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52 Apply the approach discussed in Exercise 51 to the function of Example 2.

53 Suppose that $f(x) = g(x)/h(x)$, where g and h are polynomials of degree m and n, respectively.

(a) Show that the numerator of $f'(x)$ (before any simplification) is a polynomial of degree $m + n - 1$.

(b) What is the degree of the numerator of $f''(x)$?

3.6 OPTIMIZATION PROBLEMS

In this section, we will examine applications in which we need to find the maximum or minimum values of a function. For example, a physical or geometric quantity Q is often described by means of some formula $Q = f(x)$, where f is a function. Thus, Q might represent the temperature of a substance at time x, the current in an electrical circuit when the resistance is x, or the volume of gas in a spherical balloon of radius x. Of course, we often use other symbols for variables, such as T for temperature, t for time, I for current, R for resistance, V for volume, and r for radius. If $Q = f(x)$ and f is differentiable, then the derivative $dQ/dx = f'(x)$ can be used to help find the maximum or minimum values of Q. In applications, these extreme values are sometimes called **optimal values**, because they are, in a sense, the best or most favorable values of the quantity Q. The task of finding these values is called an **optimization problem**.

If an optimization problem is stated in words, then it is often necessary to convert the statement into an appropriate formula, such as $Q = f(x)$, in order to find critical numbers. In most cases, there will be only one critical number c. If, in addition, f is continuous on a closed interval $[a, b]$ containing c, then, by Guidelines (3.9), the extrema of f are the largest and smallest of the values $f(a)$, $f(b)$, and $f(c)$. Hence, it is often unnecessary to apply a derivative test. However, if it is easy to calculate $f''(x)$, we sometimes apply the second derivative test to verify an extremum, as illustrated in the next example.

EXAMPLE 1 A long rectangular sheet of metal, 12 in. wide, is to be made into a rain gutter by turning up two sides so that they are perpendicular to the sheet. How many inches should be turned up in order to give the gutter its greatest capacity?

SOLUTION The gutter is illustrated in Figure 3.54, where x denotes the number of inches turned up on each side. The width of the base of the gutter is $12 - 2x$ inches. The capacity of the gutter will be greatest when the area of the rectangle with sides of lengths x and $12 - 2x$ has its greatest value. Letting $f(x)$ denote this area, we obtain

$$f(x) = x(12 - 2x) = 12x - 2x^2.$$

Since $0 \le 2x \le 12$, the domain of f is $0 \le x \le 6$. If $x = 0$, or $x = 6$, no gutter is formed (the area of the rectangle would be $f(0) = 0 = f(6)$).

Differentiating yields

$$f'(x) = 12 - 4x = 4(3 - x);$$

Figure 3.54

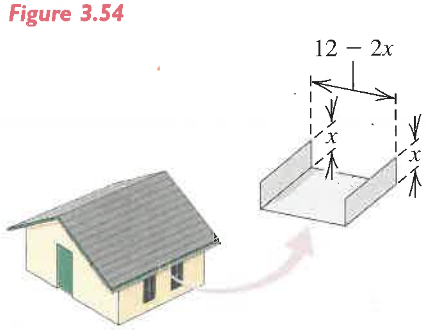
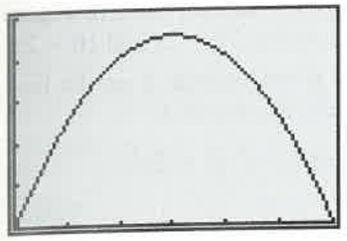


Figure 3.55
$0 \le x \le 6, 0 \le y \le 20$



thus, the only critical number of f is 3. Since $f''(x) = -4 < 0$, $f(3)$ is a local maximum for f. It follows that 3 in. should be turned up to achieve maximum capacity.

COMPUTATIONAL METHOD Once we have represented the area as a function with a prescribed domain, we can use a graphing utility to obtain a graph of the function and trace it to the maximum value. In this example, when we graph $f(x) = 12x - 2x^2$ over the interval $[0, 6]$, we obtain Figure 3.55. Using the trace option, we find that the maximum occurs at the point $(3, 18)$.

Because the types of optimization problems are unlimited, it is difficult to state specific rules for finding solutions. However, we can develop a general strategy for attacking such problems. The following guidelines are often helpful. When using the guidelines, don't become discouraged if you are unable to solve a given problem quickly. It takes a great deal of effort and practice to become proficient in solving optimization problems. Keep trying!

Guidelines for Solving Optimization Problems 3.22

1. Read the problem carefully several times, and think about the given facts as well as the unknown quantities that are to be found.
2. If possible, sketch a picture or diagram and label it appropriately, introducing variables for unknown quantities. Words such as *what, find, how much, how far,* or *when* should alert you to the unknown quantities.
3. Write down the known facts together with any relationships involving the variables.
4. Determine which variable is to be maximized or minimized, and express this variable as a function of *one* of the other variables.
5. Find the critical numbers of the function obtained in guideline (4).
6. Determine the extrema by using Guidelines (3.9) or the first or second derivative test. Check for endpoint extrema whenever appropriate.

The use of Guidelines (3.22) is illustrated in the next example.

EXAMPLE 2 An open box with a rectangular base is to be constructed from a rectangular piece of cardboard 16 in. wide and 21 in. long by cutting a square from each corner and then bending up the resulting sides. Find the size of the corner square that will produce a box having the largest possible volume. (Disregard the thickness of the cardboard.)

SOLUTION

Guideline 1 Read the problem at least one more time.

Guideline 2 Sketch the cardboard, as in Figure 3.56(a), introducing a variable x for the length of the side of the square to be cut from each corner.

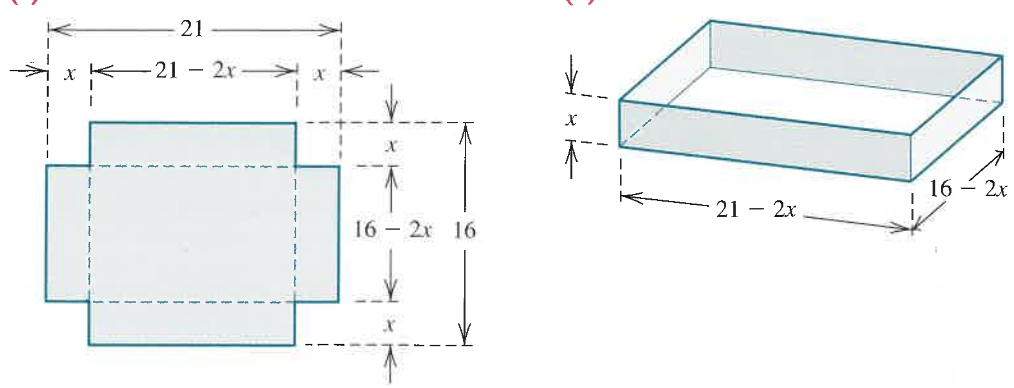
Guideline 3 If the cardboard is folded along the dashed lines in Figure 3.56(a), the base of the resulting box has dimensions $21 - 2x$ and $16 - 2x$.

Guideline 4 The quantity to be maximized is the volume V of the box. Referring to Figure 3.56(b), we express V as a function of x:

$$V = x(16 - 2x)(21 - 2x) = 2(168x - 37x^2 + 2x^3)$$

Since $0 \le 2x \le 16$, the domain of V is $0 \le x \le 8$.

Figure 3.56



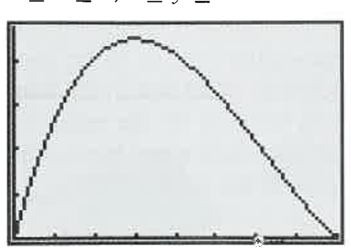
Guideline 5 To find the critical numbers for the function in guideline (4), differentiate V with respect to x:

$$\begin{aligned}\frac{dV}{dx} &= 2(168 - 74x + 6x^2)\\ &= 4(3x^2 - 37x + 84)\\ &= 4(3x - 28)(x - 3)\end{aligned}$$

Thus the possible critical numbers are $\frac{28}{3}$ and 3. Since $\frac{28}{3}$ is outside the domain of V, the only critical number is 3.

Guideline 6 Since V is continuous on [0, 8], we shall use Guidelines (3.9) to determine the extrema. The endpoints $x = 0$ and $x = 8$ of the domain yield the minimum value $V = 0$. For the critical number $x = 3$, we obtain $V = 450$, which is a maximum value. Consequently, a 3-in. square should be cut from each corner of the cardboard in order to maximize the volume of the resulting box.

Figure 3.57
$0 \le x \le 8, 0 \le y \le 475$



COMPUTATIONAL METHOD Using a graphing utility for

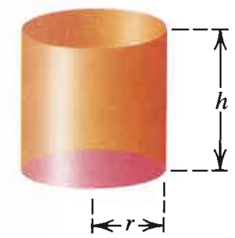
$$V(x) = 2(168x - 37x^2 + 2x^3)$$

over the interval [0, 8] yields the graph shown in Figure 3.57, from which we see that the maximum value of V occurs at $x = 3$.

In the remaining examples, we shall not always point out the guidelines used. You should be able to determine specific guidelines by studying the solutions.

EXAMPLE 3 A circular cylindrical metal container, open at the top, is to have a capacity of 24π in^3. The cost of the material used for the bottom of the container is 15 cents per in^2, and that of the material used for the curved part is 5 cents per in^2. If there is no waste of material, find the dimensions that will minimize the cost of the material.

Figure 3.58



SOLUTION We begin by sketching a typical container, as in Figure 3.58, letting r denote the radius of the base and h the altitude (both in inches). *The quantity we wish to minimize is the cost C of the material.* Since the costs per square inch for the base and the curved part are 15 cents and 5 cents, respectively, we have, in terms of cents,

$$\text{cost of container} = 15(\text{area of base}) + 5(\text{area of curved part}).$$

Thus,

$$C = 15(\pi r^2) + 5(2\pi r h),$$

or

$$C = 5\pi(3r^2 + 2rh).$$

We can express C as a function of one variable r by expressing h in terms of r. Since the volume of the container is 24π in^3, we see that

$$\pi r^2 h = 24\pi, \quad \text{or} \quad h = \frac{24}{r^2}.$$

Substituting $24/r^2$ for h in the latter formula for C gives us

$$C = 5\pi\left(3r^2 + 2r \cdot \frac{24}{r^2}\right) = 5\pi\left(3r^2 + \frac{48}{r}\right).$$

The domain of C is $(0, \infty)$.

Next, we find critical numbers by differentiating C with respect to r:

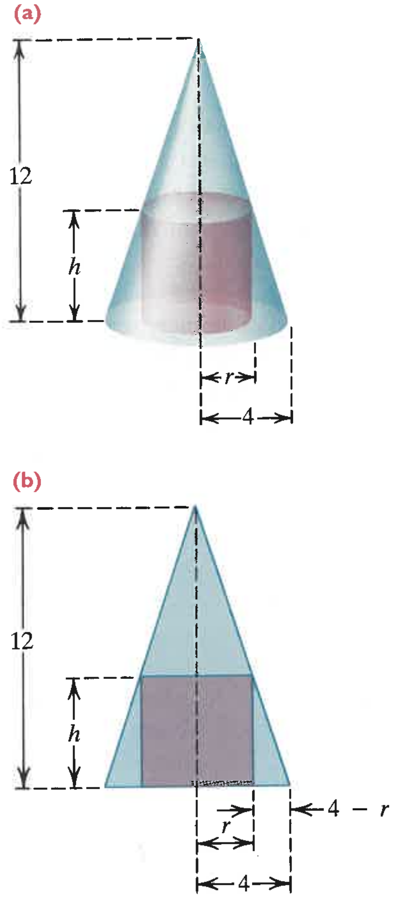
$$\frac{dC}{dr} = 5\pi\left(6r - \frac{48}{r^2}\right) = 30\pi\left(\frac{r^3 - 8}{r^2}\right)$$

Since $dC/dr = 0$ if $r = 2$, we see that 2 is the only critical number. Since $dC/dr < 0$ if $r < 2$, and $dC/dr > 0$ if $r > 2$, it follows from the first derivative test that C has its minimum value if the radius of the cylinder is 2 in. The corresponding value for the altitude (obtained from $h = 24/r^2$) is $\frac{24}{4}$, or 6 in.

EXAMPLE 4 Find the maximum volume of a right circular cylinder that can be inscribed in a cone of altitude 12 cm and base radius 4 cm, if the axes of the cylinder and cone coincide.

Figure 3.59

(a)



SOLUTION The problem is sketched in Figure 3.59, where (b) represents a cross section through the axes of the cone and the cylinder. *The quantity we wish to maximize is the volume V of the cylinder.* From geometry,

$$V = \pi r^2 h.$$

Next, we express V in terms of one variable by finding a relationship between r and h. Referring to Figure 3.59(b) and using similar triangles, we see that

$$\frac{h}{4-r} = \frac{12}{4} = 3, \quad \text{or} \quad h = 3(4-r).$$

Consequently,

$$V = \pi r^2 h = \pi r^2 \cdot 3(4-r) = 3\pi r^2(4-r).$$

The domain of V is $0 \le r \le 4$.

If either $r = 0$ or $r = 4$, we see that $V = 0$, and hence the maximum volume is not an endpoint extremum. It is sufficient, therefore, to search for local maxima. Since $V = 3\pi(4r^2 - r^3)$,

$$\frac{dV}{dr} = 3\pi(8r - 3r^2) = 3\pi r(8 - 3r).$$

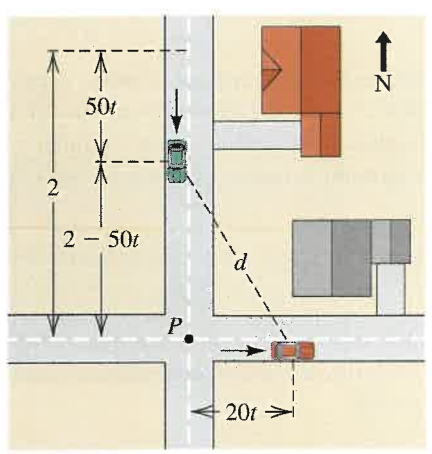
Thus the critical numbers for V are $r = 0$ and $r = \frac{8}{3}$. At $r = \frac{8}{3}$, we have

$$V = \pi\left(\frac{8}{3}\right)^2(4) = \frac{256\pi}{9} \approx 89.4 \text{ cm}^3,$$

which, by Guidelines (3.9), is a maximum value for the volume of the inscribed cylinder.

EXAMPLE ■ 5 A north–south highway intersects an east–west highway at a point P. An automobile crosses P at 10:00 A.M., traveling east at a constant speed of 20 mi/hr. At that same instant, another automobile is 2 mi north of P, traveling south at 50 mi/hr. Find the time at which they are closest to each other, and approximate the minimum distance between the automobiles.

