x)\,dx$ is less than the approximation given by the trapezoidal rule.

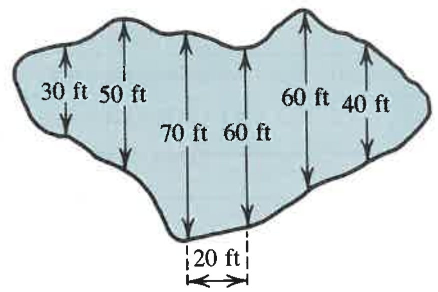
29 The graph in the figure was recorded by an instrument used to measure a physical quantity. Estimate y-coordinates of points on the graph, and approximate the area of the shaded region by using (a) the trapezoidal rule, with $n = 6$, and (b) Simpson's rule, with $n = 3$.

Exercise 29



30 An artificially created lake has the shape illustrated in the figure, with adjacent measurements 20 ft apart. Use the trapezoidal rule to estimate the surface area of the lake.

Exercise 30



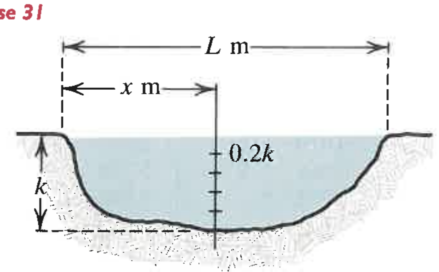
C 31 An important aspect of water management is the production of reliable data on *streamflow*, the number of cubic meters of water passing through a cross section of a stream or river per second. A first step in this computation is the determination of the average velocity $\overline{v}_x$ at a distance x meters from the river bank. If k is the depth of the stream at a point x meters from the bank and $v(y)$ is the velocity (in meters per second) at a depth of y meters (see figure), then

$$\overline{v}_x = \frac{1}{k}\int_0^k v(y)\,dy$$

(see Definition (4.29)). With the *six-point method,* velocity readings are taken at the surface; at depths $0.2k$, $0.4k$, $0.6k$, and $0.8k$; and near the river bottom. The trapezoidal rule is then used to estimate $\overline{v}_x$. Given the data in the following table, estimate $\overline{v}_x$.

y (m)	0	0.2k	0.4k	0.6k	0.8k	k
v(y) (m/sec)	0.28	0.23	0.19	0.17	0.13	0.02

Exercise 31



C 32 Refer to Exercise 31. The streamflow F (in cubic meters per second) can be approximated using the formula

$$F = \int_0^L \overline{v}_x h(x)\,dx,$$

where $h(x)$ is the depth of the stream at a distance x meters from the bank and L is the length of the cross section. Given the data in the following table, use Simpson's rule to estimate F.

x (m)	0	3	6	9	12
h(x) (m)	0	0.51	0.73	1.61	2.11
$\overline{v}_x$ (m/sec)	0	0.09	0.18	0.21	0.36

x (m)	15	18	21	24
h(x) (m)	2.02	1.53	0.64	0
$\overline{v}_x$ (m/sec)	0.32	0.19	0.11	0

C **Exer. 33–34: Use Simpson's rule, with $n = 4$, to approximate the average value of f on the given interval.**

33 $f(x) = \frac{1}{x^4 + 1}$; $[0, 4]$

34 $f(x) = \sqrt{\cos x}$; $[-1, 1]$

C **Exer. 35–36: If f is determined by the given differential equation and initial condition $f(0)$, approximate $f(1)$ using the trapezoidal rule with $n = 10$.**

35 $f'(x) = \frac{\sqrt{x}}{x^2 + 1}$; $f(0) = 1$

36 $f'(x) = \sqrt{\tan x}$; $f(0) = 2$

Exer. 37–38: Let a regular partition of $[a, b]$ be determined by $a = x_0, x_1, \ldots, x_{m-1}, x_m = b$.

37 Show that Simpson's rule can be expressed as

$$S_n = \sum_{k=1}^{n} \frac{1}{6}\left[f(x_{k-1}) + 4f(x_{k-1/2}) + f(x_k)\right]\Delta x.$$

38 If m is an even integer, show that Simpson's rule can be expressed as

$$S_{m/2} = \frac{b-a}{3m}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{m-2}) + 4f(x_{m-1}) + f(x_m)\right].$$

CHAPTER 4 REVIEW EXERCISES

Exer. 1–42: Evaluate.

