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INTRODUCTION

IN DESIGNING A DAM and projecting its cost, engineers must determine how much concrete is needed for construction. This amount depends on the volume of the dam. The volume, in turn, is a function of the shape of the dam and its thickness at various levels. The dam must be thick enough to withstand the force of the enormous amount of water held in the dam's reservoir. To compute the force of the water, the required thickness at each height, and the resulting volume of concrete, engineers set up and evaluate numerous definite integrals.

Hoover Dam, for example, which supplies much electrical power and water for a large region of the American Southwest, is one of the world's largest concrete dams. The dam is 726 ft high, the equivalent of a 50-story building, and 1244 ft wide, and its reservoir, Lake Mead, can store approximately 1.3 trillion ft^3 of water. To withstand the resulting pressure, the dam's base is 660 ft thick, and the total amount of concrete is over 118 million ft^3, enough to pave a two-lane highway from San Francisco to New York.

In this chapter, we discuss some of the many uses for the definite integral. We begin by reconsidering in Section 5.1 the application that motivated the definition of this mathematical concept: determining the area of a region in the xy-plane. Then, in turn, we use definite integrals to find volumes (Sections 5.2–5.4), lengths of graphs and surface areas of solids (Section 5.5), work done by a variable force (Section 5.6), and moments and the center of mass (the balance point) of a flat plate (Section 5.7). Definite integrals are applicable because each of these quantities can be expressed as a limit of sums.

Because of the multitude of other quantities that can be similarly expressed, the definite integral is useful in a wide variety of applications, some of which are considered in Section 5.8: finding the force exerted by a liquid against a wall (water on a dam, gasoline on one end of a storage tank, oil on the walls of an ocean tanker), measuring cardiac output and blood flow in arteries, estimating the future wealth of a corporation, calculating the thickness of the ozone layer, determining the amount of radon gas in a home, and finding the number of calories burned during a workout on an exercise bicycle.

As you proceed through this chapter and whenever you encounter definite integrals in applications, keep the following words in mind: *limit of sums, limit of sums, limit of sums.*

The design of large engineering projects such as a dam requires the calculation of many physical quantities that are most accurately described by definite integrals.

CHAPTER 5

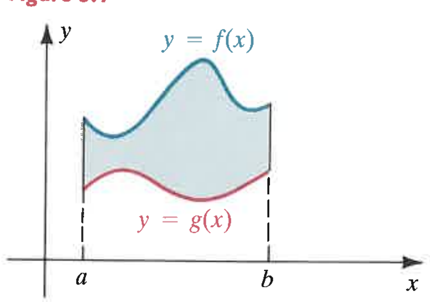
Applications of the Definite Integral

5.1 AREA

If a function f is continuous and $f(x) \geq 0$ on $[a, b]$, then, by Theorem (4.19), the area of the region under the graph of f from a to b is given by the definite integral $\int_a^b f(x)\,dx$. In this section, we consider the region that lies *between* the graphs of two functions.

If f and g are continuous and $f(x) \geq g(x) \geq 0$ for every x in $[a, b]$, then the area A of the region R bounded by the graphs of f, g, $x = a$, and $x = b$ (see Figure 5.1) can be found by subtracting the area of the region under the graph of g (the **lower boundary** of R) from the area of the region under the graph of f (the **upper boundary** of R), as follows:

Figure 5.1



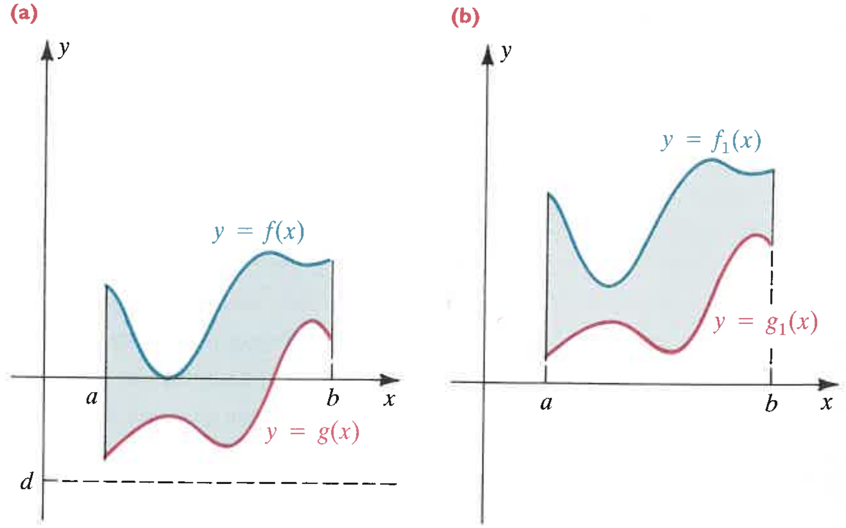
$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \int_a^b [f(x) - g(x)]\,dx.$$