Figure 3.60



SOLUTION Typical positions of the automobiles are illustrated in Figure 3.60. If t denotes the number of hours after 10:00 A.M., then the slower automobile is $20t$ miles east of P. The faster automobile is $50t$ miles south of its position at 10:00 A.M., and hence its distance from P is $2 - 50t$. By the Pythagorean theorem, the distance d between the automobiles is

$$\begin{aligned} d &= \sqrt{(2-50t)^2 + (20t)^2} \\ &= \sqrt{4 - 200t + 2500t^2 + 400t^2} = \sqrt{4 - 200t + 2900t^2}. \end{aligned}$$

We wish to find the time t at which d has its smallest value, which will occur when the expression under the radical is minimal because d increases if and only if $4 - 200t + 2900t^2$ increases. Thus, we may simplify our work by letting

$$f(t) = 4 - 200t + 2900t^2$$

and finding the value of t for which f has a minimum. Since

$$f'(t) = -200 + 5800t,$$

the only critical number for f is

$$t = \tfrac{200}{5800} = \tfrac{1}{29}.$$

Moreover, $f''(t) = 5800$, so the second derivative is always positive. Therefore, f has a local minimum at $t = \frac{1}{29}$, and $f(\frac{1}{29}) = \frac{16}{29}$. Since the domain of t is $[0, \infty)$ and since $f(0) = 4$, there is no endpoint extremum. Consequently, the automobiles will be closest at $\frac{1}{29}$ hour (or approximately 2.07 min) after 10:00 A.M. The minimum distance is

$$\sqrt{f(\tfrac{1}{29})} = \sqrt{\tfrac{16}{29}} \approx 0.74 \text{ mi}.$$

EXAMPLE ■ 6 A person in a rowboat 2 mi from the nearest point on a straight shoreline wishes to reach a house 6 mi farther down the shore. If the person can row at a rate of 3 mi/hr and walk at a rate of 5 mi/hr, find the least amount of time required to reach the house.

SOLUTION Figure 3.61 illustrates the problem: A denotes the position of the boat, B the nearest point on shore, C the house, D the point at which the boat reaches shore, and x the distance between B and D. By the Pythagorean theorem, the distance between A and D is $\sqrt{x^2+4}$, where $0 \le x \le 6$. Using the formula

$$\text{time} = \frac{\text{distance}}{\text{rate}},$$

we obtain

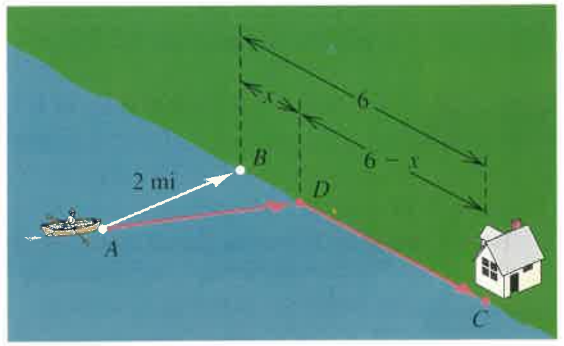
$$\text{time to row from } A \text{ to } D = \frac{\text{distance from } A \text{ to } D}{\text{rowing rate}} = \frac{\sqrt{x^2+4}}{3},$$

$$\text{time to walk from } D \text{ to } C = \frac{\text{distance from } D \text{ to } C}{\text{walking rate}} = \frac{6-x}{5}.$$

Hence the total time T for the trip is

$$T = \frac{\sqrt{x^2+4}}{3} + \frac{6-x}{5},$$

Figure 3.61



or, equivalently, $T = \frac{1}{3}(x^2+4)^{1/2} + \frac{6}{5} - \frac{1}{5}x.$

We wish to find the minimum value for T. Note that $x = 0$ corresponds to the extreme situation in which the person rows directly to B and then walks the entire distance from B to C. If $x = 6$, then the person rows directly from A to C. These numbers may be considered as endpoints of the domain of T. If $x = 0$, then, from the formula for T,

$$T = \frac{\sqrt{4}}{3} + \frac{6}{5} - 0 = \frac{28}{15},$$

which is 1 hr 52 min. If $x = 6$, then

$$T = \frac{\sqrt{40}}{3} + \frac{6}{5} - \frac{6}{5} = \frac{2\sqrt{10}}{3} \approx 2.11,$$

or approximately 2 hr 7 min.

Differentiating the general formula for T, we see that

$$\frac{dT}{dx} = \frac{1}{3}\cdot\frac{1}{2}(x^2+4)^{-1/2}(2x) - \frac{1}{5},$$

or

$$\frac{dT}{dx} = \frac{x}{3(x^2+4)^{1/2}} - \frac{1}{5}.$$

In order to find the critical numbers, we let $dT/dx = 0$, obtaining the following equations:

$$\frac{x}{3(x^2+4)^{1/2}} = \frac{1}{5}$$
$$5x = 3(x^2+4)^{1/2}$$
$$25x^2 = 9(x^2+4)$$
$$x^2 = \tfrac{36}{16}$$
$$x = \tfrac{6}{4} = \tfrac{3}{2}$$

Thus, $\frac{3}{2}$ is the only critical number. The time T that corresponds to $x = \frac{3}{2}$ is

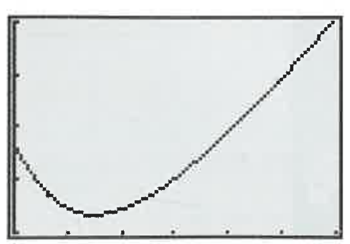
$$T = \tfrac{1}{3}(\tfrac{9}{4}+4)^{1/2} + \tfrac{6}{5} - \tfrac{3}{10} = \tfrac{26}{15},$$

or, equivalently, 1 hr 44 min.

We have already examined the values of T at the endpoints of the domain, obtaining 1 hr 52 min and approximately 2 hr 7 min, respectively. Hence the minimum time of 1 hr 44 min occurs at $x = \frac{3}{2}$. Therefore, the boat should land at D, $1\frac{1}{2}$mi from B, in order to minimize T. For a similar problem, but one in which the endpoints of the domain lead to minimum time, see Exercise 12.

Figure 3.62

$0 \le x \le 6,\ 1.7 \le y \le 2.1$



COMPUTATIONAL METHOD Examining the function

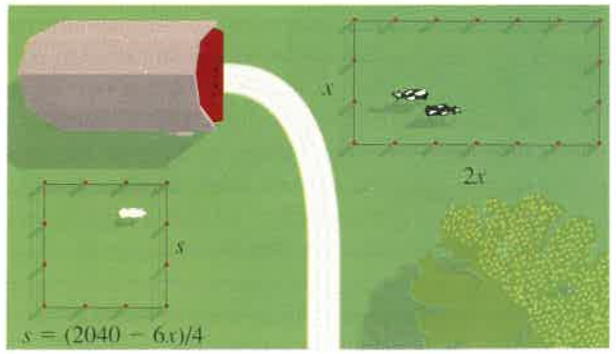
$$T(x) = \frac{\sqrt{x^2+4}}{3} + \frac{6-x}{5}$$

on [0, 6] with a graphing utility produces the graph shown in Figure 3.62. We can use the tracing operation to find that the minimum time occurs at $x = \frac{3}{2}$.

CAUTION We can err in solving optimization problems by formulating an appropriate function $f(x)$, finding where $f'(x) = 0$, and then declaring that we have located the extreme point. The critical points at which the derivative is zero may not be the extreme we are seeking. A critical point could turn out, for example, to be the minimum of the function when we needed to find the maximum. We may also have values c where $f'(c) = 0$, but the real-world constraints on the variables may put c outside the acceptable domain of f. The next two examples illustrate what can happen in such cases.

EXAMPLE 7 A farmer has 2040 ft of fencing and wishes to fence off two separate fields. As Figure 3.63 shows, one of the fields is to be a rectangle with the length twice as long as the width, while the other field is to be square. Determine the dimensions of the fields if the farmer wishes to maximize the total area of the two fields.

Figure 3.63



SOLUTION The total area A is the sum of the areas of the rectangle and the square. Let x represent the width of the rectangular field. We will write A as a function of x.

If x is the width, then the length is $2x$. The area of this rectangle is $2x^2$ square feet and its perimeter is $6x$ feet. The amount of fencing left after the rectangle has been built is $2040 - 6x$ feet. If the $2040 - 6x$ feet of fencing is used for the square, then each of its sides is $(2040 - 6x)/4 = 510 - 1.5x$ feet. Thus, the area of the square is $(510 - 1.5x)^2$ square feet.

The total area of both fields is

$$A(x) = 2x^2 + (510 - 1.5x)^2 \text{ square feet.}$$

We find the derivative $A'(x)$ to be

$$A'(x) = 4x + 2(510 - 1.5x)(-1.5) = 8.5x - 1530.$$

Solving $A'(x) = 0$ yields a unique value $x = c = 180$.

We may be tempted to advise the farmer to construct one rectangular field of dimensions 180 ft by 360 ft and one square field of sides 240 ft, which will achieve a "maximum" total area of $(180)(360) + 240^2 = 122{,}400$ ft^2.

We need to check, however, whether the critical value $c = 180$ really is a maximum for the function. We apply the second derivative test. Since

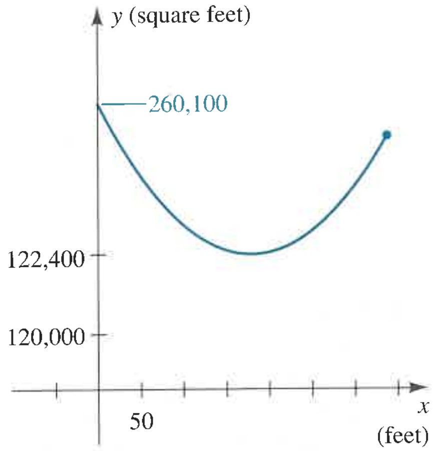
$A''(x) = 8.5$, we have $A''(c) = A''(180) = 8.5$, which is positive. Thus, there is a local *minimum* at c, not a maximum.

To determine where the maximum of $A(x)$ actually occurs, we note that A is a differentiable function, so the only other possible candidates for the local extrema are at the endpoints of the interval of the domain of A. We need to determine this interval.

Since x represents a length, we must have $x \geq 0$. The smallest possible value for x is 0, which corresponds to having no rectangle and putting all the fencing into building the square. The corresponding total area is

$$A(0) = 510^2 = 260{,}100 \text{ ft}^2.$$

Figure 3.64



On the other hand, since the amount of fencing available for the square must be nonnegative, we have $2040 - 6x \geq 0$ or, equivalently, $x \leq 340$. The largest possible value for x is 340, which corresponds to using all the fencing to construct the rectangle. The associated area is

$$A(340) = 2(340)^2 = 231{,}200 \text{ ft}^2.$$

The interval is $[0, 340]$, and the maximum value of A occurs at $x = 0$. The best advice to the farmer is to build a single square 510 ft on a side. If the farmer insists on having two separate fields, each of positive area, then the rectangular field should be made as small as possible.

The graph of $A(x)$ on the interval $[0, 340]$ is shown in Figure 3.64. If we were to fail to analyze the nature of the critical point, then we would incorrectly find the minimum value, rather than the maximum value, and give the farmer the worst possible advice.

Figure 3.65

EXAMPLE 8 A recycling company transports recyclable paper and cardboard from city A to a processing plant in city B along a highway (see Figure 3.65). Materials are carried in trucks that travel at a speed of x miles per hour. Legal speeds on the highway are between 35 and 55 mi/hr. Assume that diesel fuel costs \$1.25 per gallon and is consumed at the rate of $4 + (x^2/500)$ gallons per hour. The recycler pays its drivers \$11 per hour and reimburses them for the cost of fuel as well. At what speed should the trucks be driven in order to minimize the recycler's total cost?

SOLUTION The total cost C is the sum of the cost of wages and of fuel:

$$\text{total cost } C = \text{cost of wages} + \text{cost of fuel}$$

We note that

$$\text{cost of wages} = (\text{wages per hour})(\text{number of hours})$$

and

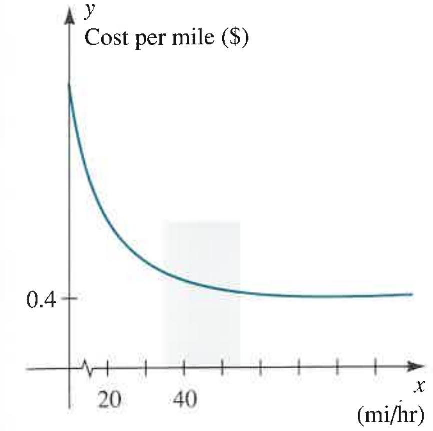
$$\begin{aligned}\text{cost of fuel} &= (\text{cost per gallon})(\text{number of gallons})\\ &= (\text{cost per gallon})(\text{gallons per hour})(\text{number of hours}).\end{aligned}$$

Since the trucks travel at x miles per hour, we have

$$\text{number of hours} = \frac{\text{distance}}{x}.$$

Figure 3.66

$$y = f(x) = \left(\frac{1}{x}\right)\left(16 + \frac{1.25x^2}{500}\right)$$



Hence, total cost $C(x)$ is given by

$$C(x) = 11\left(\frac{\text{distance}}{x}\right) + 1.25\left(4 + \frac{x^2}{500}\right)\left(\frac{\text{distance}}{x}\right),$$

which we can write as

$$C(x) = \left(\frac{\text{distance}}{x}\right)\left[11 + 1.25\left(4 + \frac{x^2}{500}\right)\right].$$

The distance traveled is not specified in the statement of the problem. We note, however, that since the distance is a positive number, the quantity $C(x)$ is minimized when the function

$$f(x) = \left(\frac{1}{x}\right)\left[11 + 1.25\left(4 + \frac{x^2}{500}\right)\right] = \left(\frac{1}{x}\right)\left(16 + \frac{1.25x^2}{500}\right)$$

is minimized.

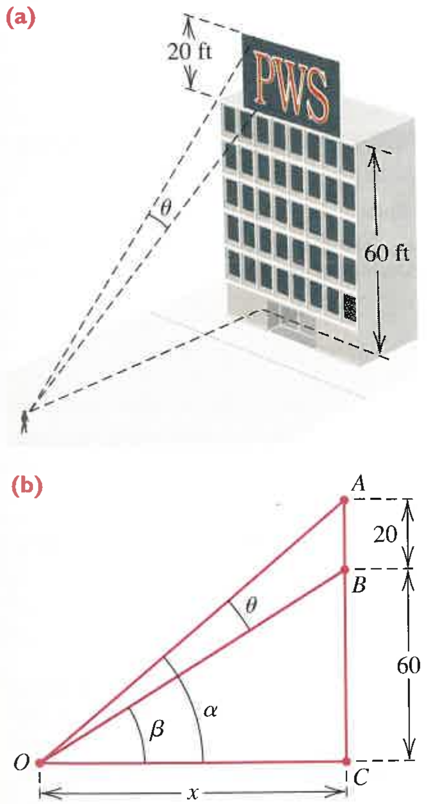
Differentiating f and simplifying yields

$$f'(x) = \frac{1.25}{500} - \frac{16}{x^2} \quad \text{and} \quad f''(x) = \frac{32}{x^3}.$$

We then solve $f'(x) = 0$ to find two roots, $x = 80$ and $x = -80$. The negative root can be ignored since x represents the speed of a truck, which must be positive. Thus we have a unique critical point at $x = 80$. We check the second derivative, $f''(80) = 32/80^3$, which is positive, so we indeed have a local minimum at $x = 80$. Another critical point at $x = 0$, where the first derivative fails to exist, can also be ignored because the trucks must move at a positive speed. (For the purposes of this problem, the domain of $C(x)$ consists only of positive numbers.)

