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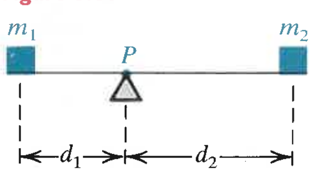
5.7 MOMENTS AND CENTERS OF MASS

In this section, we consider some topics involving the mass of an object. The terms *mass* and *weight* are sometimes confused with each other. Weight is determined by the force of gravity. For example, the weight of an object on the moon is approximately one-sixth its weight on earth, because the force of gravity is weaker. However, the mass is the same. Newton used the term *mass* synonymously with *quantity of matter* and related it to force by his *second law of motion,* $F = ma$, where F denotes the force acting on an object of mass m that has acceleration a. In the British system, we often approximate a by 32 ft/sec^2 and use the **slug** as the unit of mass. In SI units, $a \approx 9.81$ m/sec^2, and the kilogram is the unit of mass. It can be shown that

$$1 \text{ slug} \approx 14.6 \text{ kg} \quad \text{and} \quad 1 \text{ kg} \approx 0.07 \text{ slug}.$$

In applications, we generally assume that the mass of an object is concentrated at a point, and we refer to the object as a **point-mass**, regardless of its size. For example, using the earth as a frame of reference, we may regard a human being, an automobile, or a building as a point-mass.

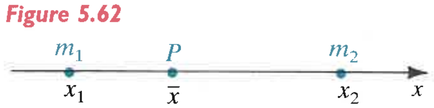
Figure 5.61



In an elementary physics experiment, we consider two point-masses m_1 and m_2 attached to the ends of a thin rod, as illustrated in Figure 5.61, and then locate the point P at which a fulcrum should be placed so that the rod balances. (This situation is similar to balancing a seesaw with a person sitting at each end.) If the distances from m_1 and m_2 to P are d_1 and d_2, respectively, then it can be shown experimentally that P is the balance point if

$$m_1 d_1 = m_2 d_2.$$

Figure 5.62



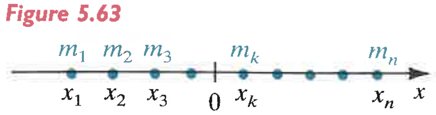
In order to generalize this concept, let us introduce an x-axis, as illustrated in Figure 5.62, with m_1 and m_2 located at points with coordinates x_1 and x_2. If the coordinate of the balance point P is $\bar{x}$, then using the formula $m_1 d_1 = m_2 d_2$ yields

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$
$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$
$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

This gives us a formula for locating the balance point P.

If a mass m is located at a point on the axis with coordinate x, then the product mx is called the *moment* M_0 *of the mass about the origin.* Our formula for $\bar{x}$ states that to find the coordinate of the balance point, we may divide the sum of the moments about the origin by the total mass. The point with coordinate $\bar{x}$ is called the *center of mass* (or *center of gravity*) of the two point-masses. The next definition extends this discussion to many point-masses located on an axis, as shown in Figure 5.63.

Figure 5.63



Definition 5.23

Let S denote a system of point-masses $m_1, m_2, \ldots, m_n$ located at $x_1, x_2, \ldots, x_n$ on a coordinate line, and let $m = \sum_{k=1}^{n} m_k$ denote the total mass.

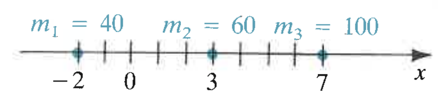
(i) The **moment of S about the origin** is $M_0 = \sum_{k=1}^{n} m_k x_k$.

(ii) The **center of mass** of S is given by $\bar{x} = M_0/m$.

The point with coordinate $\bar{x}$ is the balance point of the system S in the same sense as in our seesaw illustration.

EXAMPLE ■ 1 Three point-masses of 40, 60, and 100 kg are located at -2, 3, and 7, respectively, on an x-axis. Find the center of mass.

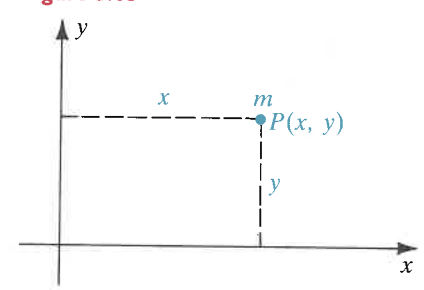
Figure 5.64



SOLUTION If we denote the three masses by m_1, m_2, and m_3, we have the situation illustrated in Figure 5.64, with $x_1 = -2$, $x_2 = 3$, and $x_3 = 7$. Applying Definition (5.23) gives us the coordinate $\bar{x}$ of the center of mass:

$$\bar{x} = \frac{40(-2) + 60(3) + 100(7)}{40 + 60 + 100} = \frac{800}{200} = 4$$

Figure 5.65



Let us next consider a point-mass m located at $P(x, y)$ in a coordinate plane (see Figure 5.65). We define the moments M_x and M_y of m about the coordinate axes as follows:

$$\textit{moment about the x-axis:} \quad M_x = my$$
$$\textit{moment about the y-axis:} \quad M_y = mx$$

In words, to find M_x we multiply m by the y-coordinate of P, and to find M_y we multiply m by the x-coordinate. To find M_x and M_y for a *system* of point-masses, we add the individual moments, as in (i) and (ii) of the next definition.

Definition 5.24

Let S denote a system of point-masses $m_1, m_2, \ldots, m_n$ located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in a coordinate plane, and let $m = \sum_{k=1}^{n} m_k$ denote the total mass.

(i) The **moment of S about the x-axis** is $M_x = \sum_{k=1}^{n} m_k y_k$.

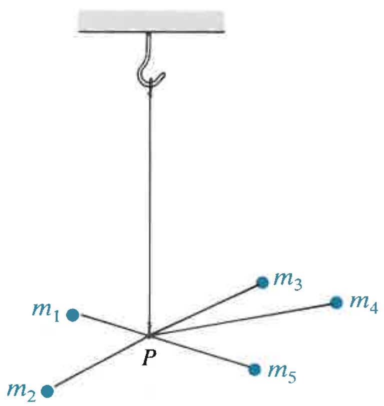
(continued)

(ii) The **moment of S about the y-axis** is $M_y = \sum_{k=1}^{n} m_k x_k$.

(iii) The **center of mass** of S is the point $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}.$$

Figure 5.66



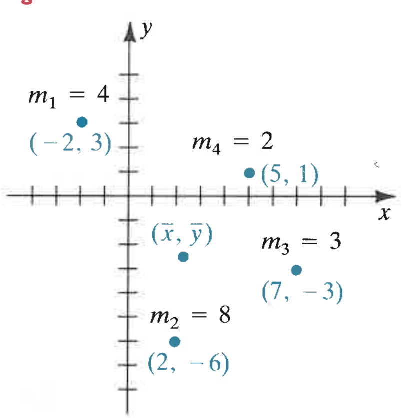
From (iii) of this definition,

$$m\bar{x} = M_y \quad \text{and} \quad m\bar{y} = M_x.$$

Since $m\bar{x}$ and $m\bar{y}$ are the moments about the y-axis and x-axis, respectively, of a single point-mass m located at $(\bar{x}, \bar{y})$, we may interpret the center of mass as the point at which the total mass can be concentrated to obtain the moments M_y and M_x of S.

We might think of the n point-masses in (5.24) as being fastened to the center of mass P by weightless rods, as spokes of a wheel are attached to the center of the wheel. The system S would balance if supported by a cord attached to P, as illustrated in Figure 5.66. The appearance would be similar to that of a mobile having all its objects in the same horizontal plane.

Figure 5.67



EXAMPLE 2 Point-masses of 4, 8, 3, and 2 kg are located at $(-2, 3)$, $(2, -6)$, $(7, -3)$, and $(5, 1)$, respectively. Find M_x, M_y, and the center of mass of the system.

SOLUTION The masses are illustrated in Figure 5.67, in which we have also anticipated the position of $(\bar{x}, \bar{y})$. Applying Definition (5.24) gives us

$$M_x = (4)(3) + (8)(-6) + (3)(-3) + (2)(1) = -43$$

$$M_y = (4)(-2) + (8)(2) + (3)(7) + (2)(5) = 39.$$

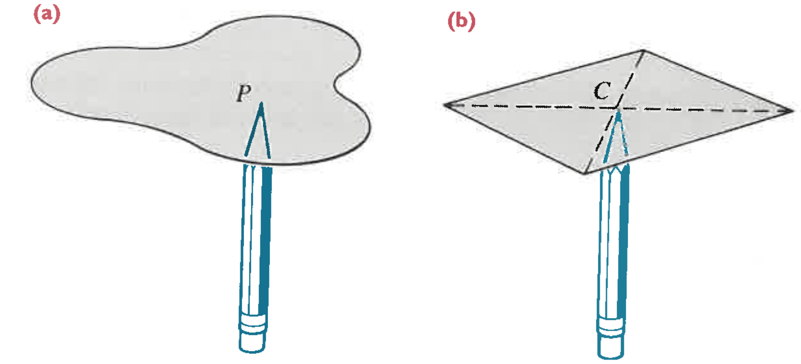
Since $m = 4 + 8 + 3 + 2 = 17$,

$$\bar{x} = \frac{M_y}{m} = \frac{39}{17} \approx 2.3 \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = -\frac{43}{17} \approx -2.5.$$

Thus, the center of mass is $(\frac{39}{17}, -\frac{43}{17})$.

Later in the text we shall consider solid objects that are **homogeneous** in the sense that the mass is uniformly distributed throughout the solid. In physics, the **density** ρ (rho) of a homogeneous solid of mass m and volume V is defined by $\rho = m/V$. Thus, *density is mass per unit volume.* The SI unit for density is kg/m^3; however, g/cm^3 is also used. The British unit is lb/ft^3 or lb/in^3.