1 $\int \frac{8x^2 - 4x + 5}{x^4}\,dx$

2 $\int (3x^5 + 2x^3 - x)\,dx$

3 $\int 100\,dx$

4 $\int x^{3/5}(2x - \sqrt{x})\,dx$

5 $\int (2x + 1)^7\,dx$

6 $\int \sqrt[3]{5x + 1}\,dx$

7 $\int (1 - 2x^2)^3 x\,dx$

8 $\int \frac{(1 + \sqrt{x})^2}{\sqrt[3]{x}}\,dx$

9 $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2}\,dx$

10 $\int (x^2 + 4)^2\,dx$

11 $\int (3 - 2x - 5x^3)\,dx$

12 $\int (x + x^{-1})^2\,dx$

13 $\int (4x + 1)(4x^2 + 2x - 7)^2\,dx$

14 $\int \frac{\sqrt[4]{1 - (1/x)}}{x^2}\,dx$

15 $\int (2x^{-3} - 3x^2)\,dx$

16 $\int (x^{3/2} + x^{-3/2})\,dx$

17 $\int_0^1 \sqrt[3]{8x^7}\,dx$

18 $\int_1^2 \frac{x^2 - x - 6}{x + 2}\,dx$

19 $\int_0^1 \frac{x^2}{(1 + x^3)^2}\,dx$

20 $\int_1^9 \sqrt{2x + 7}\,dx$

21 $\int_1^2 \frac{x + 1}{\sqrt{x^2 + 2x}}\,dx$

22 $\int_1^2 \frac{x^2 + 2}{x^2}\,dx$

23 $\int_0^2 x^2\sqrt{x^3 + 1}\,dx$

24 $\int_1^1 3x^2\sqrt{x^3 + x}\,dx$

25 $\int_0^1 (2x - 3)(5x + 1)\,dx$

26 $\int_{-1}^1 (x^2 + 1)^2\,dx$

27 $\int_0^4 \sqrt{3x}(\sqrt{x} + \sqrt{3})\,dx$

28 $\int_{-1}^{1} (x+1)(x+2)(x+3)\,dx$

29 $\int \sin(3-5x)\,dx$

30 $\int x^2 \cos(2x^3)\,dx$

31 $\int \cos 3x \sin^4 3x\,dx$

32 $\int \dfrac{\sin(1/x)}{x^2}\,dx$

33 $\int \dfrac{\cos 3x}{\sin^3 3x}\,dx$

34 $\int (3\cos 2\pi t - 5\sin 4\pi t)\,dt$

35 $\int_0^{\pi/2} \cos x\sqrt{3+5\sin x}\,dx$

36 $\int_{-\pi/4}^{0} (\sin x + \cos x)^2\,dx$

37 $\int_0^{\pi/4} \sin 2x \cos^2 2x\,dx$

38 $\int_{\pi/6}^{\pi/4} (\sec x + \tan x)(1-\sin x)\,dx$

39 $\int \dfrac{d}{dx}\sqrt[5]{x^4+2x^2+1}\,dx$

40 $\int_0^{\pi/2} \dfrac{d}{dx}(x\sin^3 x)\,dx$

41 $\dfrac{d}{dx}\int_0^1 (x^3+x^2-7)^5\,dx$

42 $\dfrac{d}{dx}\int_0^x (t^2+1)^{10}\,dt$

Exer. 43–44: Solve the differential equation subject to the given conditions.

43 $\dfrac{d^2y}{dx^2} = 6x-4; \quad y=4 \text{ and } y'=5 \text{ if } x=2$

44 $f''(x) = x^{1/3}-5; \quad f'(1)=2;\ f(1)=-8$

Exer. 45–46: Let $f(x) = 9-x^2$ for $-2 \le x \le 3$, and let P be the regular partition of $[-2,3]$ into five subintervals.

45 Find the Riemann sum R_P if f is evaluated at the midpoint of each subinterval of P.

46 Find (a) A_{IP} and (b) A_{CP}.

Exer. 47–48: Verify the inequality without evaluating the integrals.

47 $\int_0^1 x^2\,dx \ge \int_0^1 x^3\,dx$

48 $\int_1^2 x^2\,dx \le \int_1^2 x^3\,dx$

Exer. 49–50: Express as one integral.

49 $\int_c^e f(x)\,dx + \int_a^b f(x)\,dx - \int_c^b f(x)\,dx - \int_d^{d} f(x)\,dx$

50 $\int_a^d f(x)\,dx - \int_t^b f(x)\,dx - \int_g^{g} f(x)\,dx + \int_m^b f(x)\,dx + \int_t^a f(x)\,dx$

51 A stone is thrown directly downward from a height of 900 ft with an initial velocity of 30 ft/sec.

(a) Determine the stone's distance above the ground after t seconds.

(b) Find its velocity after 5 sec.

(c) Determine when it strikes the ground.

52 Is the following argument valid? Explain.
The function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational and } x>0 \\ -1 & \text{if } x \text{ is rational and } x<0 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is defined for all numbers in $[-1,1]$ and has the property that $f(-x) = -f(x)$ for all x in $[-1,1]$. Thus, $\int_{-1}^{1} f(x)\,dx = 0$.