We cannot advise the company to instruct its truckers to drive at 80 mi/hr in violation of the legal speed limits, which are on the interval $[33, 55]$. We thus conclude that the global minimum is not available. Examination of the derivative $f'(x)$ shows that it is negative for all positive x less than 80, so cost is lowest when the speed is the highest legal one, 55 mi/hr. Note that for $x = 55$, $f(x) \approx 0.4284$ (about 43 cents per mile), so the minimum total cost is about 0.4284(distance). Figure 3.66 is a graph of f, which shows the location of the endpoint extrema.

Figure 3.67



EXAMPLE 9 A billboard 20 ft tall is located on top of a building, with its lower edge 60 ft above the level of a viewer's eye, as shown in Figure 3.67(a). How far from a point directly below the sign should a viewer stand to maximize the angle θ between the lines of sight of the top and bottom of the billboard? (This angle should result in the best view of the billboard.)

SOLUTION The problem is sketched in Figure 3.67(b), using right triangles AOC and BOC having common side OC of (variable) length x. We see that

$$\tan\alpha = \frac{80}{x} \quad \text{and} \quad \tan\beta = \frac{60}{x}.$$

The angle $\theta = \alpha - \beta$ is a function of x and

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Substituting for $\tan\alpha$ and $\tan\beta$ and simplifying, we obtain

$$\tan\theta = \frac{(80/x) - (60/x)}{1 + (80/x)(60/x)} = \frac{20x}{x^2 + 4800}.$$

The extrema of θ occur if $d\theta/dx = 0$. Differentiating implicitly with respect to x and using the quotient rule gives us

$$\sec^2\theta\frac{d\theta}{dx} = \frac{(x^2 + 4800)(20) - 20x(2x)}{(x^2 + 4800)^2} = \frac{96{,}000 - 20x^2}{(x^2 + 4800)^2}.$$

Since $\sec^2\theta > 0$, it follows that $d\theta/dx = 0$ if and only if

$$96{,}000 - 20x^2 = 0, \quad \text{or} \quad x^2 = 4800.$$

Thus the only critical number of θ is

$$x = \sqrt{4800} = 40\sqrt{3}.$$

We may verify that the sign of $d\theta/dx$ changes from positive to negative at $\sqrt{4800}$, and hence a maximum value of θ occurs at $x = 40\sqrt{3}$ ft ≈ 69.3 ft.

EXERCISES 3.6

Exer. 1–6: You will need to formulate a function and find its extreme values on some interval, which you must also determine. In addition to Guidelines (3.9) and the first and second derivative tests, you may wish to use a graphing utility to examine the graph of the function on the chosen interval.

1 Find the maximum value of Z if $Z = xw$, where $x + w = 30$.

2 Find the maximum value of B if $B = st$, where $4s + 3t = 48$.

3 Find the minimum value of A if $A = 4y + x^2$, where $(x^2 + 1)y = 324$.

4 Find the maximum value of S if $S = 8x - 512y^2$, where $x(y^2 + 1) = 64$.

5 Find the minimum value of P if $P = x^2 + y^2$, where $x - y = 40$.

6 Find the minimum value of C if $C = \sqrt{x^2 + y^2}$, where $xy = 9$.

7 If a box with a square base and an open top is to have a volume of 4 ft^3, find the dimensions that require the least material. (Disregard the thickness of the material and waste in construction.)

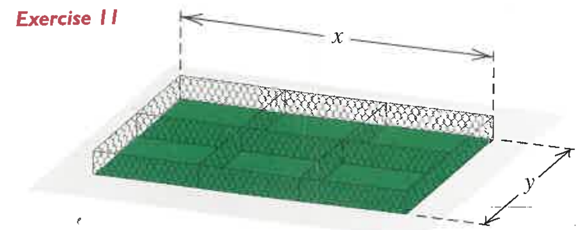
8 Work Exercise 7 if the box has a closed top.

9 A metal cylindrical container with an open top is to hold 1 ft^3. If there is no waste in construction, find the dimensions that require the least amount of material. (Compare with Example 3.)

10 If the circular base of the container in Exercise 9 is cut from a square sheet and the remaining metal is discarded, find the dimensions that require the least amount of material.

11 One thousand feet of chain link fence will be used to construct six cages for a zoo exhibit, as shown in the figure on the following page. Find the dimensions that maximize the enclosed area A. (*Hint:* First express y as a function of x, and then express A as a function of x.)

Exercise 11



12 Refer to Example 6. If the person is in a motorboat that can travel at an average rate of 15 mi/hr, what route should be taken to arrive at the house in the least amount of time?

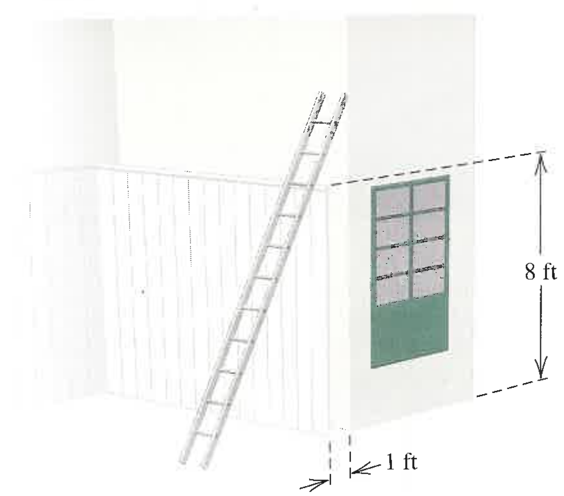
13 At 1:00 P.M., ship A is 30 mi due south of ship B and is sailing north at a rate of 15 mi/hr. If ship B is sailing west at a rate of 10 mi/hr, find the time at which the distance d between the ships is minimal (see figure).

Exercise 13

14 A window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 15 ft, find the dimensions that will allow the maximum amount of light to enter.

15 A fence 8 ft tall stands on level ground and runs parallel to a tall building (see figure). If the fence is 1 ft from the building, find the length of the shortest ladder that will extend from the ground over the fence to the wall of the building. (*Hint*: Use similar triangles.)

Exercise 15



16 A page of a book is to have an area of 90 in^2, with 1-in. margins at the bottom and sides and a $\frac{1}{2}$-in. margin at the top. Find the dimensions of the page that will allow the largest printed area.

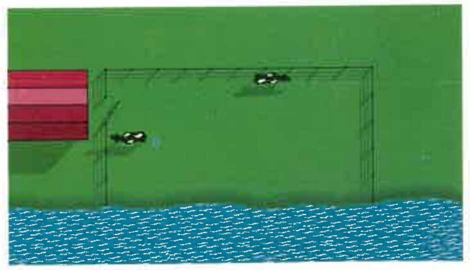
17 A builder intends to construct a storage shed having a volume of 900 ft^3, a flat roof, and a rectangular base whose width is three-fourths the length. The cost per square foot of the materials is \$4 for the floor, \$6 for the sides, and \$3 for the roof. What dimensions will minimize the cost?

18 A water cup in the shape of a right circular cone is to be constructed by removing a circular sector from a circular sheet of paper of radius a and then joining the two straight edges of the remaining paper (see figure). Find the volume of the largest cup that can be constructed.

Exercise 18

19 A farmer has 500 ft of fencing with which to enclose a rectangular field. A straight riverbank will be used as part of the fencing on one side of the field (see figure). Prove that the area of the field is greatest when the rectangle is square.

Exercise 19



20 Refer to Exercise 19. Suppose the farmer wants the area of the rectangular field to be A ft^2. Prove that the least amount of fencing is required when the rectangle is a square.

21 A hotel that charges \$80 per day for a room gives special rates to organizations that reserve between 30 and 60 rooms. If more than 30 rooms are reserved, the charge per room is decreased by \$1 times the number of rooms over 30. Under these conditions, how many rooms must be rented if the hotel is to receive the maximum income per day?

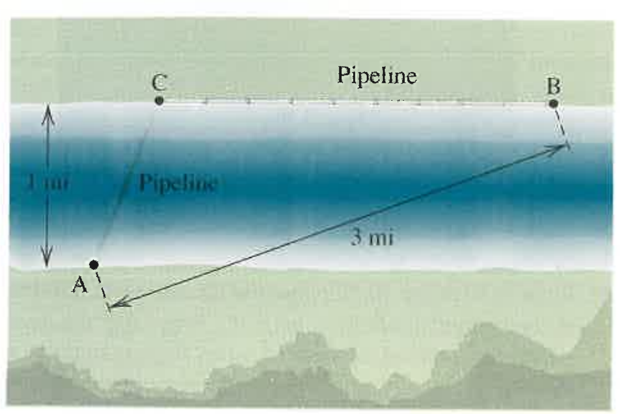
22 Refer to Exercise 21. Suppose that for each room rented it costs the hotel \$6 per day for cleaning and maintenance. In this case, how many rooms must be rented to obtain the greatest net income?

23 A steel storage tank for propane gas is to be constructed in the shape of a right circular cylinder with a hemisphere at each end (see figure). The construction cost per square foot for the end pieces is twice that for the cylindrical piece. If the desired capacity is 10π ft^3, what dimensions will minimize the cost of construction?

Exercise 23

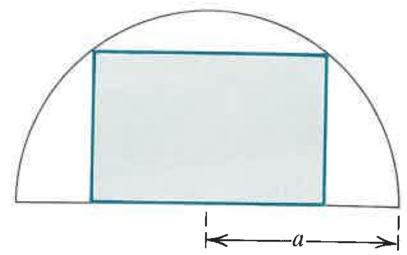
24 A pipeline for transporting oil will connect two points A and B that are 3 mi apart and on opposite banks of a straight river 1 mi wide (see figure). Part of the pipeline will run under water from A to a point C on the opposite bank, and then above ground from C to B. If the cost per mile of running the pipeline under water is four times the cost per mile of running it above ground, find the location of C that will minimize the cost (disregard the slope of the river bed).

Exercise 24



25 Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius a, if two vertices lie on the diameter (see figure).

Exercise 25



26 Find the dimensions of the rectangle of maximum area that can be inscribed in an equilateral triangle of side a, if two vertices of the rectangle lie on one of the sides of the triangle.

27 Of all possible right circular cones that can be inscribed in a sphere of radius a, find the volume of the one that has maximum volume.

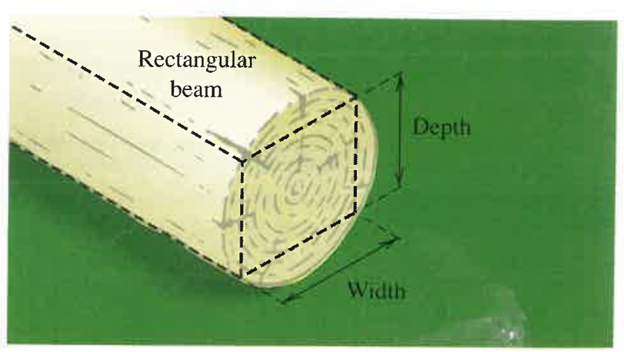
28 Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius a.

29 Find the point on the graph of $y = x^2 + 1$ that is closest to the point (3, 1).

30 Find the point on the graph of $y = x^3$ that is closest to the point (4, 0).

31 The strength of a rectangular beam is directly proportional to the product of its width and the square of the depth of a cross section. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius a (see figure).

Exercise 31



32 The illumination from a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources of strengths S_1 and S_2 are d units apart, at what point on the line segment joining the two sources is the illumination minimal?

33 A wholesaler sells running shoes at \$20 per pair if fewer than 50 pairs are ordered. If 50 or more pairs are ordered (up to 600), the price per pair is reduced by 2 cents times the number ordered. What size order will produce the maximum amount of money for the wholesaler?

34 A paper cup is to be constructed in the shape of a right circular cone. If the volume desired is 36π in^3, find the dimensions that require the least amount of paper. (Disregard any waste that may occur in the construction.)

35 A wire 36 cm long is to be cut into two pieces. One of the pieces will be bent into the shape of an equilateral triangle and the other into the shape of a rectangle whose length is twice its width. Where should the wire be cut if the combined area of the triangle and rectangle is to be **(a)** minimized? **(b)** maximized?

36 An isosceles triangle has base b and equal sides of length a. Find the dimensions of the rectangle of maximum area that can be inscribed in the triangle if one side of the rectangle lies on the base of the triangle.

37 A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 ft, find the dimensions of the rectangle that will produce the largest area for the window.

38 Two vertical poles of lengths 6 ft and 8 ft stand on level ground, with their bases 10 ft apart. Approximate the minimal length of cable that can reach from the top of one pole to some point on the ground between the poles and then to the top of the other pole.

39 Prove that the rectangle of largest area having a given perimeter p is a square.

40 A right circular cylinder is generated by rotating a rectangle of perimeter p about one of its sides. What dimensions of the rectangle will generate the cylinder of maximum volume?

41 The owner of an apple orchard estimates that if 24 trees are planted per acre, then each mature tree will yield 600 apples per year. For each additional tree planted per acre, the number of apples produced by each tree decreases by 12 per year. How many trees should be planted per acre to obtain the most apples per year?

42 A real estate company owns 180 efficiency apartments, which are fully occupied when the rent is \$300 per month. The company estimates that for each \$10 increase in rent, 5 apartments will become unoccupied. What rent should be charged in order to obtain the largest gross income?

43 A package can be sent by parcel post only if the sum of its length and girth (the perimeter of the base) is not more than 108 in. Find the dimensions of the box of maximum volume that can be sent, if the base of the box is a square.

44 A north–south highway A and an east–west highway B intersect at a point P. At 10:00 A.M., an automobile crosses P traveling north on highway A at a speed of 50 mi/hr. At that same instant, an airplane flying east at a speed of 200 mi/hr and an altitude of 26,400 ft is directly above the point on highway B that is 100 mi west of P. If the automobile and the airplane maintain the same speeds and directions, at what time will they be closest to each other?

45 Two factories A and B that are 4 mi apart emit particles in smoke that pollute the area between the factories. Suppose that the number of particles emitted from each factory is directly proportional to the amount of smoke and inversely proportional to the cube of the distance from the other factory. If factory A emits twice as much smoke as factory B, at what point between A and B is the pollution minimal?