Figure 5.68

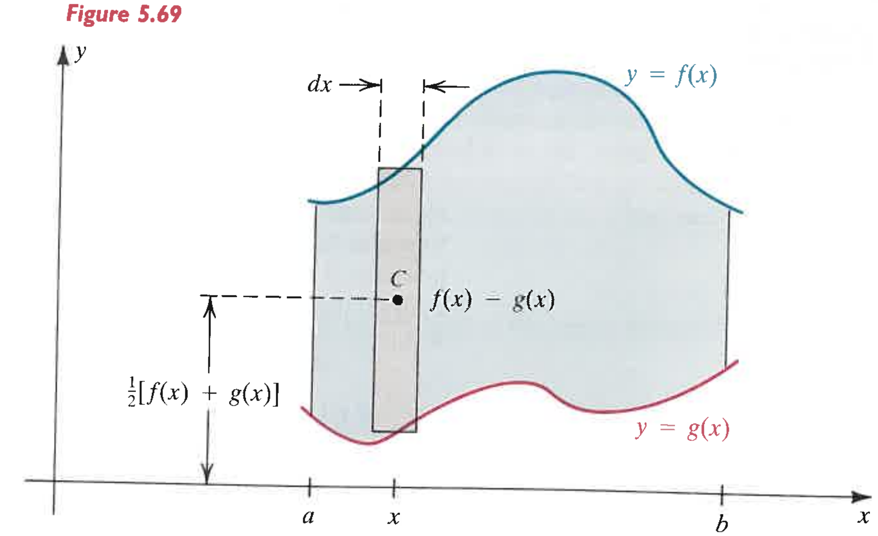


In this section, we restrict our discussion to homogeneous **laminas** (thin flat plates) that have **area density** (mass per unit area) ρ. Area density is measured in kg/m^2, lb/ft^2, and so on. If the area of one face of a lamina is A and the area density is ρ, then its mass m is given by $m = \rho A$. We wish to define the center of mass P such that if the tip of a sharp pencil were placed at P, as illustrated in Figure 5.68, the lamina would balance in a horizontal position. As in Figure 5.68(b), we shall assume that *the center of mass of a rectangular lamina is the point C at which the diagonals intersect.* We call C the *center* of the rectangle. Thus, for problems involving mass, we may assume that a rectangular lamina is a point-mass located at the center of the rectangle. This assumption is the key to our definition of the center of mass of a lamina.

Consider a lamina that has area density ρ and the shape of the R_x region in Figure 5.69. Since we have had ample experience using limits of Riemann sums for definitions in Sections 5.1–5.6, let us proceed directly to the method of representing the width of the rectangle in the figure by dx (instead of Δx_k), obtaining

$$\text{area of rectangle:} \quad [f(x) - g(x)]\,dx.$$

Figure 5.69



Since the area density of the lamina is ρ, we may write

$$\text{mass of rectangular lamina:}\quad \rho[f(x) - g(x)]\,dx.$$

If, as in previous sections, we regard $\int_a^b$ as an operator that takes limits of sums, we arrive at the following definition for the mass m of the lamina:

$$m = \int_a^b \rho[f(x) - g(x)]\,dx$$

We next assume that the rectangular lamina in Figure 5.69 is a point-mass located at the center C of the rectangle. Since, by the midpoint formula (5), on page 11, the distance from the x-axis to C is $\frac{1}{2}[f(x) + g(x)]$, we obtain the following result for the rectangular lamina:

$$\textit{moment about the x-axis:}\quad \tfrac{1}{2}[f(x) + g(x)] \cdot \rho[f(x) - g(x)]\,dx$$

Similarly, since the distance from the y-axis to C is x,

$$\textit{moment about the y-axis:}\quad x \cdot \rho[f(x) - g(x)]\,dx.$$

Taking limits of sums by applying $\int_a^b$ leads to the next definition.

Definition 5.25

Let a lamina L of area density ρ have the shape of the R_x region in Figure 5.69.

(i) The **mass** of L is $m = \int_a^b \rho[f(x) - g(x)]\,dx$.

(ii) The **moments of L about the x-axis and y-axis** are

$$M_x = \int_a^b \tfrac{1}{2}[f(x) + g(x)] \cdot \rho[f(x) - g(x)]\,dx$$

and $$M_y = \int_a^b x \cdot \rho[f(x) - g(x)]\,dx.$$

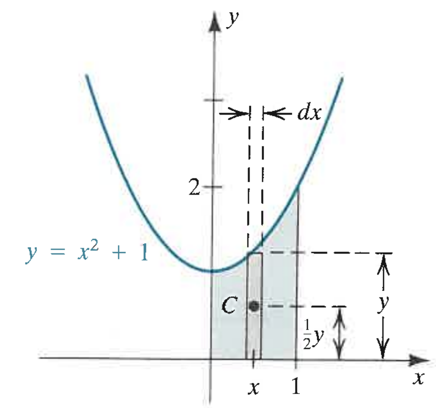
(iii) The **center of mass** of L is the point $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m} \quad\text{and}\quad \bar{y} = \frac{M_x}{m}.$$

An analogous definition can be stated if L has the shape of an R_y region and the integrations are with respect to y. We could also obtain formulas for moments with respect to lines other than the x-axis or y-axis; however, it is advisable to remember the *technique* for finding moments—multiplying a mass by a distance from an axis—instead of memorizing formulas that cover all possible cases.

EXAMPLE 3 A lamina of area density ρ has the shape of the region bounded by the graphs of $y = x^2 + 1$, $x = 0$, $x = 1$, and $y = 0$. Find the center of mass.

Figure 5.70



SOLUTION The region and a typical rectangle of width dx and height y are sketched in Figure 5.70. As indicated in the figure, the distance from the x-axis to the center C of the rectangle is $\frac{1}{2}y$, and the distance from the y-axis to C is x. Hence, *for the rectangular lamina*, we have the following:

$$\begin{aligned}
\text{mass:}&\quad \rho y\,dx = \rho(x^2+1)\,dx\\
\text{moment about } x\text{-axis:}&\quad \tfrac{1}{2}y \cdot \rho y\,dx = \tfrac{1}{2}\rho(x^2+1)^2\,dx\\
\text{moment about } y\text{-axis:}&\quad x \cdot \rho y\,dx = \rho x(x^2+1)\,dx
\end{aligned}$$

We now take a limit of sums of these expressions by applying the operator $\int_0^1$:

$$m = \int_0^1 \rho(x^2+1)\,dx = \rho\left[\tfrac{1}{3}x^3 + x\right]_0^1 = \tfrac{4}{3}\rho$$

$$M_x = \int_0^1 \tfrac{1}{2}\rho(x^2+1)^2\,dx = \tfrac{1}{2}\rho\int_0^1 (x^4 + 2x^2 + 1)\,dx$$

$$= \tfrac{1}{2}\rho\left[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x\right]_0^1 = \tfrac{14}{15}\rho$$

$$M_y = \int_0^1 \rho x(x^2+1)\,dx = \rho\int_0^1 (x^3 + x)\,dx$$

$$= \rho\left[\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right]_0^1 = \tfrac{3}{4}\rho$$

To find the center of mass $(\bar{x}, \bar{y})$, we use Definition (5.25)(iii):

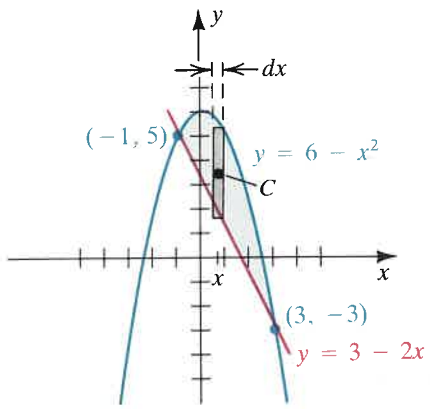
$$\bar{x} = \frac{M_y}{m} = \frac{\frac{3}{4}\rho}{\frac{4}{3}\rho} = \frac{9}{16} \quad\text{and}\quad \bar{y} = \frac{M_x}{m} = \frac{\frac{14}{15}\rho}{\frac{4}{3}\rho} = \frac{7}{10}$$

When we found $(\bar{x}, \bar{y})$ in Example 3, the constant ρ in the numerator and the denominator canceled. This will always be the case for a homogeneous lamina. Hence, the center of mass is independent of the area density ρ; that is, $\bar{x}$ and $\bar{y}$ depend only on the shape of the lamina. For this reason, the point $(\bar{x}, \bar{y})$ is sometimes referred to as the center of mass of a *region* in the plane, or as the **centroid** of the region. We can obtain formulas for moments of centroids by letting $\rho = 1$ and $m = A$ (the area of the region) in our previous work.

EXAMPLE 4 Find the centroid of the region bounded by the graphs of $y = 6 - x^2$ and $y = 3 - 2x$.

SOLUTION The region is the same as that considered in Example 2 of Section 5.1 and is resketched in Figure 5.71 on the following page. To find the moments and the centroid, we take $\rho = 1$ and $m = A$. Referring

Figure 5.71



to the typical rectangle with center C shown in Figure 5.71, we obtain the following:

area of rectangle: $[(6 - x^2) - (3 - 2x)]\,dx$

distance from x-axis to C: $\frac{1}{2}[(6 - x^2) + (3 - 2x)]$

moment about x-axis: $\frac{1}{2}[(6 - x^2) + (3 - 2x)] \times [(6 - x^2) - (3 - 2x)]\,dx$

distance from y-axis to C: x

moment about y-axis: $x[(6 - x^2) - (3 - 2x)]\,dx$

We now take a limit of sums by applying the operator $\int_{-1}^{3}$:

$$
\begin{aligned}
M_x &= \int_{-1}^{3} \tfrac{1}{2}[(6 - x^2) + (3 - 2x)] \cdot [(6 - x^2) - (3 - 2x)]\,dx \\
&= \tfrac{1}{2}\int_{-1}^{3} [(6 - x^2)^2 - (3 - 2x)^2]\,dx \\
&= \tfrac{1}{2}\int_{-1}^{3} (x^4 - 16x^2 + 12x + 27)\,dx = \tfrac{416}{15} \\
M_y &= \int_{-1}^{3} x[(6 - x^2) - (3 - 2x)]\,dx \\
&= \int_{-1}^{3} (3x + 2x^2 - x^3)\,dx = \tfrac{32}{3}
\end{aligned}
$$

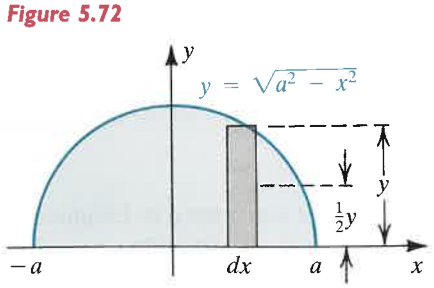
Using $A = \frac{32}{3}$ and Definition (5.25)(iii), we determine the centroid:

$$
\bar{x} = \frac{M_y}{m} = \frac{32/3}{32/3} = 1 \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{416/15}{32/3} = \frac{13}{5}
$$

We could have found the centroid by using Definition (5.25) with $f(x) = 6 - x^2$, $g(x) = 3 - 2x$, $a = -1$, and $b = 3$, but that would merely teach you how to substitute and not how to think.

If a homogeneous lamina has the shape of a region that has an axis of symmetry, then the center of mass must lie on that axis. This fact is used in the next example.

Figure 5.72



EXAMPLE ■ 5 Find the centroid of the semicircular region bounded by the x-axis and the graph of $y = \sqrt{a^2 - x^2}$ with $a > 0$.