53 Find a definite integral for which

$$\sum_{k=1}^{50} \sqrt{1+3\left(-2+\frac{k}{10}\right)^2}\left(\frac{1}{10}\right)$$

is a right rectangle rule approximation.

54 Given $\int_1^4 (x^2+2x-5)\,dx$, find

(a) a number z that satisfies the conclusion of the mean value theorem for integrals (4.28)

(b) the average value of x^2+2x-5 on $[1,4]$

c **Exer. 55–58: Approximate the definite integral using the indicated rule for the stated values of n.**

55 $\int_0^2 \sin(x^2)\,dx$; midpoint rule for $n=5$ and 10

56 $\int_0^1 \cos\sqrt{x}\,dx$; trapezoidal rule for $n=10$ and 20

57 $\int_2^4 \sqrt{x^3+x}\,dx$; Simpson's rule for $n=4$ and 8

58 $\int_0^5 \dfrac{1}{x+2}\,dx$; Simpson's rule for $n=5$ and 20

c 59 To monitor the thermal pollution of a river, a biologist takes hourly temperature readings (in °F) from 9 A.M. to 5 P.M. The results are shown in the following table.

Time of day	9	10	11	12	1
Temperature	75.3	77.0	83.1	84.8	86.5

Time of day	2	3	4	5
Temperature	86.4	81.1	78.6	75.1

Use Simpson's rule and Definition (4.29) to estimate the average water temperature between 9 A.M. and 5 P.M.

EXTENDED PROBLEMS AND GROUP PROJECTS

1 Let $f(t) = 1/(1+t^2)$.

(a) Sketch the graph of f and discuss its symmetries.

(b) Show that f is continuous for all real numbers t.

(c) Prove that the function $F(x) = \int_0^x f(t)\,dt$ exists and is differentiable for all real numbers x.

(d) Find $F(0)$.

(e) Show that $F'(x) = 1/(1+x^2)$.

(f) Show that F is a strictly increasing function.

(g) Find $F''(x)$ and determine the intervals over which F is concave upward and concave downward. Find all points of inflection.

(h) Use the information obtained so far to sketch a graph of F.

(i) Show that F must have an inverse function T. Assuming that T is differentiable, use the identity $F(T(x)) = x$, differentiation, and the chain rule to conclude that $T'(x) = 1+[T(x)]^2$.

(j) Show that the tangent function satisfies

$$(\tan x)' = 1 + [\tan x]^2.$$

(k) Discuss the similarity between the results of parts (i) and (j).

2 Let $f(t)$ be a continuous function and define

$$F(x) = \int_0^x x\,f(t)\,dt.$$

(a) Find $F'(x)$ if $f(t) = t$.

(b) Find $F'(x)$ if $f(t) = t^2$.

(c) Find $F'(x)$ if $f(t) = \cos t$.

(d) Formulate a general result about $F'(x)$.

(e) Prove that

$$\int_0^x \left(\int_0^u f(t)\,dt\right) du = \int_0^x f(u)(x-u)\,du.$$

(*Hint:* Differentiate both sides, using the result of part (d).)

(f) Prove that

$$\int_0^x f(u)(x-u)^2\,du = 2\int_0^x \left[\int_0^w \left(\int_0^v f(t)\,dt\right) dv\right] dw.$$

3 One way to measure the effectiveness of a numerical integration method is to test the method on polynomials. A method is **exact for polynomials of degree n** if it produces zero error for any polynomial of degree at most n, but does produce error for some polynomial of degree $n+1$.

(a) Show that a method is exact for polynomials of degree n if and only if it produces zero error for monomials, $f(x) = x^j$ for $j = 0, 1, \ldots, n$, but has some error for monomials x^{n+1}.

(b) Show that the midpoint rule and the trapezoidal rule are exact for polynomials of degree 1.

(c) Show that Simpson's rule is exact for polynomials of degree 3.

(d) Show that the following numerical integration rule (called a *Gaussian rule*) is exact for polynomials of degree 3:

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n}\left\{f\left[x_{k-1}+\left(1-\sqrt{\tfrac{1}{3}}\right)\frac{\Delta x}{2}\right] + f\left[x_{k-1}+\left(1+\sqrt{\tfrac{1}{3}}\right)\frac{\Delta x}{2}\right]\right\}\frac{\Delta x}{2}$$

(e) Discuss the advantages and disadvantages of implementing the Gaussian rule in part (d) over implementing Simpson's rule.

(f) Test the following numerical integration rule (called *Simpson's 3/8 rule*) to determine its polynomial exactness:

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n}\frac{3}{8}\left[f(x_{k-1}) + 3f\left(x_{k-1}+\frac{\Delta x}{3}\right) + 3f\left(x_{k-1}+2\frac{\Delta x}{3}\right) + f(x_k)\right]\frac{\Delta x}{3}$$

(g) Can a numerical integration rule that is exact for polynomials of degree 2 be designed?