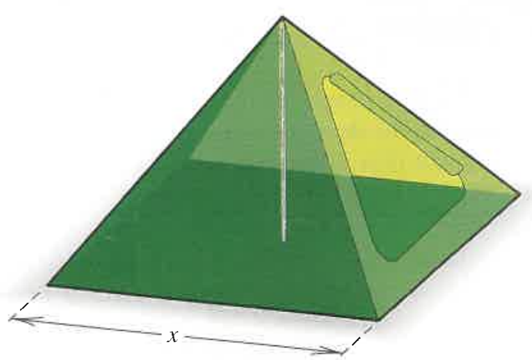
46 An oil field contains 8 wells, which produce a total of 1600 barrels of oil per day. For each additional well that is drilled, the average production per well decreases by 10 barrels per day. How many additional wells should be drilled to obtain the maximum amount of oil per day?

47 A canvas tent is to be constructed in the shape of a pyramid with a square base. A steel pole, placed in the center of the tent, will form the support (see figure). If S ft^2 of canvas is available for the four sides and x is the length of the base, show that

(a) the volume V of the tent is $V = \frac{1}{6}x\sqrt{S^2 - x^4}$

(b) V has its maximum value when x equals $\sqrt{2}$ times the length of the pole

Exercise 47



48 A boat must travel 100 mi upstream against a 10-mi/hr current. When the velocity of the boat relative to the water is v mi/hr, the number of gallons of gasoline consumed each hour is directly proportional to v^2.

(a) If a constant velocity of v mi/hr is maintained, show that the total number y of gallons of gasoline consumed is given by $y = 100kv^2/(v - 10)$ for $v > 10$ and for some positive constant k.

(b) Find the speed that minimizes the number of gallons of gasoline consumed during the trip.

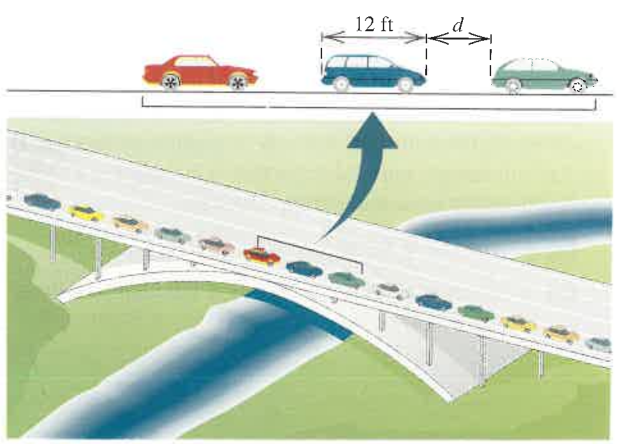
49 Cars are crossing a bridge that is 1 mi long. Each car is 12 ft long and is required to stay a distance of at least d ft from the car in front of it (see figure).

(a) Show that the greatest number of cars that can be on the bridge at one time is $[\![5280/(12 + d)]\!]$, where $[\![\,]\!]$ denotes the greatest integer function.

(b) If the velocity of each car is v mi/hr, show that the maximum traffic flow rate F (in cars per hour) is given by $F = [\![5280v/(12 + d)]\!]$.

(c) The stopping distance (in feet) of a car traveling v mi/hr is approximately $0.05v^2$. If $d = 0.025v^2$, find the speed that maximizes the traffic flow across the bridge.

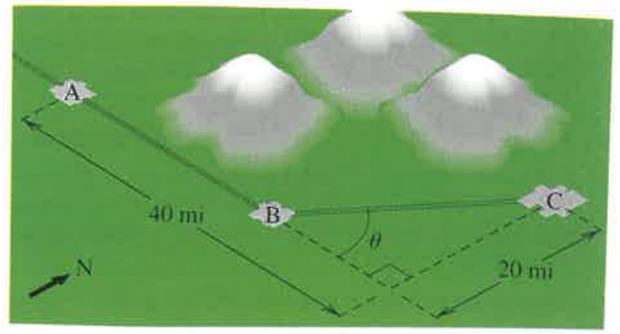
Exercise 49



50 Prove that the shortest distance from a point (x_1, y_1) to the graph of a differentiable function f is measured along a normal line to the graph—that is, a line perpendicular to the tangent line.

51 A railroad route is to be constructed from town A to town C, branching out from a point B toward C at an angle of θ degrees (see figure). Because of the mountains between A and C, the branching point B must be at least 20 mi east of A. If the construction costs are \$50,000 per mile between A and B and \$100,000 per mile between B and C, find the branching angle θ that minimizes the total construction cost.

Exercise 51



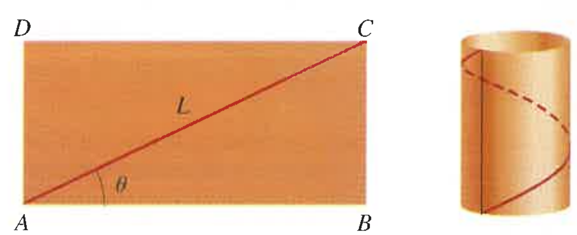
52 A long rectangular sheet of metal, 12 in. wide, is to be made into a rain gutter by turning up two sides at angles of 120° to the sheet. How many inches should be turned up to give the gutter its greatest capacity?

53 Refer to Exercise 18. Find the central angle of the sector that will maximize the volume of the cup.

54 A square picture having sides 2 ft long is hung on a wall such that the base is 6 ft above the floor. If a person whose eye level is 5 ft above the floor looks at the picture and if θ is the angle between the line of sight and the top and bottom of the picture, find the person's distance from the wall at which θ has its maximum value.

55 A rectangle made of elastic material will be made into a cylinder by joining edges AD and BC (see figure). To support the structure, a wire of fixed length L is placed along the diagonal of the rectangle. Find the angle θ that will result in the cylinder of maximum volume.

Exercise 55



56 When a person is walking, the magnitude F of the vertical force of one foot on the ground (see figure) can be approximated by $F = A(\cos bt - a\cos 3bt)$ for time t (in seconds), with $A > 0$, $b > 0$, and $0 < a < 1$.

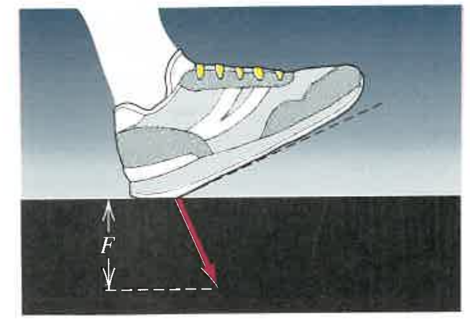
(a) Show that $F = 0$ when $t = -\pi/(2b)$ and $t = \pi/(2b)$. (The time $t = -\pi/(2b)$ corresponds to the instant when the foot first touches the ground and the weight of the body is being supported by the other foot.)

(b) Show that the maximum force occurs when $t = 0$ or when $\sin^2 bt = (9a - 1)/(12a)$.

(c) If $a = \frac{1}{3}$, express the maximum force in terms of A.

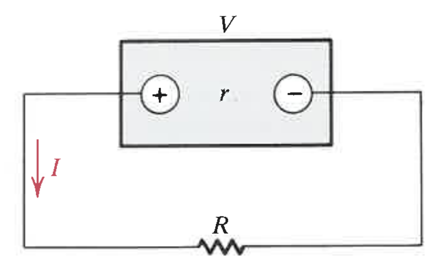
(d) If $0 < a \le \frac{1}{9}$, express the maximum force in terms of A.

Exercise 56



57 A battery having fixed voltage V and fixed internal resistance r is connected to a circuit that has variable resistance R (see figure). By Ohm's law, the current I in the circuit is $I = V/(R + r)$. If the power output P is given by $P = I^2R$, show that the maximum power occurs if $R = r$.

Exercise 57



58 The power output P of an automobile battery is given by $P = VI - I^2r$ for voltage V, current I, and internal resistance r of the battery. What current corresponds to the maximum power?

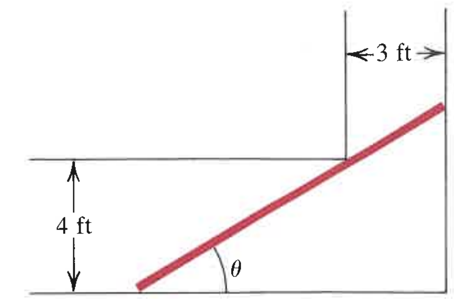
59 Two corridors 3 ft and 4 ft wide, respectively, meet at a right angle. Find the length of the longest nonbendable rod that can be carried horizontally around the corner, as shown in the figure. (Disregard the thickness of the rod.)

60 Light travels from one point to another along the path that requires the least amount of time. Suppose that light has velocity v_1 in air and v_2 in water, where $v_1 > v_2$. If light travels from a point P in air to a point Q in water (see figure), show that the path requires the least amount of time if

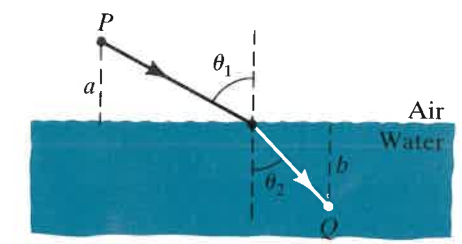
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

(This is an example of *Snell's law of refraction.*)

Exercise 59



Exercise 60



61 A circular cylinder of fixed radius R is surmounted by a cone (see figure). The ends of the cylinder are open, and the total volume is to be a specified constant V.

(a) Show that the total surface area S is given by

$$S = \frac{2V}{R} + \pi R^2\left(\csc\theta - \frac{2}{3}\cot\theta\right).$$

(b) Show that S is minimized when $\theta \approx 48.2°$.

Exercise 61

62 In the classic honeycomb-structure problem, a hexagonal prism of fixed radius (and side) R is surmounted by adding three identical rhombuses that meet in a common vertex (see figure). The bottom of the prism is open, and the total volume is to be a specified constant V. A more elaborate geometric argument than that in Exercise 61 establishes that the total surface area S is given by

$$S = \frac{4}{3}\sqrt{3}\frac{V}{R} - \frac{3}{2}R^2\cot\theta + \frac{3\sqrt{3}}{2}R^2\csc\theta.$$

Show that S is minimized when $\theta \approx 54.7°$. (Remarkably, bees construct their honeycombs so that the amount of wax S is minimized.)

Exercise 62

3.7 VELOCITY AND ACCELERATION

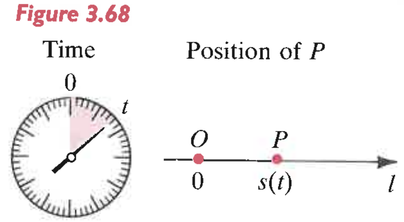
In this section, we use derivatives to describe and analyze several important types of motion that occur in physical situations. One of the greatest early achievements in the history of calculus was Newton's derivation in the seventeenth century of Kepler's laws of planetary motions (see Section 11.6 for more details). In the succeeding three hundred years, calculus has repeatedly helped scientists study moving objects. Our focus will be objects moving along a straight line.

RECTILINEAR MOTION

As we saw in Section 2.1, the term *rectilinear motion* is used to describe movement of a point along a line. In the mathematical model of rectilinear motion in this section, we will represent the car as a point P and the highway as a straight line l. If l is a vertical or horizontal coordinate line and if the coordinate of point P at time t is $s(t)$, then s is considered the *position function* of P (see Figure 3.68). Recall from Definition (2.3) that the *velocity* of P at time t, the rate of change of P with respect to t, is the derivative $s'(t)$. The velocity is also denoted as $v(t)$.

The *acceleration* $a(t)$ of P at time t is defined as the rate of change of velocity with respect to time: $a(t) = v'(t)$. Thus the acceleration is the second derivative $(d/dt)(s'(t)) = s''(t)$. The next definition summarizes this discussion and also introduces the notion of the *speed* of P.

Figure 3.68



Definition 3.23

Let $s(t)$ be the coordinate of a point P on a coordinate line l at time t.

(i) The **velocity** of P is $v(t) = s'(t)$.

(ii) The **speed** of P is $|v(t)|$.

(iii) The **acceleration** of P is $a(t) = v'(t) = s''(t)$.

We shall call v the **velocity function** of P and a the **acceleration function** of P. We sometimes use the notation

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt}.$$

If t is in seconds and $s(t)$ is in centimeters, then $v(t)$ is in cm/sec and $a(t)$ is in cm/sec^2 (centimeters per second per second). If t is in hours and $s(t)$ is in miles, then $v(t)$ is in mi/hr and $a(t)$ is in mi/hr^2 (miles per hour per hour).

If $v(t)$ is positive in a time interval, then $s'(t) > 0$, and, by Theorem (3.15), $s(t)$ is increasing—that is, the point P is moving in the positive direction on l. If $v(t)$ is negative, the motion is in the negative direction. The velocity is zero at a point where P changes direction. If the acceleration $a(t) = v'(t)$ is positive, the velocity is increasing. If $a(t)$ is negative, the velocity is decreasing.

NOTE We make a distinction between the *velocity* $v(t)$ and the *speed* $|v(t)|$ of a moving object. The speed conveys only how fast the object is moving; it contains no information about the direction of motion. We will make use of the speed in Chapter 5 in determining the total distance that an object moves. The velocity conveys not only the speed of motion but also whether the object is moving in a positive or a negative direction along the coordinate line.

EXAMPLE ■ 1 The position function s of a point P on a coordinate line is given by

$$s(t) = t^3 - 12t^2 + 36t - 20,$$

with t in seconds and $s(t)$ in centimeters. Describe the motion of P during the time interval $[-1, 9]$.

SOLUTION Differentiating, we obtain

$$v(t) = s'(t) = 3t^2 - 24t + 36 = 3(t-2)(t-6),$$
$$a(t) = v'(t) = 6t - 24 = 6(t-4).$$

Let us determine when $v(t) > 0$ and when $v(t) < 0$, since this will tell us when P is moving to the right and to the left, respectively. Since $v(t) = 0$ at $t = 2$ and $t = 6$, we examine the following time subintervals of $[-1, 9]$:

$$(-1, 2), \quad (2, 6), \quad \text{and} \quad (6, 9)$$

We may determine the sign of $v(t)$ by using test values, as indicated in the table (check each entry):

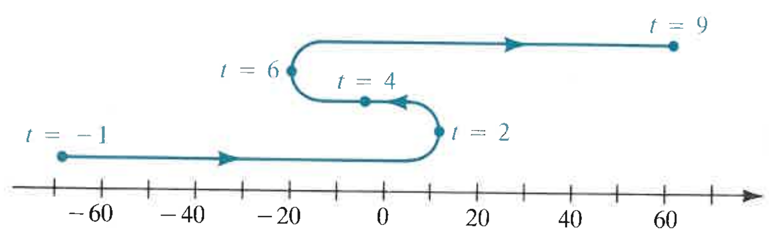
Time interval	$(-1, 2)$	$(2, 6)$	$(6, 9)$
k	0	3	7
Test value $v(k)$	36	-9	15
Sign of $v(t)$	+	−	+
Direction of motion	right	left	right

The next table lists the values of the position, velocity, and acceleration functions at the endpoints of the time interval $[-1, 9]$ and the times at which the velocity or acceleration is zero.

t	-1	2	4	6	9
$s(t)$	-69	12	-4	-20	61
$v(t)$	63	0	-12	0	63
$a(t)$	-30	-12	0	12	30

It is convenient to represent the motion of P schematically, as in Figure 3.69. *The curve above the coordinate line is not the path of the point*, but rather a scheme for showing the manner in which P moves on the line l.