SOLUTION The region is sketched in Figure 5.72. By symmetry, the centroid is on the y-axis; that is, $\bar{x} = 0$. Hence, we need find only $\bar{y}$. Referring to the rectangle in Figure 5.72 and using $\rho = 1$ gives us the following result:

moment about x-axis: $\frac{1}{2}y \cdot y\,dx = \frac{1}{2}y^2\,dx = \frac{1}{2}(a^2 - x^2)\,dx$

We now take a limit of sums by applying the operator $\int_{-a}^{a}$:

$$
\begin{aligned}
M_x &= \int_{-a}^{a} \tfrac{1}{2}(a^2 - x^2)\,dx = 2\int_{0}^{a} \tfrac{1}{2}(a^2 - x^2)\,dx \\
&= \left[a^2x - \tfrac{1}{3}x^3\right]_0^a = \tfrac{2}{3}a^3
\end{aligned}
$$

Using $m = A = \frac{1}{2}\pi a^2$ gives us

$$
\bar{y} = \frac{M_x}{m} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \approx 0.42a.
$$

Thus, the centroid is the point $\left(0, \dfrac{4}{3\pi}a\right)$.

We conclude this section by stating a useful theorem about solids of revolution. To illustrate a special case of the theorem, consider an R_x region R of the type shown in Figure 5.69. Using $\rho = 1$ and $m = A$ (the area of R), we find that the moment of R about the y-axis is given by

$$
M_y = \int_a^b x[f(x) - g(x)]\,dx.
$$

If R is revolved about the y-axis, then using cylindrical shells, we find that the volume V of the resulting solid is given by

$$
V = \int_a^b 2\pi x[f(x) - g(x)]\,dx.
$$

Comparing these two equations, we see that

$$
M_y = \frac{V}{2\pi}.
$$

If $(\bar{x}, \bar{y})$ is the centroid of R, then, by Definition (5.25)(iii),

$$
\bar{x} = \frac{M_y}{m} = \frac{(V/2\pi)}{A} = \frac{V}{2\pi A}
$$

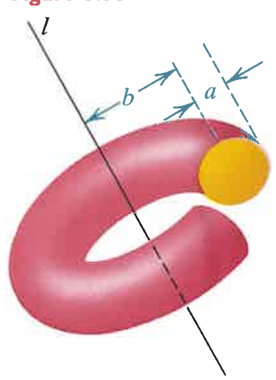
and hence, $\qquad V = 2\pi\bar{x}A.$

Since $\bar{x}$ is the distance from the y-axis to the centroid of R, the last formula states that the volume V of the solid of revolution may be found by multiplying the area A of R by the distance $2\pi\bar{x}$ that the centroid travels when R is revolved once about the y-axis. A similar statement is true if R is revolved about the x-axis. In Chapter 13, we shall prove the following more general theorem, named after the mathematician Pappus of Alexandria (ca. A.D. 300).

Theorem of Pappus 5.26

> Let R be a region in a plane that lies entirely on one side of a line l in the plane. If R is revolved once about l, the volume of the resulting solid is the product of the area of R and the distance traveled by the centroid of R.

Figure 5.73



EXAMPLE ■ 6 The region bounded by a circle of radius a is revolved about a line l, in the plane of the circle, that is a distance b from the center of the circle, where $b > a$ (see Figure 5.73). Find the volume V of the resulting solid. (The surface of this doughnut-shaped solid is called a **torus**.)

SOLUTION The region bounded by the circle has area πa^2, and the distance traveled by the centroid is $2\pi b$. Hence, by the theorem of Pappus,

$$V = (2\pi b)(\pi a^2) = 2\pi^2 a^2 b.$$

EXERCISES 5.7

Exer. 1–2: The table lists point-masses (in kilograms) and their coordinates (in meters) on an x-axis. Find m, M_0, and the center of mass.

1

Mass	100	80	70
Coordinate	−3	2	4

2

Mass	50	100	50
Coordinate	−10	2	3

Exer. 3–4: The table lists point-masses (in kilograms) and their locations (in meters) in an xy-plane. Find m, M_x, M_y, and the center of mass of the system.

3

Mass	2	7	5
Location	(4, −1)	(−2, 0)	(−8, −5)

4

Mass	10	3	4	1	8
Location	(−5, −2)	(3, 7)	(0, −3)	(−8, −3)	(0, 0)

Exer. 5–14: Sketch the region bounded by the graphs of the equations, and find m, M_x, M_y, and the centroid.

5 $y = x^3$, $y = 0$, $x = 1$

6 $y = \sqrt{x}$, $y = 0$, $x = 9$

7 $y = 4 - x^2$, $y = 0$

8 $2x + 3y = 6$, $y = 0$, $x = 0$

9 $y^2 = x$, $2y = x$

10 $y = x^2$, $y = x^3$

11 $y = 1 - x^2$, $x - y = 1$

12 $y = x^2$, $x + y = 2$

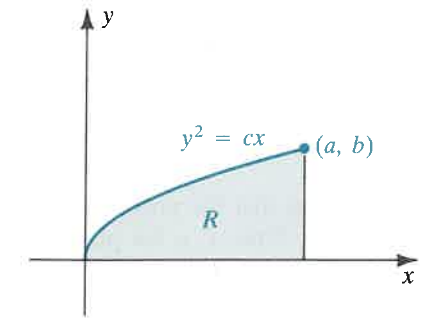
13 $x = y^2$, $x - y = 2$

14 $x = 9 - y^2$, $x + y = 3$

15 Find the centroid of the region in the first quadrant bounded by the circle $x^2 + y^2 = a^2$ and the coordinate axes.

16 Let R be the region in the first quadrant bounded by part of the parabola $y^2 = cx$ with $c > 0$, the x-axis, and the vertical line through the point (a, b) on the parabola, as shown in the figure on the following page. Find the centroid of R.

Exercise 16



17 A region has the shape of a square of side $2a$ surmounted by a semicircle of radius a. Find the centroid. (*Hint:* Use Example 5 and the fact that the moment of the region is the sum of the moments of the square and the semicircle.)

18 Let the points P, Q, R, and S have coordinates $(-b, 0)$, $(-a, 0)$, $(a, 0)$, and $(b, 0)$, respectively, with $0 < a < b$. Find the centroid of the region bounded by the graphs of $y = \sqrt{b^2 - x^2}$, $y = \sqrt{a^2 - x^2}$, and the line segments PQ and RS. (*Hint:* Use Example 5.)

19 Prove that the centroid of a triangle coincides with the intersection of the medians. (*Hint:* Take the vertices at the points $(0, 0)$, (a, b), and $(0, c)$, with a, b, and c positive.)

20 A region has the shape of a square of side a surmounted by an equilateral triangle of side a. Find the centroid. (*Hint:* See Exercise 19 and the hint given for Exercise 17.)

Exer. 21–24: Use the theorem of Pappus.

21 Let R be the rectangular region with vertices (1, 2), (2, 1), (5, 4), and (4, 5). Find the volume of the solid generated by revolving R about the y-axis.

22 Let R be the triangular region with vertices (1, 1), (2, 2), and (3, 1). Find the volume of the solid generated by revolving R about the y-axis.

23 Find the centroid of the region in the first quadrant bounded by the graph of $y = \sqrt{a^2 - x^2}$ and the coordinate axes.

24 Find the centroid of the triangular region with vertices $O(0, 0)$, $A(0, a)$, and $B(b, 0)$ for positive numbers a and b.

c **25** A lamina of area density ρ has the shape of the region bounded by the graphs of $f(x) = \sqrt{|\cos x|}$ and $g(x) = x^2$. Graph f and g on the same coordinate axes.

(a) Set up an integral that can be used to approximate the mass of the lamina.

(b) Use Simpson's rule, with $n = 2$, to approximate the integral in part (a).

c **26** Use Simpson's rule, with $n = 2$, to approximate the centroid of the region bounded by the graphs of $y = 0$, $y = (\sin x)/x$, $x = 1$, and $x = 2$.

5.8 OTHER APPLICATIONS

It should be evident from our work in this chapter that if a quantity can be approximated by a sum of many terms, then it is a candidate for representation as a definite integral. The main requirement is that as the number of terms increases, the sums approach a limit. In this section, we consider several miscellaneous applications of the definite integral. Let us begin with the force exerted by a liquid on a submerged object.

In physics, the **pressure** p at a depth h in a fluid is defined as the weight of fluid contained in a column that has a cross-sectional area of one square unit and an altitude h. Pressure may also be regarded as the force per unit area exerted by the fluid. If a fluid has density ρ, then the pressure p at depth h is given by

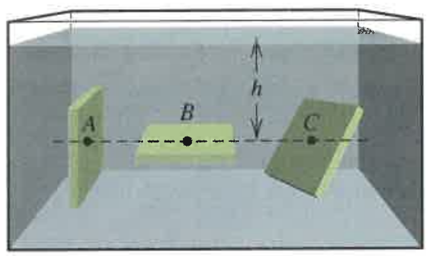
$$p = \rho h.$$

The following illustration is for water, with $\rho = 62.5$ lb/ft^3.

ILLUSTRATION

Density ρ (lb/ft^3)	Depth h (ft)	Pressure $p = \rho h$ (lb/ft^2)
62.5	2	125
62.5	4	250
62.5	6	375

Figure 5.74



Pascal's principle in physics states that the pressure at a depth h in a fluid is the same in every direction. Thus, if a flat plate is submerged in a fluid, then the pressure on one side of the plate at a point that is h units below the surface is ρh, regardless of whether the plate is submerged vertically, horizontally, or obliquely (see Figure 5.74, where the pressure at points A, B, and C is ρh).

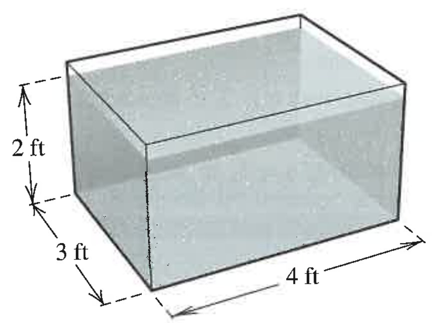
If a rectangular tank, such as a fish aquarium, is filled with water (see Figure 5.75), the total force exerted by the water on the base may be calculated as follows:

$$\text{force on base} = (\text{pressure at base}) \cdot (\text{area of base})$$

For the tank in Figure 5.75, we use $\rho = 62.5$ lb/ft^3 and $h = 2$ ft to obtain

$$\text{force on base} = (125 \text{ lb/ft}^2) \cdot (12 \text{ ft}^2) = 1500 \text{ lb}.$$

Figure 5.75

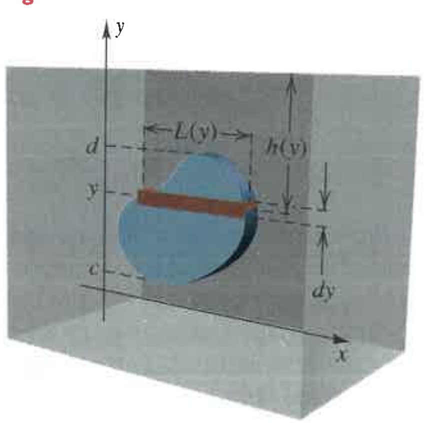


This corresponds to 12 columns of water, each having cross-sectional area 1 ft^2 and each weighing 125 lb.