Figure 3.69



As indicated by the tables and Figure 3.69, at $t = -1$ the point is 69 cm to the left of the origin and is moving to the right with a velocity of 63 cm/sec. The negative acceleration -30 cm/sec^2 indicates that the velocity is decreasing at a rate of 30 cm/sec, each second. The point continues to move to the right, slowing down until it has zero velocity at $t = 2$, 12 cm to the right of the origin. The point P then reverses direction and moves until, at $t = 6$, it is 20 cm to the left of the origin. It then again reverses direction and moves to the right for the remainder of the time interval, with increasing velocity. The direction of motion is indicated by the arrows on the curve in Figure 3.69.

Since the point began at position -69 and ended at position 61, the net change in its position is $61 - (-69) = 130$ units. The total distance traveled by the point, however, is more. In the time interval $[-1, 2]$, it moved $12 - (-69) = 81$ units. In the time interval $[2, 6]$, it moved a distance of 32 units, and in the interval $[6, 9]$, it moved an additional 81 units. The total distance traveled was $81 + 32 + 81 = 194$ units.

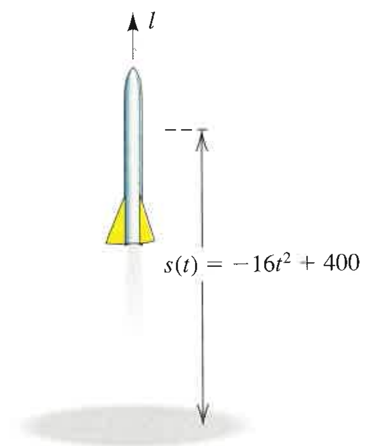
EXAMPLE ■ 2 A projectile is fired straight upward with a velocity of 400 ft/sec. From physics, its distance above the ground after t seconds is $s(t) = -16t^2 + 400t$.

(a) Find the time and the velocity at which the projectile hits the ground.

(b) Find the maximum altitude achieved by the projectile.

(c) Find the acceleration at any time t.

Figure 3.70



SOLUTION

(a) Let us represent the path of the projectile on a vertical coordinate line l with origin at ground level and positive direction upward, as illustrated in Figure 3.70. The projectile is on the ground when $-16t^2 + 400t = 0$—that is, when $-16t(t - 25) = 0$. This gives us $t = 0$ and $t = 25$. Hence the projectile hits the ground after 25 sec. The velocity at time t is

$$v(t) = s'(t) = -32t + 400.$$

In particular, at $t = 25$, we obtain the *impact velocity*:

$$v(25) = -32(25) + 400 = -400 \text{ ft/sec}.$$

The negative velocity indicates that the projectile is moving in the negative direction on l (downward) at the instant that it strikes the ground. Note that the *speed* at this time is

$$|v(25)| = |-400| = 400 \text{ ft/sec}.$$

(b) The maximum altitude occurs when the velocity is zero—that is, when $s'(t) = -32t + 400 = 0$. Solving for t gives us $t = \frac{400}{32} = \frac{25}{2}$, and hence the maximum altitude is

$$s(\tfrac{25}{2}) = -16(\tfrac{25}{2})^2 + 400(\tfrac{25}{2}) = 2500 \text{ ft}.$$

(c) The acceleration at any time t is

$$a(t) = v'(t) = -32 \text{ ft/sec}^2.$$

This constant acceleration is caused by the force of gravity.

SIMPLE HARMONIC MOTION

Simple harmonic motion takes place in waves. It involves trigonometric functions and is defined as follows.

Definition 3.24

A point P moving on a coordinate line l is in **simple harmonic motion** if its distance $s(t)$ from the origin at time t is given by either

$$s(t) = k\sin(\omega t + b) \quad \text{or} \quad s(t) = k\cos(\omega t + b),$$

where k, ω, and b are constants, with $\omega > 0$.

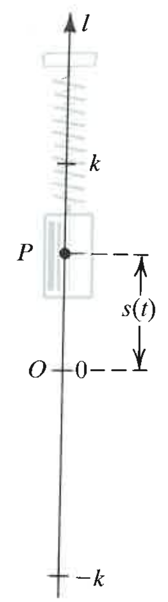
Simple harmonic motion may also be defined by requiring that the acceleration $a(t)$ satisfy the condition

$$a(t) = -\omega^2 s(t)$$

for every t. It can be shown that this condition is equivalent to Definition (3.24).

In simple harmonic motion, the point P oscillates between the points on l with coordinates $-k$ and k. The **amplitude** of the motion is the

Figure 3.71



maximum displacement $|k|$ of the point from the origin. The **period** is the time $2\pi/\omega$ required for one complete oscillation. The **frequency** $\omega/2\pi$ is the number of oscillations per unit of time.

Simple harmonic motion takes place in many different types of waves, such as water waves, sound waves, radio waves, light waves, and distortional waves present in vibrating bodies. Functions of the type defined in (3.24) also occur in the analysis of electrical circuits that contain an alternating electromotive force and current.

As another example of simple harmonic motion, consider a spring with an attached weight that is oscillating vertically relative to a coordinate line, as illustrated in Figure 3.71. The number $s(t)$ represents the coordinate of a fixed point P in the weight, and we assume that the amplitude $|k|$ of the motion is constant. In this case, there is no frictional force retarding the motion. If friction is present, then the amplitude decreases with time, and the motion is *damped*.

EXAMPLE 3 Suppose that the weight shown in Figure 3.71 is oscillating and

$$s(t) = 10\cos\frac{\pi}{6}t,$$

where t is in seconds and $s(t)$ is in centimeters. Discuss the motion of the weight.

SOLUTION Comparing the given equation with the general form $s(t) = k\cos(\omega t + b)$ in Definition (3.24), we obtain $k = 10$, $\omega = \pi/6$, and $b = 0$, which gives us the following:

$$\text{amplitude:} \quad k = 10 \text{ cm}$$

$$\text{period:} \quad \frac{2\pi}{\omega} = \frac{2\pi}{\pi/6} = 12 \text{ sec}$$

$$\text{frequency:} \quad \frac{\omega}{2\pi} = \frac{1}{12} \text{ oscillation/sec}$$

Let us examine the motion during the time interval [0, 12]. The velocity and acceleration functions are given by the following:

$$v(t) = s'(t) = 10\left(-\sin\frac{\pi}{6}t\right)\cdot\frac{\pi}{6} = -\frac{5\pi}{3}\sin\frac{\pi}{6}t,$$

$$a(t) = v'(t) = -\frac{5\pi}{3}\left(\cos\frac{\pi}{6}t\right)\cdot\frac{\pi}{6} = -\frac{5\pi^2}{18}\cos\frac{\pi}{6}t.$$

The velocity is 0 at $t = 0$, $t = 6$, and $t = 12$, since $\sin[(\pi/6)t] = 0$ for these values of t. The acceleration is 0 at $t = 3$ and $t = 9$, since in these cases, $\cos[(\pi/6)t] = 0$. The times at which the velocity and acceleration are 0 lead us to examine the time intervals (0, 3), (3, 6), (6, 9), and (9, 12). The following table displays the main characteristics of the motion. The signs of $v(t)$ and $a(t)$ in the intervals can be determined using test values (verify each entry).

Time interval	Sign of $v(t)$	Direction of motion	Sign of $a(t)$	Variation of $v(t)$	Speed $\lvert v(t)\rvert$
(0, 3)	−	downward	−	decreasing	increasing
(3, 6)	−	downward	+	increasing	decreasing
(6, 9)	+	upward	+	increasing	increasing
(9, 12)	+	upward	−	decreasing	decreasing

Note that if $0 < t < 3$, the velocity $v(t)$ is negative and decreasing; that is, $v(t)$ becomes *more* negative. Hence the absolute value $|v(t)|$, the speed, is *increasing*. If $3 < t < 6$, the velocity is negative and increasing ($v(t)$ becomes *less* negative); that is, the speed of P is *decreasing* in the time interval (3, 6). Similar remarks can be made for the intervals (6, 9) and (9, 12).

We may summarize the motion of P as follows: At $t = 0$, $s(0) = 10$ and the point P is 10 cm above the origin O. It then moves downward, gaining speed until it reaches the origin O at $t = 3$. It then slows down until it reaches a point 10 cm below O at the end of 6 sec. The direction of motion is then reversed, and the weight moves upward, gaining speed until it reaches O at $t = 9$, after which it slows down until it returns to its original position at the end of 12 sec. The direction of motion is then reversed again, and the same pattern is repeated indefinitely.

FREE FALL

According to Newton's second law of motion, the product of the mass and acceleration of an object is equal to the sum of the forces acting on it. Using this as a first model for the fall of an object toward the surface of a planet from a starting position not far from the surface, we ignore all forces except the gravitational attraction of the planet, which we take to be constant.

With this model, Newton's second law becomes

$$m\,a(t) = mg,$$

where m is the mass of the object and g is the gravitational constant. This equation simplifies to

$$a(t) = g.$$

Figure 3.72

If we set up our coordinate line as in Figure 3.72 with 0 at the surface of the planet and the positive side above the surface, then the force of gravity is in the negative direction; thus g will have a negative value. A useful convention is to assign time t to be 0 at the instant the object begins to move.

Since $a(t) = v'(t)$ is the constant function whose value is g, the velocity function $v(t)$ must be of the form

$$v(t) = gt + C$$

for some constant C. If we evaluate this equation at the initial time $t = 0$, we have

$$v(0) = g \cdot 0 + C = C,$$

so we see that C is the *initial velocity* $v(0)$, which we will denote as v_0.

Now, using the relationship that velocity is the derivative of the position function, we have

$$v(t) = s'(t) = gt + v_0.$$

One possible candidate for the position function is an expression of the form $(g/2)t^2 + v_0 t$, since this expression has derivative $gt + v_0$. By Corollary (3.14), any other function with the same derivative, $gt + v_0$, must differ from $(g/2)t^2 + v_0 t$ by a constant. Hence, the position function must have the form

$$s(t) = \frac{g}{2}(t^2) + v_0 t + C$$

for some constant C. Evaluation again at the initial time $t = 0$ gives $s(0) = 0 + 0 + C$, so the constant C is just the *initial position* $s(0)$, which we will denote by s_0.

Putting these results together, we find that the rectilinear motion of an object falling near the surface of a planet is given by the position function

$$s(t) = \frac{gt^2}{2} + v_0 t + s_0.$$

For an object near the earth's surface, the value of g is approximately -32 ft/sec^2, or -9.8 m/sec^2.

EXAMPLE ■ 4 A student accidentally drops her calculus book from an upper-story window of her dormitory 144 ft above the ground. How fast is the book moving when it strikes the ground?

SOLUTION Taking the instant the book is dropped to be $t = 0$, we have an initial position $s_0 = 144$ and an initial velocity $v_0 = 0$ (since the book is dropped rather than thrown downward or hurled upward). Since the distance is measured in feet, we use $g = -32$. The equations for position and velocity thus take the form

$$s(t) = -16t^2 + 144 \quad \text{and} \quad v(t) = -32t.$$

The book hits the ground when $s(t) = 0$, so the value of t at the instant of impact satisfies

$$-16t^2 + 144 = 0$$

or

$$t^2 = \tfrac{144}{16},$$

which yields

$$t = \pm 3.$$

We can ignore the negative value because all the action takes place after $t = 0$. We conclude that the book strikes the ground in 3 sec. At that moment, it is traveling with a velocity of $v(3) = -32(3) = -96$ ft/sec.

EXAMPLE ■ 5 To practice his skill catching fly balls, a baseball player hurls the ball straight up in the air, releasing it from his outstretched arm at a point 8 ft above the ground. How high does the ball go if he can impart an initial velocity of 60 mi/hr?

SOLUTION Since 60 mi/hr translates into 88 ft/sec, we have $v_0 = 88$ and $s_0 = 8$. The position and velocity functions become

$$s(t) = -16t^2 + 88t + 8 \quad \text{and} \quad v(t) = -32t + 88.$$

The ball reaches its maximum height when the velocity is 0, which occurs when

$$-32t + 88 = 0; \quad \text{that is, } t = \tfrac{11}{4} \text{ sec.}$$

The height of the ball at this time is

$$s(\tfrac{11}{4}) = -16(\tfrac{11}{4})^2 + 88(\tfrac{11}{4}) + 8 = 129 \text{ ft.}$$

The model we have discussed in this section, one of constant acceleration, is a relatively simple one in that it ignores other forces that are often quite important. Air resistance, for example, often plays a crucial role, especially for a relatively light object with a relatively large size. The model also treats the gravitational force as constant; a more realistic one would take into account that the force due to gravity varies with the distance between the object and the center of attracting planetary mass. Still, this simple model gives reasonably good predictions. You may wish to explore some generalizations of our model.

CONSTANT ACCELERATION

In many situations, the motion of an object is governed by constant acceleration or deceleration, as the next examples illustrate.

EXAMPLE ■ 6 An automobile manufacturer claims that its new model can accelerate "from 0 to 60 miles per hour in 11 seconds." Find the constant acceleration that makes this rate of speed possible.

SOLUTION Let the unknown constant acceleration be a ft/sec^2. At time $t = 0$, the velocity v_0 is 0, so the velocity at time t is given by

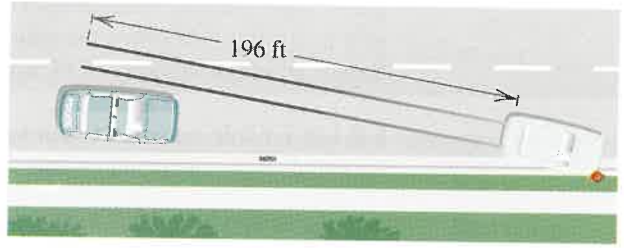
$$v(t) = gt + v_0 = at.$$

We are given that $v(11) = 88$ ft/sec. Hence,

$$88 = 11a, \quad \text{or, equivalently,} \quad a = 8 \text{ ft/sec}^2.$$

EXAMPLE ■ 7 Suppose that the automobile of Example 6 is involved in an accident on a quiet residential street. Police officers investigating the incident conclude from physical evidence and reports of witnesses that the car traveled 196 ft after the driver slammed on the brakes

Figure 3.73



before it stopped (see Figure 3.73). Assuming that the car decelerated at a rate of 8 ft/sec^2, determine how fast the car was going when the brakes were applied.

SOLUTION Let $t = 0$ be the moment when the brakes are applied. When the car comes to rest at unknown time t^*, its velocity $v(t^*) = 0$ and its position $s(t^*) = 196$. We wish to find v_0.

If we use $g = -8$, the velocity and position functions become

$$v(t) = -8t + v_0 \quad \text{and} \quad s(t) = -4t^2 + v_0 t.$$

Evaluating these functions at time t^*, we find

$$0 = -8t^* + v_0 \quad \text{and} \quad 196 = -4(t^*)^2 + v_0 t^*.$$

From the first equation, we have the relationship $t^* = v_0/8$, which we substitute into the second equation to obtain

$$196 = -4\left(\frac{v_0}{8}\right)^2 + v_0\left(\frac{v_0}{8}\right).$$

Simplifying gives

$$\frac{v_0^2}{16} = 196, \quad \text{so} \quad v_0 = 56 \text{ ft/sec}.$$

Converting to miles per hour, we find that the car was traveling at a speed slightly above 38 mi/hr.

EXERCISES 3.7

Exer. 1–8: A point moving on a coordinate line has position function s. Find the velocity and the acceleration at time t, and describe the motion of the point during the indicated time interval. Illustrate the motion by means of a diagram of the type shown in Figure 3.69.

1 $s(t) = 3t^2 - 12t + 1$; $[0, 5]$

2 $s(t) = t^2 + 3t - 6$; $[-2, 2]$

3 $s(t) = t^3 - 9t + 1$; $[-3, 3]$

4 $s(t) = 24 + 6t - t^3$; $[-2, 3]$

5 $s(t) = -2t^3 + 15t^2 - 24t - 6$; $[0, 5]$

6 $s(t) = 2t^3 - 12t^2 + 6$; $[-1, 6]$

7 $s(t) = 2t^4 - 6t^2$; $[-2, 2]$

8 $s(t) = 2t^3 - 6t^5$; $[-1, 1]$

Exer. 9–10: An automobile rolls down an incline, traveling $s(t)$ feet in t seconds. (a) Find its velocity at $t = 3$. (b) After how many seconds will the velocity be k ft/sec?

9 $s(t) = 5t^2 + 2$; $k = 28$

10 $s(t) = 3t^2 + 7$; $k = 88$

Exer. 11–12: A projectile is fired directly upward with an initial velocity of v_0 ft/sec, and its height (in feet) above the ground after t seconds is given by $s(t)$. Find (a) the velocity and acceleration after t seconds, (b) the maximum height, and (c) the duration of the flight.

11 $v_0 = 144$; $s(t) = 144t - 16t^2$

12 $v_0 = 192$; $s(t) = 100 + 192t - 16t^2$

Exer. 13–16: A particle in simple harmonic motion has position function s, and t is the time in seconds. Find the amplitude, the period, and the frequency.

13 $s(t) = 5\cos\frac{\pi}{4}t$

14 $s(t) = 4\sin\pi t$

15 $s(t) = 6\sin\frac{2\pi}{3}t$

16 $s(t) = 3\cos 2t$

17 The electromotive force V and current I in an alternating-current circuit are given by

$$V = 220\sin 360\pi t,$$
$$I = 20\sin\left(360\pi t - \frac{\pi}{4}\right).$$

Find the rates of change of V and I with respect to time at $t = 1$.

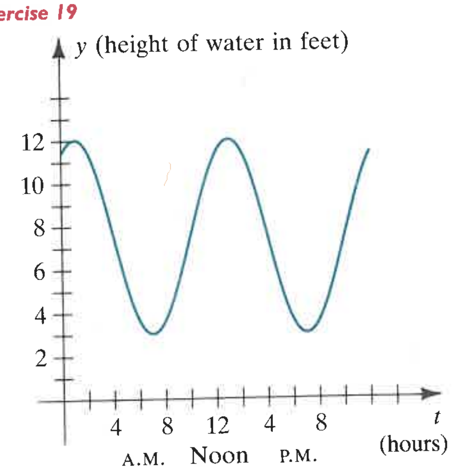
18 The annual variation in temperature T (in °C) in Vancouver, B.C., may be approximated by the formula

$$T = 14.8\sin\left[\frac{\pi}{6}(t - 3)\right] + 10,$$

where t is in months, with $t = 0$ corresponding to January 1. Approximate the rate at which the temperature is changing at time $t = 3$ (April 1) and at time $t = 10$ (November 1). At what time of the year is the temperature changing most rapidly?

19 The graph in the figure shows the rise and fall of the water level in Boston Harbor during a particular 24-hr period.

Exercise 19



(a) Approximate the water level y by means of an expression of the form

$$y = a\sin(bt + c) + d,$$

with $t = 0$ corresponding to midnight.

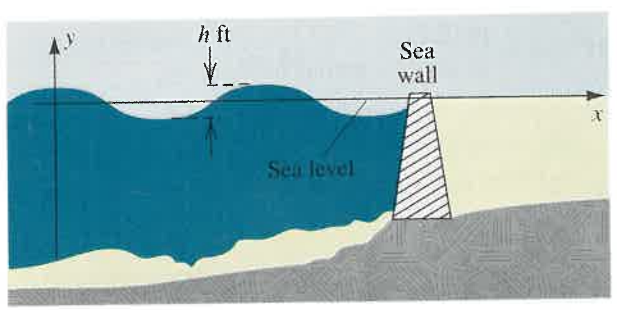
(b) Approximately how fast is the water level rising at 12 noon?

20 A *tsunami* is a tidal wave caused by an earthquake beneath the sea. These waves can be more than 100 ft in height and can travel at great speeds. Engineers sometimes represent tsunamis by an equation of the form $y = a\cos bt$. Suppose that a wave has a height $h = 25$ ft and period 30 min and is traveling at the rate of 180 ft/sec.

(a) Let (x, y) be a point on the wave represented in the figure. Express y as a function of t if $y = 25$ ft when $t = 0$.

(b) How fast is the wave rising (or falling) when $y = 10$ ft?

Exercise 20

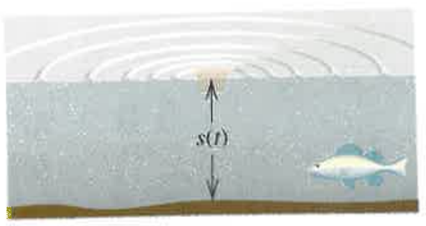


21 A cork bobs up and down in a lake. The distance from the bottom of the lake to the center of the cork at time $t \geq 0$ is given by $s(t) = \cos\pi t + 12$, where $s(t)$ is in inches and t is in seconds (see figure).

(a) Find the velocity of the cork at $t = 0, \frac{1}{2}, 1, \frac{3}{2}$, and 2.

(b) During what time intervals is the cork rising?

Exercise 21



22 A particle in a vibrating spring is moving vertically such that its distance $s(t)$ from a fixed point on the line of vibration is given by $s(t) = 4 + \frac{1}{25}\sin 100\pi t$, where $s(t)$ is in centimeters and t is in seconds.

(a) How long does it take the particle to make one complete vibration?

(b) Find the velocity of the particle at $t = 1, 1.005, 1.01$, and 1.015.

Exer. 23–24: Show that $s''(t) = -\omega^2 s(t)$.

23 $s(t) = k\cos(\omega t + b)$ **24** $s(t) = k\sin(\omega t + b)$

25 A point $P(x, y)$ is moving at a constant rate around the circle $x^2 + y^2 = a^2$. Prove that the projection $Q(x, 0)$ of P onto the x-axis is in simple harmonic motion.

26 If a point P moves on a coordinate line such that

$$s(t) = a\cos\omega t + b\sin\omega t,$$

show that P is in simple harmonic motion by

(a) using the remark following Definition (3.24)

(b) using only trigonometric methods (*Hint:* Show that $s(t) = A\cos(\omega t - c)$ for some constants A and c.)

c **Exer. 27–28: A point moving on a coordinate line has position $s(t)$. (a) Graph $y = s(t)$ for $0 \le t \le 5$. (b) Approximate the point's position, velocity, and acceleration at $t = 0, 1, 2, 3, 4$, and 5.**

27 $s(t) = \dfrac{10\sin t}{t^2 + 1}$ **28** $s(t) = \dfrac{5\tan(\frac{1}{4}t)}{2t + 1}$

29 Emergency food supplies are dropped from a helicopter and hit the ground 10 sec later.

(a) What is the height h of the helicopter?

Exercise 29

(b) The box in which the supplies are packed is strong enough to withstand a speed of 180 mi/hr on impact. Will the supplies be intact?

(c) What is the maximum height at which the helicopter can be positioned to guarantee that the box will not break up when it hits the ground?

30 A golf ball projected vertically from the ground lands back on the surface in 8 sec. What was the initial velocity?

31 Judy's roommate drops Judy's car keys from the dormitory window, which is 144 ft above the ground, to Judy, who is standing on the ground below the window.

(a) How long must Judy wait for the keys?

(b) If Judy needs the keys in 2 sec, with what initial speed should her roommate throw the keys?

32 If a stone is hurled vertically upward, show that it takes the same amount of time for the stone to achieve its maximum height as it takes to drop from that spot back to the ground.

33 A truck driver speeding down a narrow street suddenly sees a bicyclist s feet ahead. The driver slams on the brakes, imparting a constant deceleration a to the truck. If the bicycle is moving at a rate of v_1 mi/hr in the same direction as the truck, find the maximum value for the initial speed v_0 of the truck so that the vehicles will not collide.

34 In Vermont, a straight stretch of U.S. highway 7 connects Burlington to Vergennes. A car begins in Burlington at $t = 0$ and heads toward Vergennes with a velocity given by $v(t) = 60t - 12t^2$, measured in miles per hour. When the automobile arrives at the Vergennes city limit, it is clocked at a speed of 48 mi/hr and it is speeding up.

(a) How far apart are Burlington and Vergennes?

(b) What are the minimum and maximum velocities experienced during the trip? When do they occur?

(c) How would the answer be affected if the car were observed to be slowing down rather than speeding up when it reached Vergennes?

3.8 APPLICATIONS TO ECONOMICS, SOCIAL SCIENCES, AND LIFE SCIENCES

A primary focus of calculus is **change** in functional relationships. Since change is a characteristic property of most natural and social systems, calculus provides powerful techniques to understand these systems. In this section, we examine some applications of the derivative to the social and

life sciences. We concentrate first on some applications of the derivative in economics. We then examine how the calculus we have developed so far can help us make some qualitative conclusions about complex mathematical systems that we cannot quantitatively solve.

ECONOMICS

Calculus has become an important tool for solving problems that occur in economics because of its power to analyze functional relationships. Revenues and profits, for example, are functions of fluctuating costs and prices, and these in turn depend on varying supply and demand. Economists often face optimizing problems that involve making the most efficient use of scarce resources to achieve societal goals.

If x is the number of units of a commodity, economists often use the functions C, c, R, and P, defined as follows:

Cost function:	$C(x) =$ cost of producing x units
Average cost function:	$c(x) = C(x)/x$ $=$ average cost of producing one unit
Revenue function:	$R(x) =$ revenue received for selling x units
Profit function:	$P(x) = R(x) - C(x)$ $=$ profit in selling x units

To use the techniques of calculus, we regard x as a real number, even though this variable may take on only integer values. We always assume that $x \ge 0$, since the production of a negative number of units has no practical significance.

EXAMPLE ■ 1 A manufacturer of miniature tape decks has a monthly fixed cost of \$10,000, a production cost of \$12 per unit, and a selling price of \$20 per unit.

(a) Find $C(x)$, $c(x)$, $R(x)$, and $P(x)$.

(b) Find the function values in part (a) if $x = 1000$.

(c) How many units must be manufactured in order to break even?

SOLUTION

(a) The production cost of manufacturing x units is $12x$. Since there is also a fixed monthly cost of \$10,000, the total monthly cost of manufacturing x units is

$$C(x) = 12x + 10{,}000.$$

The remaining functions are given by

$$c(x) = \frac{C(x)}{x} = 12 + \frac{10{,}000}{x},$$

$$R(x) = 20x,$$

$$P(x) = R(x) - C(x) = 8x - 10{,}000.$$

(b) Substituting $x = 1000$ in part (a) gives us the following values:

$C(1000) = 22{,}000$ cost of manufacturing 1000 units

$c(1000) = 22$ average cost of manufacturing one unit

$R(1000) = 20{,}000$ total revenue received for 1000 units

$P(1000) = -2000$ profit in manufacturing 1000 units

Note that the manufacturer incurs a loss of \$2000 per month if only 1000 units are produced and sold.

(c) The break-even point corresponds to zero profit—that is, when we have $8x - 10{,}000 = 0$. This result gives us

$$8x = 10{,}000, \quad \text{or} \quad x = 1250.$$

Thus to break even, it is necessary to produce and sell 1250 units per month.

If a function f is used to describe some economic entity, the adjective *marginal* is used to specify the derivative f'. The derivatives C', c', R', and P' are called the **marginal cost function,** the **marginal average cost function,** the **marginal revenue function**, and the **marginal profit function,** respectively. The number $C'(x)$ is referred to as the **marginal cost** associated with the production of x units. If we interpret the derivative as a rate of change, then $C'(x)$ is the rate at which the cost changes with respect to the number x of units produced. Similar statements can be made for $c'(x)$, $R'(x)$, and $P'(x)$.

If C is a cost function and n is a positive integer, then, by Definition (2.5),

$$C'(n) = \lim_{h \to 0} \frac{C(n+h) - C(n)}{h}.$$

Hence, if h is small, then

$$C'(n) \approx \frac{C(n+h) - C(n)}{h}.$$

If the number n of units produced is large, economists often let $h = 1$ in the preceding formula to approximate the marginal cost, obtaining

$$C'(n) \approx C(n+1) - C(n).$$

In this context, *the marginal cost associated with the production of n units is (approximately) the cost of producing one more unit.*

Some companies find that the cost $C(x)$ of producing x units of a commodity is given by a formula such as

$$C(x) = a + bx + dx^2 + kx^3.$$

The constant a represents a fixed overhead charge for items like rent, heat, and light that are independent of the number of units produced. If the cost of producing one unit were b dollars and no other factors were

involved, then the second term bx in the formula would represent the cost of producing x units. If x becomes very large, then the terms dx^2 and kx^3 may significantly affect production costs.

EXAMPLE ■ 2 An electronics company estimates that the cost (in dollars) of producing x components used in electronic toys is given by

$$C(x) = 200 + 0.05x + 0.0001x^2.$$

(a) Find the cost, the average cost, and the marginal cost of producing 500 units, 1000 units, and 5000 units.

(b) Compare the marginal cost of producing 1000 units with the cost of producing the 1001st unit.

SOLUTION

(a) The average cost of producing x components is

$$c(x) = \frac{C(x)}{x} = \frac{200}{x} + 0.05 + 0.0001x.$$

The marginal cost is

$$C'(x) = 0.05 + 0.0002x.$$

You should verify the entries in the following table, where numbers in the last three columns represent dollars.