It is more complicated to find the force exerted on one of the sides of the aquarium, since the pressure is not constant there but increases as the depth increases. Instead of investigating this particular problem, let us consider the following more general situation.

Suppose a flat plate is submerged in a fluid of density ρ such that the face of the plate is perpendicular to the surface of the fluid. Let us introduce a coordinate system as shown in Figure 5.76, where the width of the plate extends over the interval $[c, d]$ on the y-axis. Assume that for each y in $[c, d]$, the corresponding depth of the fluid is $h(y)$ and the length of the plate is $L(y)$, where h and L are continuous functions.

Figure 5.76



We shall use our standard technique of considering a typical horizontal rectangle of width dy and length $L(y)$, as illustrated in Figure 5.76. If dy is small, then the pressure at any point in the rectangle is approximately $\rho h(y)$. Thus, the force on one side of the rectangle can be approximated by

$$\text{force on rectangle} \approx (\text{pressure}) \cdot (\text{area of rectangle}),$$

or

$$\text{force on rectangle} \approx \rho h(y) \cdot L(y)\, dy.$$

Taking a limit of sums of these forces by applying the operator $\int_c^d$ leads to the following definition.

***Definition* 5.27**

The **force F exerted by a fluid** of constant density ρ on one side of a submerged region of the type illustrated in Figure 5.76 is

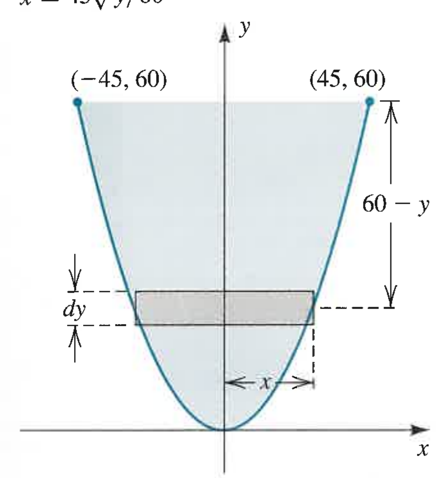
$$F = \int_c^d \rho h(y) \cdot L(y)\, dy.$$

If a more complicated region is divided into subregions of the type illustrated in Figure 5.76, we apply Definition (5.27) to each subregion and add the resulting forces.

The coordinate system may be introduced in various ways, as the next two examples illustrate. In Example 1, we choose the x-axis at the base of the liquid and the positive direction of the y-axis upward. In Example 2, we choose the x-axis along the surface of the liquid and the positive direction of the y-axis downward.

EXAMPLE ■ 1 One end of a reservoir presses against the wall of a small dam. The wall follows the depth contours of the reservoir and is generally in the shape of a parabola. If the wall of the dam is 60 ft deep at its center and 90 ft across at the water level, find the total force of the water in the reservoir against this wall of the dam.

Figure 5.77
$x = 45\sqrt{y/60}$



SOLUTION Figure 5.77 illustrates the end of the dam superimposed on a rectangular coordinate system. An equation for the parabola is $y = (60/45^2)x^2$, or, equivalently, $x = \pm 45\sqrt{y/60}$. Referring to Figure 5.77 gives us the following, for a horizontal rectangle of width dy:

$$\begin{aligned}
\text{length: } & 2x = 2 \cdot 45\sqrt{y/60} = 90\sqrt{y/60} \\
\text{area: } & 90\sqrt{y/60}\, dy \\
\text{depth: } & 60 - y \\
\text{pressure: } & 62.5(60 - y) \\
\text{force: } & 62.5(60 - y)\, 90\sqrt{y/60}\, dy
\end{aligned}$$

Taking a limit of sums by applying the operator $\int_0^{60}$, we obtain, as in Definition (5.27),

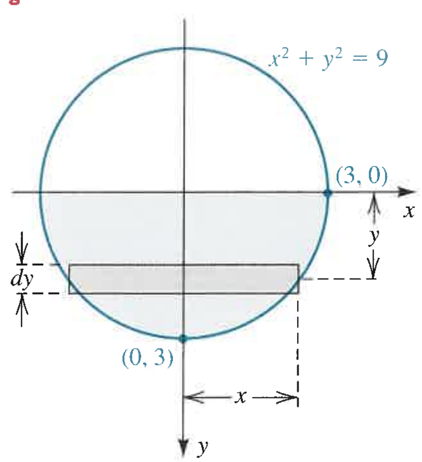
$$\begin{aligned}
F &= \int_0^{60} 62.5(60 - y)\, 90\sqrt{\frac{y}{60}}\, dy \\
&= \frac{(62.5)(90)}{\sqrt{60}} \int_0^{60} [(60 - y)y^{1/2}]\, dy \\
&= \frac{(62.5)(90)}{\sqrt{60}} \int_0^{60} [60y^{1/2} - y^{3/2}]\, dy \\
&= \frac{(62.5)(90)}{\sqrt{60}} \left[\frac{2}{3}(60y^{3/2}) - \frac{2}{5}y^{5/2}\right]_0^{60} \\
&= \frac{(62.5)(90)}{\sqrt{60}} \left[y^{1/2}\left(40y - \frac{2}{5}y^2\right)\right]_0^{60} \\
&= \frac{(62.5)(90)}{\sqrt{60}} \left\{\sqrt{60}\left[(40)(60) - \frac{2}{5}(60)^2\right]\right\} \\
&= (62.5)(90)[(40)(60) - (24)(60)] \\
&= (62.5)(90)(16)(60) = 5{,}400{,}000 \text{ lb.}
\end{aligned}$$

In the preceding example, the *length* of the reservoir is irrelevant when we consider the force on the dam. The same is true for the length of the oil tank in the next example.

EXAMPLE ▪ 2 A cylindrical oil storage tank 6 ft in diameter and 10 ft long is lying on its side. If the tank is half full of oil that weighs 58 lb/ft^3, set up an integral for the force exerted by the oil on one end of the tank.

SOLUTION Let us introduce a coordinate system such that the end of the tank is a circle of radius 3 ft with the center at the origin. The equation of the circle is $x^2 + y^2 = 9$. If we choose the positive direction of the y-axis *downward*, then referring to the horizontal rectangle in Figure 5.78 gives us the following:

Figure 5.78



$$\begin{aligned} \text{length:} &\quad 2x = 2\sqrt{9 - y^2} \\ \text{area:} &\quad 2\sqrt{9 - y^2}\, dy \\ \text{depth:} &\quad y \\ \text{pressure:} &\quad 58y \\ \text{force:} &\quad 58y \cdot 2\sqrt{9 - y^2}\, dy \end{aligned}$$

Taking a limit of sums by applying $\int_0^3$, we obtain

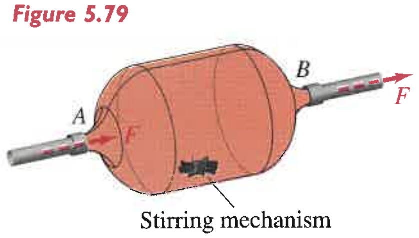
$$F = \int_0^3 116y\sqrt{9 - y^2}\, dy.$$

Evaluating the integral by using the method of substitution gives us

$$F = 1044 \text{ lb}.$$

Definite integrals can be applied to dye-dilution or tracer methods used in physiological tests and elsewhere. One example involves the measurement of cardiac output—that is, the rate at which blood flows through the aorta. A simple model for tracer experiments is sketched in Figure 5.79, where a liquid (or gas) flows into a tank at A and exits at B, with a constant flow rate F (in liters per second). Suppose that at time $t = 0$, Q_0 grams of tracer (or dye) are introduced into the tank at A and that a stirring mechanism thoroughly mixes the solution at all times. The concentration $c(t)$ (in grams per liter) of tracer at time t is monitored at B. Thus, the amount of tracer passing B at time t is given by

Figure 5.79



$$(\text{flow rate}) \cdot (\text{concentration}) = F \cdot c(t) \text{ g/sec}.$$

If the amount of tracer in the tank at time t is $Q(t)$, where Q is a differentiable function, then the rate of change $Q'(t)$ of Q is given by

$$Q'(t) = -F \cdot c(t)$$

(the negative sign indicates that Q is decreasing).

If T is a time at which all the tracer has left the tank, then $Q(T) = 0$ and, by the fundamental theorem of calculus,

$$\int_0^T Q'(t)\, dt = Q(t)\Big]_0^T = Q(T) - Q(0) = 0 - Q_0 = -Q_0.$$

We may also write

$$\int_0^T Q'(t)\, dt = \int_0^T [-F \cdot c(t)]\, dt = -F \int_0^T c(t)\, dt.$$

Equating the two forms for the integral gives us the following formula.

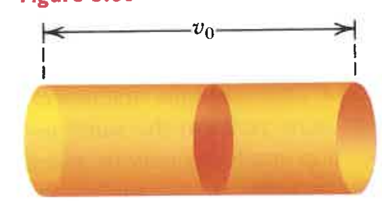
Flow Concentration Formula 5.28

$$Q_0 = F \int_0^T c(t)\, dt$$

Usually an explicit form for $c(t)$ will not be known, but, instead, a table of function values will be given. By using numerical integration, we may find an approximation to the flow rate F (see Exercises 11 and 12).

Let us next consider another aspect of the flow of liquids. If a liquid flows through a cylindrical tube and if the velocity is a constant v_0, then the volume of liquid passing a fixed point per unit time is given by $v_0 A$, where A is the area of a cross section of the tube (see Figure 5.80).

Figure 5.80



A more complicated formula is required to study the flow of blood in an arteriole. In this case, the flow is in layers, as illustrated in Figure 5.81. In the layer closest to the wall of the arteriole, the blood tends to stick to the wall, and its velocity may be considered zero. The velocity increases as the layers approach the center of the arteriole.

For computational purposes, we may regard the blood flow as consisting of thin cylindrical shells that slide along, with the outer shell fixed and the velocity of the shells increasing as the radii of the shells decrease (see Figure 5.81). If the velocity in each shell is considered constant, then from the theory of liquids in motion, the velocity $v(r)$ in a shell having average radius r is

Figure 5.81

$$v(r) = \frac{P}{4\nu l}(R^2 - r^2),$$

where R is the radius of the arteriole (in centimeters), l is the length of the arteriole (in centimeters), P is the pressure difference between the two ends of the arteriole (in dyn/cm^2), and ν is the viscosity of the blood (in dyn-sec/cm^2). Note that the formula gives zero velocity if $r = R$ and maximum velocity $PR^2/(4\nu l)$ as r approaches 0. If the radius of the kth shell is r_k and the thickness of the shell is Δr_k, then, by (5.10), the volume of blood in this shell is

$$2\pi r_k v(r_k)\Delta r_k = \frac{2\pi r_k P}{4\nu l}(R^2 - r_k^2)\Delta r_k.$$

If there are n shells, then the total flow in the arteriole per unit time may be approximated by

$$\sum_{k=1}^{n} \frac{2\pi r_k P}{4\nu l}(R^2 - r_k^2)\Delta r_k.$$

To estimate the total flow F (the volume of blood per unit time), we consider the limit of these sums as n increases without bound. This leads to the following definite integral:

$$\begin{aligned} F &= \int_0^R \frac{2\pi r P}{4\nu l}(R^2 - r^2)\,dr \\ &= \frac{2\pi P}{4\nu l}\int_0^R (R^2 r - r^3)\,dr \\ &= \frac{\pi P}{2\nu l}\left[\frac{1}{2}R^2 r^2 - \frac{1}{4}r^4\right]_0^R \\ &= \frac{\pi P R^4}{8\nu l}\ \text{cm}^3 \end{aligned}$$

This formula for F is not exact, because the thickness of the shells cannot be made arbitrarily small. The lower limit is the width of a red blood cell, or approximately 2×10^{-4} cm. We may assume, however, that the formula gives a reasonable estimate. It is interesting to observe that a small change in the radius of an arteriole produces a large change in the flow, since F is directly proportional to the fourth power of R. A small change in pressure difference has a lesser effect, since P appears to the first power.

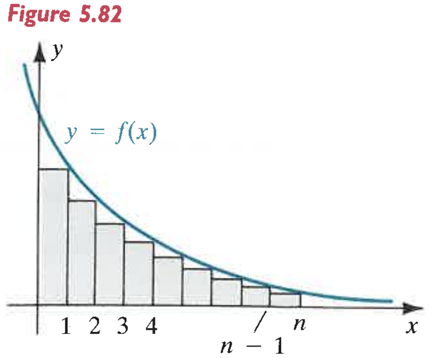
In many types of employment, a worker must perform the same assignment repeatedly. For example, a bicycle shop employee may be asked to assemble new bicycles. As more and more bicycles are assembled, the time required for each assembly should decrease until a certain minimum assembly time is reached. Another example of this process of learning by repetition is that of a data processor who must keyboard information from written forms into a computer system. The time required to process each entry should decrease as the number of entries increases. As a final illustration, the time required for a person to trace a path through a maze should improve with practice.

Let us consider a general situation in which a certain task is to be repeated many times. Suppose experience has shown that the time required to perform the task for the kth time can be approximated by $f(k)$ for a continuous decreasing function f on a suitable interval. The total time required to perform the task n times is given by the sum

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

If we consider the graph of f, then, as illustrated in Figure 5.82, the preceding sum equals the area of the pictured inscribed rectangular polygon and, therefore, may be approximated by the definite integral $\int_0^n f(x)\,dx$. Evidently, the approximation will be close to the actual sum if f decreases slowly on $[0, n]$. If f changes rapidly per unit change in x, then an integral should not be used as an approximation.

Figure 5.82



EXAMPLE 3 A company that conducts polls via telephone interviews finds that the time required by an employee to complete one interview depends on the number of interviews that the employee has completed previously. Suppose it is estimated that, for a certain survey, the number of minutes required to complete the kth interview is given by $f(k) = 6(1+k)^{-1/5}$ for $0 \le k \le 500$. Use a definite integral to approximate the time required for an employee to complete 100 interviews and 200 interviews. If an interviewer receives \$4.80 per hour, estimate how much more expensive it is to have two employees each conduct 100 interviews than to have one employee conduct 200 interviews.

SOLUTION From the preceding discussion, the time required for 100 interviews is approximately

$$\int_0^{100} 6(1+x)^{-1/5}\,dx = 6 \cdot \tfrac{5}{4}(1+x)^{4/5}\Big]_0^{100} \approx 293.5\ \text{min.}$$

The time required for 200 interviews is approximately

$$\int_0^{200} 6(1+x)^{-1/5}\,dx \approx 514.4\ \text{min.}$$

Since an interviewer receives \$0.08 per minute, the cost for one employee to conduct 200 interviews is roughly (\$0.08)(514.4), or \$41.15. If two employees each conduct 100 interviews, the cost is about 2(\$0.08)(293.5), or \$46.96, which is \$5.81 more than the cost of one employee. Note, however, that the time saved in using two people is approximately 221 min.

Using a computer, we have

$$\sum_{k=1}^{100} 6(1+k)^{-1/5} \approx 291.75$$

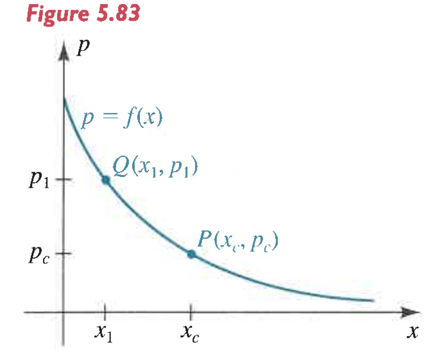
and

$$\sum_{k=1}^{200} 6(1+k)^{-1/5} \approx 512.57.$$

Hence, the results obtained by integration (the area under the graph of f) are roughly 2 min more than the value of the corresponding sum (the area of the inscribed rectangular polygon).

In economics, the price p at which there is a demand for x units of a particular product may be given by a function, $p = f(x)$. Figure 5.83 illustrates the graph of such a function, which is called the *price–demand curve*. It reflects the assumption that decreases in price correspond to increases in demand. Point $P(x_c, p_c)$ represents the current price p_c (in dollars) at any point in time and the corresponding current demand of x_c units. Point $Q(x_1, p_1)$ is the higher price ($p_1 > p_c$) consumers are hypothetically willing to pay for the same product when the demand is smaller ($x_1 < x_c$).

Figure 5.83



We can use a definite integral to determine the *consumer's surplus*, which is the savings or total difference between what they are willing to pay at higher prices and what they actually pay at the current price. We need to consider all possible prices greater than p_c dollars. From the price–demand curve, we see that the prices exceeding p_c dollars correspond to demands for fewer than x_c units. We partition the interval $[0, x_c]$ into n equal subintervals of width Δx and choose a point w_k in each subinterval. The corresponding price is $f(w_k)$, so the savings per unit is $[f(w_k) - p_c]$. If the price remained constant on the kth subinterval, then the savings to consumers over this subinterval would be

$$(\text{savings per unit}) \cdot (\text{number of units}) = [f(w_k) - p_c]\Delta x.$$

Thus, we can approximate the total savings by

$$\sum_{k=1}^{n} [f(w_k) - p_c]\Delta x.$$

This approximation improves as Δx approaches zero. But this sum is also a Riemann sum for $\int_0^{x_c} [f(x) - p_c]\,dx$ and so its limit as Δx approaches zero is the definite integral. We summarize our discussion in the next definition.

Definition 5.29

If (x_c, p_c) is a point representing current demand of x_c units of a particular good or service and current price p_c on the graph of a continuous price–demand function $p = f(x)$, then the **consumers' surplus** is given by

$$\int_0^{x_c} [f(x) - p_c]\,dx,$$

which represents the consumers' savings or total difference between what they are hypothetically willing to pay and what they actually pay.

EXAMPLE 4 The price–demand function for a particular product is given by $p = f(x) = 50 - \frac{1}{10}x$. Determine the consumers' surplus for this price–demand function at a price level of \$10.

SOLUTION For the price–demand function $p = 50 - (x/10)$, we note that when $x = 0$, $p = 50$. Thus, at a price of \$50, there is no demand for the product. When $x = 200$, $p = f(200) = 50 - (200/10) = 30$. At \$30 per unit, there is a demand for 200 units. To find the consumers' surplus, we first determine the demand x_c at the current price $p_c = 10$: Solving

$$10 = 50 - \tfrac{1}{10}x_c$$

for x_c yields $x_c = 400$. Thus, the consumers' surplus is given by the definite integral

$$\int_0^{400} [(50 - \tfrac{1}{10}x) - 10]\,dx = \int_0^{400} (40 - \tfrac{1}{10}x)\,dx$$
$$= \left[40x - \frac{x^2}{20}\right]_0^{400} = 8000,$$

and the consumers' surplus is \$8000.

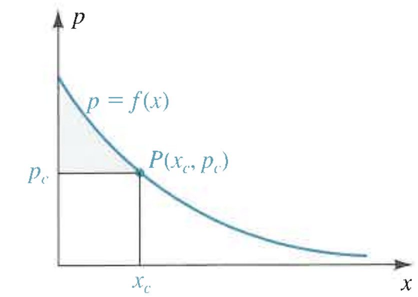


Figure 5.84

Note that by Theorem (4.23)(ii), the consumers' surplus is equal to

$$\int_0^{x_c} [f(x) - p_c]\,dx = \int_0^{x_c} f(x)\,dx - \int_0^{x_c} p_c\,dx.$$

By Theorem (4.21),

$$\int_0^{x_c} p_c\,dx = p_c x_c.$$

The product $p_c x_c$ is the total amount paid by consumers for x_c units at the current price of p_c. Since $f(x) > 0$ for $0 < x < x_c$, the definite integral

$$\int_0^{x_c} f(x)\,dx$$

is the area under the price–demand curve between 0 and x_c, and we can interpret the consumers' surplus at a price level of p_c to be the amount by which the area under the price–demand curve exceeds the total amount paid for demanded goods at the current price level. The area of the shaded region in Figure 5.84 represents the consumers' surplus.

Any quantity that can be interpreted as an area of a region in a plane may be investigated by means of a definite integral. Conversely, definite integrals allow us to represent physical quantities as areas. In the following illustrations, a quantity is *numerically equal* to an area of a region; that is, we *disregard units of measurement*, such as centimeter, foot-pound, and so on.