Units x	Cost $C(x)$	Average cost $c(x) = \frac{C(x)}{x}$	Marginal cost $C'(x)$
500	250.00	0.50	0.15
1000	350.00	0.35	0.25
5000	2950.00	0.59	1.05

(b) Using the cost function yields

$$C(1001) = 200 + 0.05(1001) + (0.0001)(1001)^2$$
$$\approx 350.25.$$

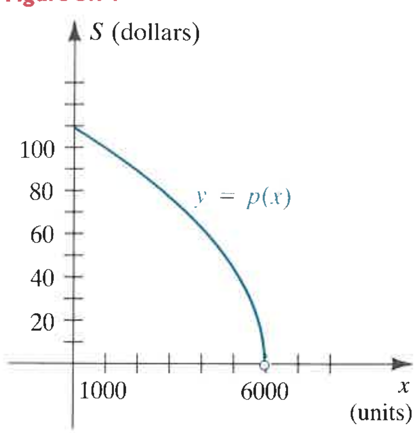
Hence the cost of producing the 1001st unit is

$$C(1001) - C(1000) \approx 350.25 - 350.00$$
$$= 0.25,$$

which is approximately the same as the marginal cost $C'(1000)$.

A company must consider many factors in order to determine a selling price for each product. In addition to the cost of production and the profit desired, the company should be aware of the manner in which consumer demand will vary if the price increases. For some products, there is a

Figure 3.74



constant demand, and changes in price have little effect on sales. For items that are not necessities of life, a price increase will probably lead to a decrease in the number of units sold. Suppose a company knows from past experience that it can sell x units when the price per unit is given by $p(x)$ for some function p. We sometimes say that $p(x)$ is the price per unit when there is a **demand** for x units, and we refer to p as the **demand function** for the commodity. The total income, or revenue, is the number of units sold times the price per unit—that is, $x \cdot p(x)$. Thus,

$$R(x) = xp(x).$$

The derivative p' is called the **marginal demand function**.

If $S = p(x)$, then S is the selling price per unit associated with a demand of x units. Since a decrease in S would ordinarily be associated with an increase in x, a demand function p is usually decreasing; that is, $p'(x) < 0$ for every x. Demand functions are sometimes defined implicitly by an equation involving S and x, as in the next example.

EXAMPLE ■ 3 The demand for x units of a product is related to a selling price of S dollars per unit by the equation $2x + S^2 - 12{,}000 = 0$.

(a) Find the demand function, the marginal demand function, the revenue function, and the marginal revenue function.

(b) Find the number of units and the price per unit that yield the maximum revenue.

(c) Find the maximum revenue.

SOLUTION

(a) Since $S^2 = 12{,}000 - 2x$ and S is positive, we see that the demand function p is given by

$$S = p(x) = \sqrt{12{,}000 - 2x}.$$

The domain of p consists of every x such that $12{,}000 - 2x > 0$, or, equivalently, $2x < 12{,}000$. Thus, $0 \le x < 6000$. The graph of p is sketched in Figure 3.74. In theory, there are no sales if the selling price is $\sqrt{12{,}000}$, or approximately \$109.54, and when the selling price is close to \$0, the demand is close to 6000.

The marginal demand function p' is given by

$$p'(x) = \frac{-1}{\sqrt{12{,}000 - 2x}}.$$

The negative sign indicates that a decrease in price is associated with an increase in demand.

The revenue function R is given by

$$R(x) = xp(x) = x\sqrt{12{,}000 - 2x}.$$

Differentiating and simplifying gives us the marginal revenue function R':

$$R'(x) = \frac{12{,}000 - 3x}{\sqrt{12{,}000 - 2x}}.$$

(b) A critical number for the revenue function R is $x = 12{,}000/3 = 4000$. Since $R'(x)$ is positive if $0 \le x < 4000$ and negative if $4000 < x < 6000$, the maximum revenue occurs when 4000 units are produced and sold. This corresponds to a selling price per unit of

$$p(4000) = \sqrt{12{,}000 - 2(4000)} \approx \$63.25.$$

(c) The maximum revenue, obtained from selling 4000 units at \$63.25 per unit, is

$$4000(63.25) = \$253{,}000.$$

A MODEL FROM SOCIOLOGY AND GEOGRAPHY

Sociologists and geographers often study a phenomenon called **social diffusion**; that is, the spreading of a piece of information, technological innovation, or cultural fad among a population. The individuals in the population can be divided into those who have the information and those who do not.

In a fixed population, it is reasonable to assume that the rate of diffusion is proportional to the number who have the information and the number yet to receive it. The rate of diffusion should be proportional to the number of encounters between individuals of the two groups. If both populations are large, then there will be a relatively large number of such contacts, while if either population is small, there will be relatively few such meetings.

If x represents the number of people in a population of N individuals who have the information, then the rate of diffusion r is the rate of change of x—that is, $r(x) = dx/dt = x'(t)$. A mathematical model for the rate $r(x)$ at which the information is spreading is the equation

$$(*) \qquad r(x) = kx(N - x) = kNx - kx^2,$$

where k is a positive proportionality constant.

If the rate of diffusion in a population of N people is given by $(*)$, then we may want to be able to find when the rate is zero and interpret the result. We are also interested in determining when the information is spreading most rapidly.

The rate of information spread is zero when $r(x) = 0$—that is, when $kx(N - x) = 0$. The possible values of x are $x = 0$ and $x = N$. We can draw the following conclusion: *If no one has the information, it cannot diffuse; if everyone has it, it cannot spread any further. So long as there are some people who have the information and some who do not, the information will continue to spread.*

To determine when the information is spreading most rapidly, we compute the derivative of r as

$$r'(x) = kN - 2kx.$$

Thus, $r'(x) > 0$ when $0 < x < N/2$ and $r'(x) < 0$ when $N/2 < x < N$. Accordingly, the rate of information spread increases until half the population is informed, and then it begins to decrease. Information is spreading most rapidly when $x = N/2$.

A MODEL FROM EPIDEMIOLOGY

Scientists in different fields often use essentially the same mathematical model to represent the dynamics of what appear to be widely different situations. As an example, the diffusion-of-information model that we have just examined assumes that the rate of change of the informed population is jointly proportional to the numbers of informed and uninformed individuals. Precisely the same concept lies at the heart of many models for the transmission of a communicable disease.

In such a model, two important subgroups of the population are the *infectives* (those who are currently infected with the disease and are capable of spreading it) and the *susceptibles* (those who are currently uninfected but could contract the illness). One of the important equations in this model is

$$S'(t) = -\beta S(t)\, I(t),$$

where β is a positive constant and $S = S(t)$ and $I = I(t)$ are the number of susceptibles and infectives at time t, respectively. Note that since the populations of susceptibles and infectives must remain nonnegative and β is positive, we have $S'(t) \leq 0$, so the susceptible population is nonincreasing.

We want to use the model to show that if the population is constant and is made up entirely of susceptibles S and infectives I, then the rate of change of the susceptible population has the same form as the rate of information spread.

We let N represent the constant population size. Then

$$S + I = N \quad \text{or, equivalently,} \quad I = N - S.$$

The equation for the rate of change of susceptibles has the form

$$S' = -\beta SI = -\beta S(N - S),$$

which has the same form as the social diffusion equation

$$x' = kx(N - x).$$

A MODEL FROM ECOLOGY

We now examine a model from ecology that uses the same central concept. Imagine a simple ecosystem with two animal species, one of which preys on the other. To enliven the model, think of rabbits as the prey species and foxes as the predators. We assume that the rabbits live on clover, which is in abundant supply, but the foxes have only a single source of food, the rabbits. The classic predator–prey model represents the growth rate of both the rabbit and the fox populations over time by a linked system of equations:

$$R'(t) = a\,R(t) - b\,R(t)\,F(t),$$
$$F'(t) = m\,R(t)\,F(t) - n\,F(t),$$

where $R(t)$ and $F(t)$ are the rabbit and the fox populations at time t, respectively, and a, b, m, and n are positive constants.

We write this system in a slightly condensed notation, suppressing the explicit mention of the variable t:

$$R' = aR - bRF \quad \text{and} \quad F' = mRF - nF.$$

Although the full solution of this pair of equations lies beyond our current understanding of calculus, we can still do quite a bit of fruitful analysis of the model. First, let us note that in the absence of foxes ($F = 0$), the growth function for the rabbits becomes

$$R' = aR.$$

Since $R'/R = a$ in this situation, the rabbits would grow at a constant percentage rate. We will study the consequences of such growth systematically in Chapter 6. As you might imagine, the rabbits will experience rapid increases in numbers.

Second, if there are no rabbits ($R = 0$), then the dynamics of the fox population reduces to

$$F' = -nF,$$

so the foxes would experience a constant percentage *decline* in numbers. It is not surprising that the foxes would face extinction. (Chapter 6 provides the tools for the quantitative analysis.)

Let us turn then to the more interesting case in which both rabbits and foxes are running around. Each of the growth equations involves the product RF. We are making the not unreasonable assumption that the number of kills of rabbits by foxes is proportional to the frequency of encounters between the two species, which, in turn, is proportional to the product of the two populations. There will be few kills if there are few rabbits or few foxes, and many kills only when both populations are large. Each kill diminishes the rabbit population and enhances the likelihood for growth in the number of foxes.

If we rewrite the equations as

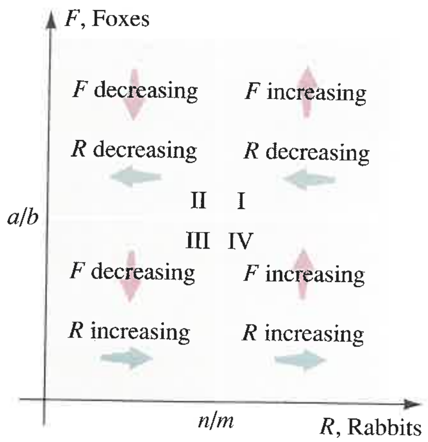
$$R' = R(a - bF) \quad \text{and} \quad F' = F(mR - n),$$

then we see (since $R > 0$ and $F > 0$) that the sign of R' is the same as the sign of $(a - bF)$ and the sign of F' is the sign of $(mR - n)$. Since positive signs for the derivative correspond to increases in the function and negative signs to decreases, we have

$$\begin{cases} R \text{ increases if } F < a/b \\ R \text{ decreases if } F > a/b \end{cases} \quad \text{and} \quad \begin{cases} F \text{ increases if } R > n/m \\ F \text{ decreases if } R < n/m \end{cases}$$

We can gain further insight into the dynamics of the predator–prey relationship by constructing a Cartesian coordinate system with the rabbit population graphed along the horizontal axis and the fox population along the vertical axis, as in Figure 3.75 on the following page.

Figure 3.75



We have pictured the first quadrant only since the variables R and F represent nonnegative numbers. Along the horizontal line $F = a/b$, we have $R' = 0$, and along the vertical line $R = n/m$, we have $F' = 0$. The point of intersection of these two lines $((n/m), (a/b))$ represents a **stable point**. If the populations were to reach this point, both rates of change would be zero and the populations would remain at this level. At every other point in the first quadrant, at least one of the populations would be changing.

What happens if the initial population levels are at some other point? Suppose we begin in region I. Here both fox and rabbit populations are relatively large, which is at first good for the foxes since there will be lots of prey. The fox population will increase as the rabbit population decreases. As time goes by, we will find that the population level has moved to a point to the northwest of the initial point. This northwest movement continues until we reach the critical vertical line $R = n/m$. As we cross this line, the rabbits become scarcer and the fox population also begins to decrease. For a while, there will be dwindling numbers of both foxes and rabbits. The population level moves in a southwesterly direction until it ultimately hits the critical horizontal line $F = a/b$. Now the number of foxes has dropped and there is less danger for the rabbits. The rabbit population begins to increase, but since it is relatively small, the fox population will continue to decrease. The population level moves in a southeasterly direction, continuing this path until the rabbit population passes the vertical line $R = n/m$. Now there are sufficiently many rabbits to support a growing fox population. Both species continue to prosper and the population level moves in a northeasterly direction. Eventually, however, it passes the horizontal line at height a/b and we are back in region I. The process then proceeds as before.

We can see from this analysis that the population point moves in a counterclockwise direction, visiting all four regions in turn. We wish to determine what will happen in the long run—that is, whether the population point will spiral in toward the stable point, spiral out, or form some sort of elliptical closed orbit. Answering these questions must be deferred until we have developed sufficiently powerful tools of calculus.

EXERCISES 3.8

Exer. 1–4: If C is the cost function for a particular product, find (a) the cost of producing 100 units and (b) the average and the marginal cost functions and their values at $x = 100$.

1 $C(x) = 800 + 0.04x + 0.0002x^2$

2 $C(x) = 6400 + 6.5x + 0.003x^2$

3 $C(x) = 250 + 100x + 0.001x^3$

4 $C(x) = 200 + 0.01x + (100/x)$

5 A manufacturer of small motors estimates that the cost (in dollars) of producing x motors per day is given by $C(x) = 100 + 50x + (100/x)$. Compare the marginal cost of producing five motors with the cost of producing the sixth motor.

6 A company conducts a pilot test for production of a new industrial solvent and finds that the cost of producing x liters of each pilot run is given by the formula $C(x) = 3 + x + (10/x)$. Compare the marginal cost of producing 10 liters with the cost of producing the 11th liter.

Exer. 7–8: For the given demand and cost functions, find (a) the marginal demand function, (b) the revenue function, (c) the profit function, (d) the marginal profit function, (e) the maximum profit, and (f) the marginal cost when the demand is 10 units.

7 $p(x) = 50 - 0.1x$; $\quad C(x) = 10 + 2x$

8 $p(x) = 80 - \sqrt{x-1}$; $\quad C(x) = 75x + 2\sqrt{x-1}$

9 A travel agency estimates that, in order to sell x package-deal vacations, it must charge a price per vacation of $1800 - 2x$ dollars for $1 \le x \le 100$. If the cost to the agency for x vacations is $1000 + x + 0.01x^2$ dollars, find

(a) the revenue function

(b) the profit function

(c) the number of vacations that will maximize the profit

(d) the maximum profit

10 A manufacturer determines that x units of a product will be sold if the selling price is $400 - 0.05x$ dollars for each unit. If the production cost for x units is $500 + 10x$, find

(a) the revenue function

(b) the profit function

(c) the number of units that will maximize the profit

(d) the price per unit when the marginal revenue is 300

11 A kitchen specialty company determines that the cost of manufacturing and packaging x pepper mills per day is $500 + 0.02x + 0.001x^2$. If each mill is sold for \$8.00, find

(a) the rate of production that will maximize the profit

(b) the maximum daily profit

12 A company that conducts bus tours found that when the price was \$9.00 per person, the average number of customers was 1000 per week. When the company reduced the price to \$7.00 per person, the average number of customers increased to 1500 per week. Assuming that the demand function is linear, what price should be charged to obtain the greatest weekly revenue?

Exer. 13–14: Analyze each model using the technique developed in the investigation of the predator–prey model.