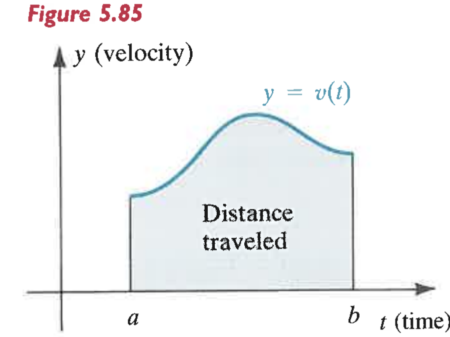


Figure 5.85

Suppose $v(t)$ is the velocity, at time t, of an object that is moving on a coordinate line. If s is the position function, then $s'(t) = v(t)$ and

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(t)\Big]_a^b = s(b) - s(a).$$

If $v(t) > 0$ throughout the time interval $[a, b]$, this tells us that the area under the graph of the function v from a to b represents the distance that the object travels, as illustrated in Figure 5.85. This observation is useful to an engineer or physicist, who may not have an explicit form for $v(t)$ but merely a graph (or table) indicating the velocity at various times. The distance traveled may then be estimated by approximating the area under the graph.

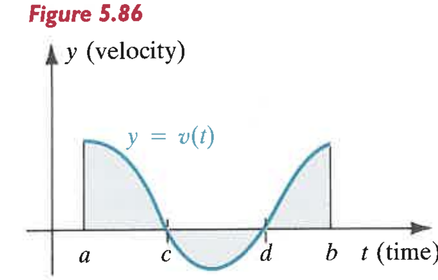


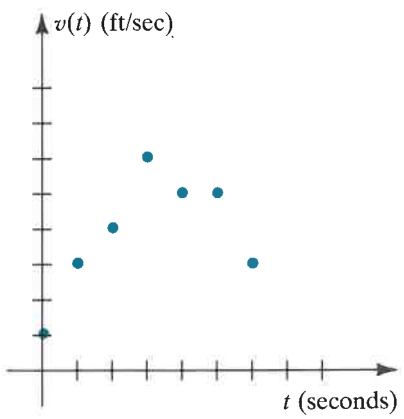
Figure 5.86

If $v(t) < 0$ at certain times in $[a, b]$, the graph of v may resemble that in Figure 5.86. The figure indicates that the object moved in the negative direction from $t = c$ to $t = d$. The distance that it traveled during that time is given by $\int_c^d |v(t)|\,dt$. It follows that $\int_a^b |v(t)|\,dt$ is the *total* distance traveled in $[a, b]$, whether $v(t)$ is positive or negative.

EXAMPLE 5 As an object moves along a straight path, its velocity $v(t)$ (in feet per second) at time t is recorded each second for 6 sec. The results are given in the following table.

t	0	1	2	3	4	5	6
$v(t)$	1	3	4	6	5	5	3

Figure 5.87

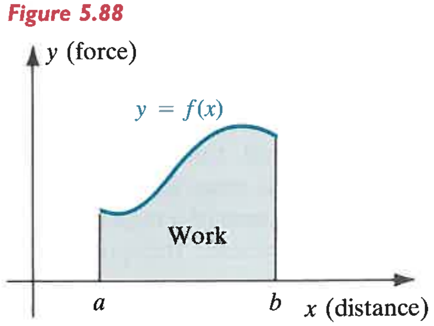


Approximate the distance traveled by the object.

SOLUTION The points $(t, v(t))$ are plotted in Figure 5.87. If we assume that v is a continuous function, then, as in the preceding discussion, the distance traveled during the time interval $[0, 6]$ is $\int_0^6 v(t)\,dt$. Let us approximate this definite integral by means of Simpson's rule, with $n = 3$:

$$\begin{aligned}\int_0^6 v(t)\,dt &\approx \frac{6-0}{6\cdot 3}[v(0) + 4v(1) + 2v(2) + 4v(3) \\ &\quad + 2v(4) + 4v(5) + v(6)] \\ &= \tfrac{1}{3}[1 + 4\cdot 3 + 2\cdot 4 + 4\cdot 6 + 2\cdot 5 + 4\cdot 5 + 3] = 26 \text{ ft}\end{aligned}$$

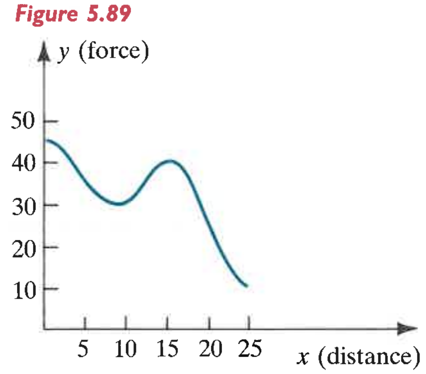
Figure 5.88



In (5.21), we defined the work W done by a variable force $f(x)$ that acts along a coordinate line from $x = a$ to $x = b$ by $W = \int_a^b f(x)\,dx$. Suppose that $f(x) \geq 0$ throughout $[a, b]$. If we sketch the graph of f as illustrated in Figure 5.88, then the work W is numerically equal to the area under the graph from a to b.

EXAMPLE 6 An engineer obtains the graph in Figure 5.89, which shows the force (in pounds) acting on a small cart as it moves 25 ft along horizontal ground. Estimate the work done.

Figure 5.89



SOLUTION If we assume that the force is a continuous function f for $0 \leq x \leq 25$, then the work done is

$$W = \int_0^{25} f(x)\,dx.$$

We do not have an explicit form for $f(x)$; however, we may estimate function values from the graph and approximate W by means of numerical integration.

Let us apply the trapezoidal rule with $a = 0$, $b = 25$, and $n = 5$. Referring to the graph to estimate function values gives us the table on the following page.

k	x_k	$f(x_k)$
0	0	45
1	5	35
2	10	30
3	15	40
4	20	25
5	25	10

Since $(b - a)/(2n) = (25 - 0)/10 = 2.5$, the trapezoidal rule (4.37) gives

$$\begin{aligned}T_5 &= (2.5)[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)] \\ &= (2.5)[45 + 70 + 60 + 80 + 50 + 10] = (2.5)(315) = 790.\end{aligned}$$

It follows that

$$W = \int_0^{25} f(x)\,dx \approx 790 \text{ ft-lb.}$$

Suppose that the amount of a physical entity, such as oil, water, electric power, money supply, bacteria count, or blood flow, is increasing or decreasing in some manner, and that $R(t)$ is the rate at which it is changing at time t. If $Q(t)$ is the amount of the entity present at time t and if Q is differentiable, then $Q'(t) = R(t)$. If $R(t) > 0$ (or $R(t) < 0$) in a time interval $[a, b]$, then the amount that the entity increases (or decreases) between $t = a$ and $t = b$ is

$$Q(b) - Q(a) = \int_a^b Q'(t)\,dt = \int_a^b R(t)\,dt.$$

This number may be represented as the area of the region in a ty-plane bounded by the graphs of R, $t = a$, $t = b$, and $y = 0$.

EXAMPLE 7 Starting at 9:00 A.M., oil is pumped into a storage tank at a rate of $(150t^{1/2} + 25)$ gal/hr, for time t (in hours) after 9:00 A.M. How many gallons will have been pumped into the tank at 1:00 P.M.?

SOLUTION Letting $R(t) = 150t^{1/2} + 25$ in the preceding discussion, we obtain the following:

$$\begin{aligned}\int_0^4 (150t^{1/2} + 25)\,dt &= \Big[100t^{3/2} + 25t\Big]_0^4 \\ &= 900 \text{ gal}\end{aligned}$$

We have given only a few illustrations of the use of definite integrals. The interested reader may find many more in books on the physical and biological sciences, economics, and business, and even such areas as political science and sociology.

EXERCISES 5.8

1 A glass aquarium tank is 3 ft long and has square ends of width 1 ft. If the tank is filled with water, find the force exerted by the water on

(a) one end **(b)** one side

2 If one of the square ends of the tank in Exercise 1 is divided into two parts by means of a diagonal, find the force exerted on each part.

3 The ends of a water trough 6 ft long have the shape of isosceles triangles with equal sides of length 2 ft and the third side of length $2\sqrt{3}$ ft at the top of the trough. Find the force exerted by the water on one end of the t if the trough is

(a) full of water **(b)** half full of water

4 The ends of a water trough have the shape of the region bounded by the graphs of $y = x^2$ and $y = 4$, with x and y measured in feet. If the trough is full of water, find the force on one end.

5 A cylindrical oil storage tank 4 ft in diameter and 5 ft long is lying on its side. If the tank is half full of oil weighing 60 lb/ft^3, find the force exerted by the oil on one end of the tank.

6 A rectangular gate in a dam is 5 ft long and 3 ft high. If the gate is vertical, with the top of the gate parallel to the surface of the water and 6 ft below it, find the force of the water against the gate.

7 A plate having the shape of an isosceles trapezoid with upper base 4 ft long and lower base 8 ft long is submerged vertically in water such that the bases are parallel to the surface. If the distances from the surface of the water to the lower and upper bases are 10 ft and 6 ft, respectively, find the force exerted by the water on one side of the plate.

8 A circular plate of radius 2 ft is submerged vertically in water. If the distance from the surface of the water to the center of the plate is 6 ft, find the force exerted by the water on one side of the plate.

9 A rectangular plate 3 ft wide and 6 ft long is submerged vertically in oil weighing 50 lb/ft^3, with its short side parallel to, and 2 ft below, the surface.

(a) Find the total force exerted on one side of the plate.

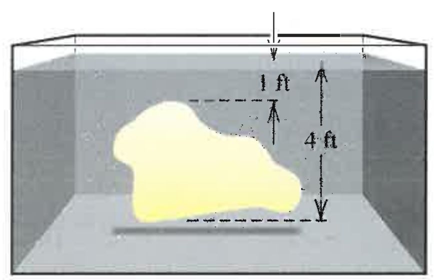
(b) If the plate is divided into two parts by means of a diagonal, find the force exerted on each part.

c **10** A flat, irregularly shaped plate is submerged vertically in water (see figure). Measurements of its width, taken at successive depths at intervals of 0.5 ft, are compiled in the following table.

Water depth (ft)	1	1.5	2	2.5	3	3.5	4
Width of plate (ft)	0	2	3	5.5	4.5	3.5	0

Estimate the force of the water on one side of the plate by using **(a)** the trapezoidal rule, with $n = 6$, and **(b)** Simpson's rule, with $n = 3$.