13 The competitive-hunters model represents an ecosystem with two species, each of which requires the same resource for survival. If x and y are the populations of two species at time t, then the model has the form

$$x'(t) = ax - bxy,$$
$$y'(t) = my - nxy,$$

where a, b, m, and n are positive constants.

14 The Richardson arms race model (Lewis F. Richardson, *Arms and Insecurity*, Pittsburgh: Boxwood Press, 1960) represents the arms expenditures x and y of two nations as the system of equations

$$x'(t) = ay - mx + r,$$
$$y'(t) = bx - ny + s,$$

where a, b, m, n, r, and s are constants, the first four of which are positive.

CHAPTER 3 REVIEW EXERCISES

Exer. 1–2: Find the extrema of f on the given interval.

1 $f(x) = -x^2 + 6x - 8$; $\quad [1, 6]$

2 $f(x) = 3x^3 + x^2$; $\quad (-1, 0]$

Exer. 3–4: Find the critical numbers of f.

3 $f(x) = (x+2)^3(3x-1)^4$

4 $f(x) = \sqrt{x-1}(x-2)^3$

Exer. 5–8: Use the first derivative test to find the local extrema of f. Find the intervals on which f is increasing or is decreasing, and sketch the graph of f.

5 $f(x) = -4x^3 + 9x^2 + 12x$

6 $f(x) = \dfrac{1}{x^2+1}$

7 $f(x) = (4-x)x^{1/3}$ $\qquad$ 8 $f(x) = \sqrt[3]{x^2-9}$

Exer. 9–12: Use the second derivative test (whenever applicable) to find the local extrema of f. Find the intervals on which the graph of f is concave upward or is concave downward, and find the x-coordinates of the points of inflection. Sketch the graph of f.

9 $f(x) = \sqrt[3]{8 - x^3}$

10 $f(x) = -x^3 + 4x^2 - 3x$

11 $f(x) = \dfrac{1}{x^2 + 1}$

12 $f(x) = 40x^3 - x^6$

13 If $f(x) = 2\sin x - \cos 2x$, find the local extrema, and sketch the graph of f for $0 \le x \le 2\pi$.

14 If $f(x) = 2\sin x - \cos 2x$, find equations of the tangent and normal lines to the graph of f at the point $(\pi/6, 1/2)$.

Exer. 15–16: Sketch the graph of a continuous function f that satisfies all the stated conditions.

15 $f(0) = 2$; $f(-2) = f(2) = 0$;
$f'(-2) = f'(0) = f'(2) = 0$;
$f'(x) > 0$ if $-2 < x < 0$;
$f'(x) < 0$ if $x < -2$ or $x > 0$;
$f''(x) > 0$ if $x < -1$ or $1 < x < 2$;
$f''(x) < 0$ if $-1 < x < 1$ or $x > 2$

16 $f(0) = 4$; $f(-3) = f(3) = 0$;
$f'(-3) = 0$; $f'(0)$ is undefined;
$f'(x) > 0$ if $-3 < x < 0$;
$f'(x) < 0$ if $x < -3$ or $x > 0$;
$f''(x) > 0$ if $x < 0$ or $0 < x < 2$;
$f''(x) < 0$ if $x > 2$

Exer. 17–22: Find the extrema and sketch the graph of f.

17 $f(x) = \dfrac{3x^2}{9x^2 - 25}$

18 $f(x) = \dfrac{x^2}{(x-1)^2}$

19 $f(x) = \dfrac{x^2 + 2x - 8}{x + 3}$

20 $f(x) = \dfrac{x^4 - 16}{x^3}$

21 $f(x) = \dfrac{x - 3}{x^2 + 2x - 8}$

22 $f(x) = \dfrac{x}{\sqrt{x + 4}}$

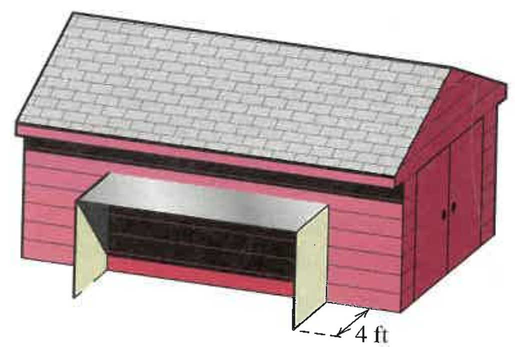
23 If $f(x) = x^3 + x^2 + x + 1$, find a number c that satisfies the conclusion of the mean value theorem on the interval $[0, 4]$.

24 The posted speed limit on a 125-mi toll highway is 65 mi/hr. When an automobile enters the toll road, the driver is issued a ticket on which is printed the exact time. If the driver completes the trip in 1 hr 40 min or less, a speeding citation is issued when the toll is paid. Use the mean value theorem to explain why this citation is justified.

25 A man wishes to put a fence around a rectangular field and then subdivide this field into three smaller rectangular plots by placing two fences parallel to one of the sides. If he can afford only 1000 yd of fencing, what dimensions will give him the maximum area?

26 An open rectangular storage shelter 4 ft deep, consisting of two vertical sides and a flat roof, is to be attached to an existing structure, as illustrated in the figure. The flat roof is made of tin and costs \$5 per ft^2. The two sides are made of plywood costing \$2 per ft^2. If \$400 is available for construction, what dimensions will maximize the volume of the shelter?

Exercise 26



27 A V-shaped water gutter is to be constructed from two rectangular sheets of metal 10 in. wide. Find the angle between the sheets that will maximize the carrying capacity of the gutter.

28 Find the altitude of the right circular cylinder of maximum curved surface area that can be inscribed in a sphere of radius a.

29 The interior of a half-mile race track consists of a rectangle with semicircles at two opposite ends. Find the dimensions that will maximize the area of the rectangle.

30 A cable television firm presently serves 5000 households and charges \$20 per month. A marketing survey indicates that each decrease of \$1 in monthly charge will result in 500 new customers. Find the monthly charge that will result in the maximum monthly revenue.

31 A wire 5 ft long is to be cut into two pieces. One piece is to be bent into the shape of a circle and the other into the shape of a square. Where should the wire be cut so that the sum of the areas of the circle and square is

(a) a maximum (b) a minimum

32 In biochemistry, the general threshold-response curve is given by $R = kS^n/(S^n + a^n)$, where R is the chemical response that corresponds to a concentration S of a substance for positive constants k, n, and a. An example is the rate R at which the liver removes alcohol from the bloodstream when the concentration of alcohol is S. Show that R is an increasing function of S and that $R = k$ is a horizontal asymptote for the curve.

33 The position function of a point moving on a coordinate line is given by $s(t) = (t^2 + 3t + 1)/(t^2 + 1)$. Find the velocity and the acceleration at time t, and describe the motion of the point during the time interval $[-2, 2]$.

34 The position of a moving point on a coordinate line is given by

$$s(t) = a\sin(kt + m) + b\cos(kt + m)$$

for constants a, b, k, and m. Prove that the magnitude of the acceleration is directly proportional to the distance from the origin.

35 A manufacturer of microwave ovens determines that the cost of producing x units is given by

$$C(x) = 4000 + 100x + 0.05x^2 + 0.0002x^3.$$

Compare the marginal cost of producing 100 ovens with the cost of producing the 101st oven.

36 The cost function for producing a microprocessor component is given by $C(x) = 1000 + 2x + 0.005x^2$. If 2000 units are produced, find the cost, the average cost, the marginal cost, and the marginal average cost.

37 An electronics company estimates that the cost of producing x calculators per day is

$$C(x) = 500 + 6x + 0.02x^2.$$

If each calculator is sold for \$18, find

(a) the revenue function

(b) the profit function

(c) the daily production that will maximize the profit

(d) the maximum daily profit

38 A small office building is to contain 500 ft^2 of floor space. Simplified floor plans are shown in the figure. If the walls cost \$100 per running foot and if the wall space above the doors is disregarded,

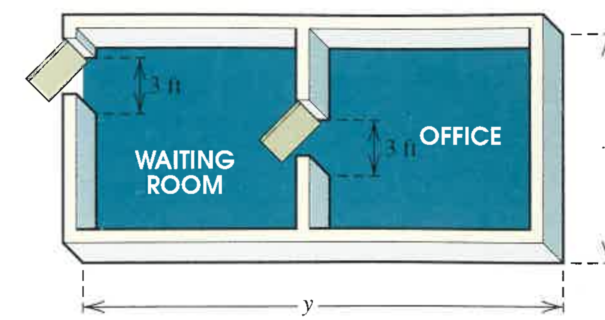
(a) show that the cost $C(x)$ of the walls is

$$C(x) = 100[3x - 6 + (1000/x)]$$

(b) find the vertical and oblique asymptotes, and sketch the graph of $C(x)$ for $x > 0$

(c) find the design that minimizes the cost

Exercise 38



39 A NASA rocket is propelled straight upward from the ground. If the acceleration is constant and the rocket achieves a height of 49 ft in 1 sec, what is the rate of acceleration?

40 The "Alpine Slide" is a way to enjoy sledding when there is no snow. The slide itself is a long concrete trough that runs downhill parallel to a ski trail. The rider takes a chair lift to the top of the slide and then races down the slide on a small sled. The sled has metal rollers on the bottom and a control stick that permits the rider to slow down while moving along the track.

A sled is initially moving at a rate of 44 ft/sec. It decelerates to 32 ft/sec over a distance of 114 ft at an unknown constant rate. It continues to decelerate at that same rate until it comes to a full stop.

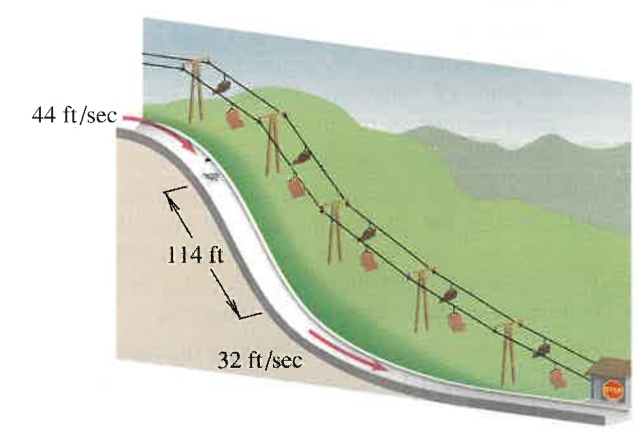
(a) How long does it take to reduce the speed to 32 ft/sec?

(b) What is the acceleration of the sled?

(c) How long does it take before the sled comes to a complete stop?

(d) How many feet does the sled travel before it comes to a stop?

Exercise 40



C Exer. 41–44: Approximate a number c that satisfies the conclusion of the mean value theorem on the interval given.

41 $f(x) = x^3 + x^2 + x + 1;$ $[0, 4]$

42 $f(x) = x^3 + 2x^2 - 3x - 4;$ $[-3, 2]$

43 $f(x) = \sin(\sin x);$ $[-\pi/2, \pi/2]$

44 $f(x) = \sin(\cos x);$ $[0, \pi]$

C Exer. 45–50: Graph f on the given interval and approximate the extrema and the points of inflection to four decimal places.

45 $f(x) = x^2 - 3\sqrt{x} - \sqrt{x^3} - 7;$ $[0, 7]$

46 $f(x) = \sin x - x \cos x;$ $[-5, 10]$

47 $f(x) = |x^3 + 24x^2 - 18x + 3|;$ $[-7, 5]$

48 $f(x) = x^5 - 5x^3 + 3x + 4;$ $[-25, 2.5]$

49 $f(x) = 5 \cos(\cos x) - 2x;$ $[-2, 2]$

50 $f(x) = 2 \sin x + 3 \sin \pi x;$ $[-2, 2]$

EXTENDED PROBLEMS AND GROUP PROJECTS

1 Here is an important generalization of the mean value theorem: *If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a number c in (a, b) such that* $[f(b) - f(a)]/[g(b) - g(a)] = f'(c)/g'(c)$.

(a) Show that the mean value theorem is a special case of this generalization. (*Hint:* Use $g(x) = x$.)

(b) Prove this theorem by applying the mean value theorem to the function $h(x)$ defined by $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$.

C **(c)** Suppose that f and g are differentiable on an open interval (a, b) containing c, except possibly at c itself. If $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$, then show that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

provided $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists.

Obtain first some numerical and graphical evidence that this result is true by examining various examples of pairs of functions f and g. Then use the generalization of the mean value theorem to prove the result.

2 Suppose that f is a function with the property that $f'(x) = 1/x$ for all $x > 0$ and $f(1) = 0$. (We have not yet seen this function, but in this problem, we will investigate what the mean value theorem and its consequences imply about such a function.)

(a) Show that f is a strictly increasing function.

(b) Find $f''(x)$ and show that the graph of f is concave downward.

(c) Let c be any positive number. Define the function $g(x)$ by $g(x) = f(cx)$. By the chain rule, show that $g'(x) = 1/x$. Why does this result imply that $g(x) = f(x) + C$ for some constant C? Use the fact that $f(1) = 0$ to find C.

(d) Show that f satisfies one of the properties of logarithmic functions—namely, $f(cx) = f(c) + f(x)$ for all positive numbers c and x.

(e) Let n be a nonzero rational number. Use the chain rule to show that the functions $g(x) = f(x^n)$ and $h(x) = nf(x)$ have the same derivative. Thus, $g(x) = h(x) + C$ for some constant C. Use the fact that $f(1) = 0$ to determine C.

(f) Show that f satisfies another property of logarithmic functions: $f(x^n) = nf(x)$.

(g) Show that if $1 < c < 2$, then $f'(c) > 1/2$. Use the mean value theorem to show that $f(2) > 1/2$. By part (f), show that this implies that $f(2^n) > n/2$ if n is a positive integer. Conclude that

$$\lim_{x\to\infty} f(x) = \infty.$$

(h) Determine $\lim_{x\to 0^+} f(x)$.

(i) Using the properties of f derived above, sketch a graph of f.

3 Suppose that f is a function with the property that $f'(x) = f(x)$ for all x and $f(0) = 1$. (We have not yet seen this function, but in this problem, we will investigate what the mean value theorem and its consequences imply about such a function.)

(a) Show that f is a strictly increasing function.

(b) Find $f''(x)$ and show that the graph of f is concave upward.

(c) Use the chain rule to show that if $h(x) = f(x + c)$, then $h'(x) = h(x)$. Apply the quotient rule to the function $g(x) = f(x + c)/f(x)$ to show that $g'(x) = 0$ for all x. Conclude that f has an important property of exponential functions: $f(x + c) = f(x)f(c)$, for all numbers x and c.

(d) Show that f also has another important property of exponential functions: $(f(x))^n = f(xn)$.

(e) Use the mean value theorem to conclude that $f(n) > 1 + n$ if $n > 0$ and hence show that $\lim_{x\to\infty} f(x) = \infty$.

(f) Determine $\lim_{x\to -\infty} f(x)$.

(g) Using the properties of f derived above, sketch a graph of f.