Exercise 10



c **11** Refer to (5.28). To estimate cardiac output F (the number of liters of blood per minute that the heart pumps through the aorta), a 5-mg dose of the tracer indocyanine-green is injected into a pulmonary artery, and dye concentration measurements $c(t)$ are taken every minute from a peripheral artery near the aorta. The results are given in the following table. Use Simpson's rule, with $n = 6$, to estimate the cardiac output.

t (min)	$c(t)$ (mg/L)
0	0
1	0
2	0.15
3	0.48
4	0.86
5	0.72
6	0.48
7	0.26
8	0.15
9	0.09
10	0.05
11	0.01
12	0

c **12** Refer to (5.28). Suppose that 1200 kg of sodium dichromate is mixed into a river at point A, and sodium dichromate samples are taken every 30 sec at a point B downstream. The concentration $c(t)$ at time t is recorded in the following table. Use the trapezoidal rule, with $n = 12$, to estimate the river flow rate F.

t (sec)	$c(t)$ (mg/L or g/m^3)
0	0
30	2.14
60	3.89
90	5.81
120	8.95
150	7.31
180	6.15
210	4.89
240	2.98
270	1.42
300	0.89
330	0.29
360	0

13 A manufacturer estimates that the time required for a worker to assemble a certain item depends on the number of this item the worker has previously assembled. If the time (in minutes) required to assemble the kth item is given by $f(k) = 20(k+1)^{-0.4} + 3$, use a definite integral to approximate the time, to the nearest minute, required to assemble

(a) 1 item **(b)** 4 items

(c) 8 items **(d)** 16 items

14 A data processor keyboards registration data for college students from written forms to electronic files. The number of minutes required to process the kth registration is estimated to be approximately $f(k) = 6(1+k)^{-1/3}$. Use a definite integral to estimate the time required for

(a) one person to keyboard 600 registrations

(b) two people to keyboard 300 registrations each

15 The number of minutes needed for a person to trace a path through a certain maze without error is estimated to be $f(k) = 5(1+k)^{-1/2}$, where k is the number of trials previously completed. Use a definite integral to approximate the time required to complete 10 trials.

16 Anne has found that if she is making string necklaces, it takes her $7(2+k)^{-2/3}$ minutes to complete the kth necklace. Use a definite integral to estimate the time that she needs to finish 10 necklaces.

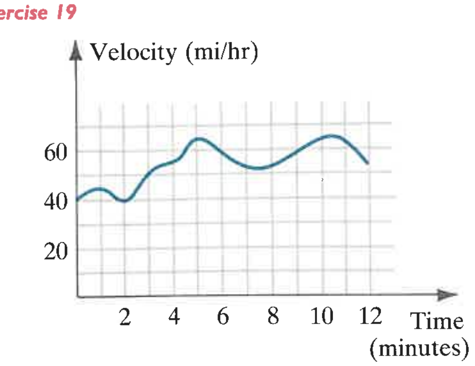
Exer. 17–18: Use a definite integral to approximate the sum, and round the answer to the nearest integer.

17 $\displaystyle\sum_{k=1}^{100} k(k^2+1)^{-1/4}$

18 $\displaystyle\sum_{k=1}^{200} 5k(k^2+10)^{-1/3}$

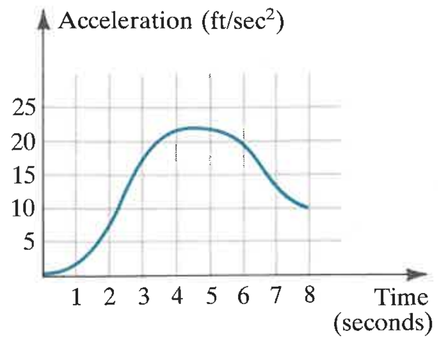
c **19** The velocity (in miles per hour) of an automobile as it traveled along a freeway over a 12-min interval is indicated in the figure. Use the trapezoidal rule to approximate the distance traveled to the nearest mile.

Exercise 19



c **20** The acceleration (in feet per second per second) of an automobile over a period of 8 sec is indicated in the figure. Use the trapezoidal rule to approximate the net change in velocity in this time period.

Exercise 20



21 The following table was obtained by recording the force $f(x)$ (in Newtons) acting on a particle as it moved 6 m along a coordinate line from $x = 1$ to $x = 7$. Estimate the work done using

(a) the trapezoidal rule, with $n = 6$

(b) Simpson's rule, with $n = 3$

x	1	2	3	4	5	6	7
$f(x)$	20	23	25	22	26	30	28

22 A bicyclist pedals directly up a hill, recording the velocity $v(t)$ (in feet per second) at the end of every 2 sec. Referring to the results recorded in the following table, use the trapezoidal rule to approximate the distance traveled.

t	0	2	4	6	8	10
$v(t)$	24	22	16	10	2	0

23 A motorboat uses gasoline at the rate of $t\sqrt{9 - t^2}$ gal/hr. If the motor is started at $t = 0$, how much gasoline is used in 2 hr?

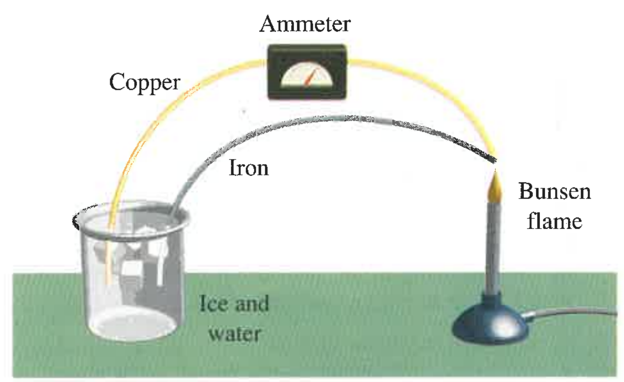
24 The population of a city has increased since 1985 at a rate of $1.5 + 0.3\sqrt{t} + 0.006t^2$ thousand people per year, where t is the number of years after 1985. Assuming that this rate continues and that the population was 50,000 in 1985, estimate the population in 1994.

25 A simple thermocouple, in which heat is transformed into electrical energy, is shown in the figure. To determine the total charge Q (in coulombs) transferred to the copper wire, current readings (in amperes) are recorded every $\frac{1}{2}$ sec, and the results are shown in the following table.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0
I (amp)	0	0.2	0.6	0.7	0.8	0.5	0.2

Use the fact that $I = dQ/dt$ and the trapezoidal rule, with $n = 6$, to estimate the total charge transferred to the copper wire during the first 3 sec.

Exercise 25



c **26** Suppose that $\rho(x)$ is the density (in centimeters per kilometer) of ozone in the atmosphere at a height of x kilometers above the ground. For example, if $\rho(6) = 0.0052$, then at a height of 6 km there is effectively a thickness of 0.0052 cm of ozone for each kilometer of atmosphere. If ρ is a continuous function, the thickness of the ozone layer between heights a and b can be found by evaluating $\int_a^b \rho(x)\,dx$. Values for $\rho(x)$ found experimentally are shown in the following table.

x (km)	$\rho(x)$ (spring)	$\rho(x)$ (autumn)
0	0.0034	0.0038
6	0.0052	0.0043
12	0.0124	0.0076
18	0.0132	0.0104
24	0.0136	0.0109
30	0.0084	0.0072
36	0.0034	0.0034
42	0.0017	0.0016

(a) Use the trapezoidal rule to estimate the thickness of the ozone layer between the altitudes of 6 and 42 km during both spring and autumn.

(b) Work part (a) using Simpson's rule.

c **27** Radon gas can pose a serious health hazard if inhaled. If $V(t)$ is the volume of air (in cubic centimeters) in an adult's lungs at time t (in minutes), then the rate of change of V can often be approximated by

$$V'(t) = 12{,}450\pi \sin(30\pi t).$$

Inhaling and exhaling correspond to $V'(t) > 0$ and $V'(t) < 0$, respectively. Suppose an adult lives in a home that has a radioactive energy concentration due to radon of 4.1×10^{-12} joule/cm^3.

(a) Approximate the volume of air inhaled by the adult with each breath.

(b) If inhaling more than 0.02 joule of radioactive energy in one year is considered hazardous, is it safe for the adult to remain at home?

28 A stationary exercise bicycle is programmed so that it can be set for different intensity levels L and workout times T. It displays the elapsed time t (in minutes), for $0 \le t \le T$, and the number of calories $C(t)$ that are being burned per minute at time t, where

$$C(t) = 5 + 3L - 6\frac{L}{T}\left|t - \frac{1}{2}T\right|.$$

Suppose that an individual exercises for 16 min, with $L = 3$ for $0 \le t \le 8$ and with $L = 2$ for $8 \le t \le 16$. Find the total number of calories burned during the workout.

29 The rate of growth R (in centimeters per year) of an average boy who is t years old is shown in the following table for $10 \le t \le 15$.

t (yr)	10	11	12	13	14	15
R (cm/yr)	5.3	5.2	4.9	6.5	9.3	7.0

Use the trapezoidal rule, with $n = 5$, to approximate the number of centimeters the boy grows between his tenth and fifteenth birthdays.

30 To determine the number of zooplankton in a portion of an ocean 80 m deep, marine biologists take samples at successive depths of 10 m, obtaining the following table, where $\rho(x)$ is the density (in number per cubic meter) of zooplankton at a depth of x meters.

x	0	10	20	30	40	50	60	70	80
$\rho(x)$	0	10	25	30	20	15	10	5	0

Use Simpson's rule, with $n = 4$, to estimate the total number of zooplankton in a *water column* (a column of water) having a cross section 1 m square extending from the surface to the ocean floor.

Exer. 31–34: Find the consumers' surplus for the given demand function $f(x)$ and the given price level p_c.

31 $f(x) = 20 - \frac{1}{20}x; \quad p_c = 4$

32 $f(x) = 30 - \frac{1}{5}x; \quad p_c = 10$

33 $f(x) = 400 - \frac{3}{8}x; \quad p_c = 100$

34 $f(x) = 60 - \frac{2}{7}x; \quad p_c = 40$

CHAPTER 5 REVIEW EXERCISES

Exer. 1–2: Sketch the region bounded by the graphs of the equations, and find the area by integrating with respect to (a) x and (b) y.

1 $y = -x^2, \quad y = x^2 - 8$

2 $y^2 = 4 - x, \quad x + 2y = 1$

Exer. 3–4: Find the area of the region bounded by the graphs of the equations.

3 $x = y^2, \quad x + y = 1$

4 $y = -x^3, \quad y = \sqrt{x}, \quad 7x + 3y = 10$

5 Find the area of the region between the graphs of the equations $y = \cos\frac{1}{2}x$ and $y = \sin x$, from $x = \pi/3$ to $x = \pi$.

6 The region bounded by the graph of $y = \sqrt{1 + \cos 2x}$ and the x-axis, from $x = 0$ to $x = \pi/2$, is revolved about the x-axis. Find the volume of the resulting solid.

Exer. 7–10: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated by revolving R about the indicated axis.

7 $y = \sqrt{4x + 1}, \quad y = 0, \quad x = 0, \quad x = 2; \quad x$-axis

8 $y = x^4$, $y = 0$, $x = 1$; y-axis

9 $y = x^3 + 1$, $x = 0$, $y = 2$; y-axis

10 $y = \sqrt[3]{x}$, $y = \sqrt{x}$; x-axis

Exer. 11–12: The region bounded by the x-axis and the graph of the given equation, from $x = 0$ to $x = b$, is revolved about the y-axis. Find the volume of the resulting solid.

11 $y = \cos x^2$; $b = \sqrt{\pi/2}$

12 $y = x \sin x^3$; $b = 1$

13 Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 4x^2$ and $4x + y = 8$ about

(a) the x-axis **(b)** $x = 1$ **(c)** $y = 16$

14 Find the volume of the solid generated by revolving the region bounded by the graphs of $y = x^3$, $x = 2$, and $y = 0$ about

(a) the x-axis **(b)** the y-axis **(c)** $x = 2$
(d) $x = 3$ **(e)** $y = 8$ **(f)** $y = -1$

15 Find the arc length of the graph of $(x + 3)^2 = 8(y - 1)^3$ from $A(-2, \frac{3}{2})$ to $B(5, 3)$.

16 A solid has for its base the region in the xy-plane bounded by the graphs of $y^2 = 4x$ and $x = 4$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles right triangle with one of its equal sides on the base of the solid.

17 An above-ground swimming pool has the shape of a right circular cylinder of diameter 12 ft and height 5 ft. If the depth of the water in the pool is 4 ft, find the work required to empty the pool by pumping the water out over the top.

18 As a bucket is raised a distance of 30 ft from the bottom of a well, water leaks out at a uniform rate. Find the work done if the bucket originally contains 24 lb of water and one-third leaks out. Assume that the weight of the empty bucket is 4 lb, and disregard the weight of the rope.

19 A square plate of side 4 ft is submerged vertically in water such that one of the diagonals is parallel to the surface. If the distance from the surface of the water to the center of the plate is 6 ft, find the force exerted by the water on one side of the plate.

20 Use differentials to approximate the arc length of the graph of $y = 2 \sin \frac{1}{3}x$ between the points with x-coordinates π and $91\pi/90$.

Exer. 21–22: Sketch the region bounded by the graphs of the equations, and find m, M_x, M_y, and the centroid.

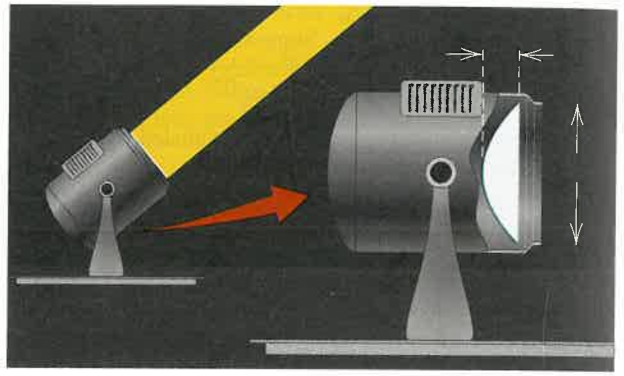
21 $y = x^3 + 1$, $x + y = -1$, $x = 1$

22 $y = x^2 + 1$, $y = x$, $x = -1$, $x = 2$

23 The graph of the equation $12y = 4x^3 + (3/x)$ from $A(1, \frac{7}{12})$ to $B(2, \frac{67}{24})$ is revolved about the x-axis. Find the area of the resulting surface.

24 The shape of a reflector in a searchlight is obtained by revolving a parabola about its axis. If, as shown in the figure, the reflector is 4 ft across at the opening and 1 ft deep, find its surface area.

Exercise 24



25 The velocity $v(t)$ of a rocket that is traveling directly upward is given in the following table. Use the trapezoidal rule to approximate the distance that the rocket travels from $t = 0$ to $t = 5$.

t (sec)	0	1	2	3	4	5
$v(t)$ (ft/sec)	100	120	150	190	240	300

c **26** An electrician suspects that a meter showing the total consumption Q in kilowatt hours (kWh) of electricity is not functioning properly. To check the accuracy, the electrician measures the consumption rate R directly every 10 min, obtaining the results in the following table.

t (min)	0	10	20	30
R (kWh/min)	1.31	1.43	1.45	1.39

t (min)	40	50	60
R (kWh/min)	1.36	1.47	1.29

(a) Use Simpson's rule to estimate the total consumption during this 1-hr period.

(b) If the meter read 48,792 kWh at the beginning of the experiment and 48,953 kWh at the end, what should the electrician conclude?

27 Interpret $\int_0^1 2\pi x^4\,dx$ in the following ways:

(a) as the area of a region in the xy-plane

(b) as the volume of a solid obtained by revolving a region in the xy-plane about

(i) the x-axis

(ii) the y-axis

(c) as the work done by a force

28 Let R be the semicircular region in the xy-plane with endpoints of its diameter at (4, 0) and (10, 0). Use the theorem of Pappus to find the volume of the solid obtained by revolving R about the y-axis.

c **Exer. 29–32: Plot the graphs of the equations. (a) Approximate the points of intersection. (b) Approximate the area bounded by the graphs of the equations.**

29 $y = \sqrt{1 + x^3}$; $y = x^2$

30 $y = 5e^{-x^2}$; $y = \ln(x + 4)$

31 $y = x^3 - 4x^2 - x + 3$; $y = \sqrt{20x}$

32 $y = \sin(\sin x)$; $y = \sin(\cos x)$;
$x = -\frac{3\pi}{4}$; $x = \frac{\pi}{4}$

EXTENDED PROBLEMS AND GROUP PROJECTS

1 Explore an alternative approach to determining the length of the graph of a function between points A and B using the notation that we developed in Section 5.5. In particular, Q_k is the point with coordinates $(x_k, f(x_k))$. Now let P_k be the point $(x_k, f(x_{k+1}))$, and let

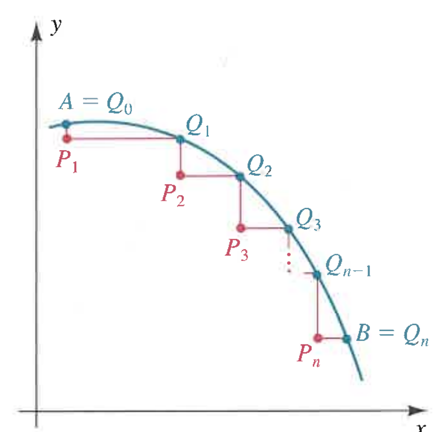
$$T_P = \sum_{k=1}^{n} [d(Q_{k-1}, P_k) + d(P_k, Q_k)].$$

(a) Show that P_k lies on a vertical line through Q_{k-1} and a horizontal line through Q_k (see figure).

(b) Discuss why T_P appears to be a good approximation for the length of the graph between A and B. In particular, show that as $\|P\|$ decreases, the distance between each P_k and the curve approaches zero.

(c) Discuss why T_P is not a good approximation for the length of the graph.

Problem 1



c **2** The model for a football using the approximate shape of the solid generated by revolving the arc of a circle $x^2 + (y + k)^2 = r^2$, where $y \geq 0$ and $0 < k < r$, does not quite match measurable dimensions. For a full-sized football, the distance from endpoint to endpoint along the axis of revolution is about 11 in. and the arc on the surface from endpoint to endpoint along a seam is about 14 in. long. Around the widest part, the circumference measures about 22 in. Explore using the shape of the solid generated by revolving the arc of an ellipse $a^2x^2 + (y + k)^2 = r^2$, where $y \geq 0$ and $0 < k < r$. Approximate

(a) the volume

(b) the surface area for this new model for a football

3 Let f be a smooth function with $f(x) \geq 0$ on $[a, b]$. Partition the interval $[a, b]$ into n subintervals of equal width, and inscribe a rectangle under the graph of f over each subinterval. Then revolve each rectangle about the x-axis. Determine the surface area of the resulting solid, and let R_n be the sum of these surface areas. Let S be the area of the surface generated by revolving the graph of f about the x-axis. In what sense is R_n a good approximation to S? Will the limit of R_n as $n \to \infty$ be equal to S? Will we do better by taking *circumscribed* rather than inscribed rectangles?

INTRODUCTION

IN 1948, THE FINNISH-BORN AMERICAN ARCHITECT Eero Saarinen (1910–1961) submitted the winning design for a new national park, the Thomas Jefferson Westward Expansion Memorial in St. Louis. The center of his design was a great gleaming stainless-steel arch. Saarinen wanted "to create a monument which would have lasting significance and would be a landmark of our time. An absolutely simple shape . . . seemed to be the basis of the great memorials that have kept their significance and dignity over time." Saarinen designed his arch to be the purest expression of the forces within. This arch . . . is a catenary curve—the curve of a hanging chain—a curve in which the forces of thrust are continuously kept within the center of the legs of the arch. The mathematical precision seemed to enhance the timelessness of the form, but at the same time its dynamic quality seemed to link it to our own time.

To understand the mathematics of Saarinen's Gateway Arch to the West, we need to examine the natural exponential function. This function and its inverse, the natural logarithm, are perhaps the most important functions in applications of calculus to the natural world. They are examples of *transcendental functions*, the main topic of this chapter. We begin in Section 6.1 with a brief review of inverse functions and develop a formula for the derivative of an inverse function that will be useful throughout the entire chapter. Next, we employ a definite integral to introduce in Section 6.2 the *natural logarithm function*, which is then used to define in Section 6.3 the *natural exponential function* as the inverse of the natural logarithm. The natural logarithmic and exponential functions occur in many indefinite integral problems, a number of which are studied in Section 6.4. There are many other pairs of exponential and logarithmic functions; we analyze the general case in Section 6.5. After developing the theory of logarithms and exponentials, we explore in Section 6.6 a number of applications that involve these functions as solutions to first-order separable differential equations, an important modeling tool.

In Sections 6.7 and 6.8, we introduce other important transcendental functions: the inverse trigonometric functions and the hyperbolic functions and their inverses. We derive the equation for the catenary curve as an application of the hyperbolic functions. The chapter concludes with l'Hôpital's rule, which provides a direct way to evaluate limits of quotients in which both the numerator and the denominator approach 0 or both approach ∞ or $-\infty$. Such limits often occur when dealing with transcendental functions.

CHAPTER 6

Transcendental functions frequently occur in the descriptions of curves that possess both aesthetic appeal and important structural properties of stability.

Transcendental Functions