This formula for A is also true if f or g is negative for some x in $[a, b]$. To verify this fact, choose a *negative* number d that is less than the minimum value of g on $[a, b]$, as illustrated in Figure 5.2(a). Next, consider the functions f_1 and g_1, defined as follows:

$$f_1(x) = f(x) - d = f(x) + |d|$$
$$g_1(x) = g(x) - d = g(x) + |d|$$

The graphs of f_1 and g_1 can be obtained by vertically shifting the graphs of f and g a distance $|d|$. If A is the area of the region in Figure 5.2(b),

Figure 5.2



then

$$A = \int_a^b [f_1(x) - g_1(x)]\,dx$$
$$= \int_a^b [(f(x) - d) - (g(x) - d)]\,dx$$
$$= \int_a^b [f(x) - g(x)]\,dx.$$

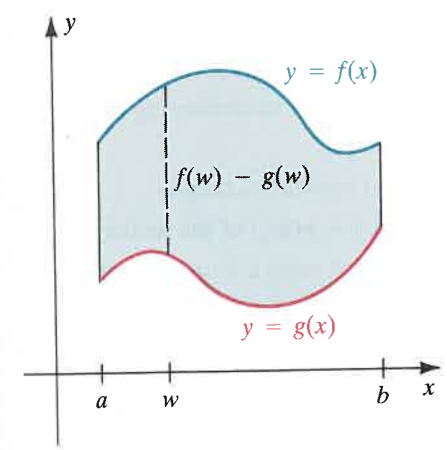
We may summarize our discussion as follows.

Theorem 5.1

If f and g are continuous and $f(x) \geq g(x)$ for every x in $[a, b]$, then the area A of the region bounded by the graphs of f, g, $x = a$, and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Figure 5.3

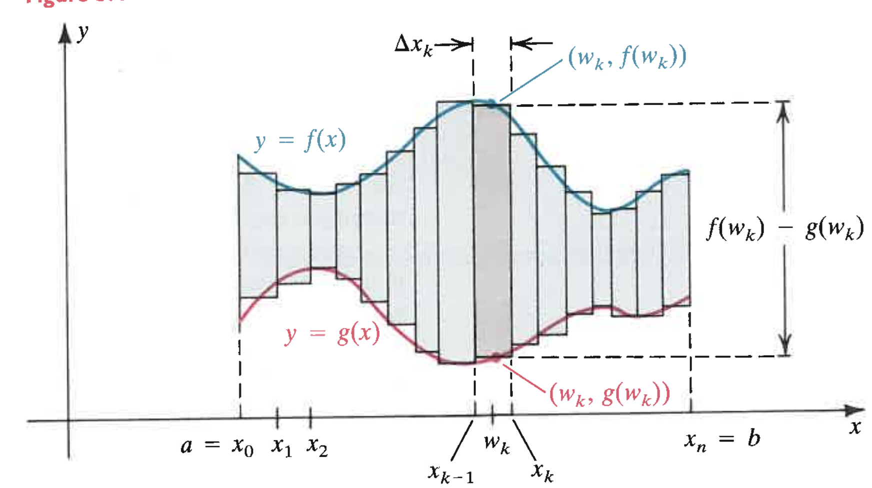


We may interpret the formula for A in Theorem (5.1) as a limit of sums. If we let $h(x) = f(x) - g(x)$ and if w is in $[a, b]$, then $h(w)$ is the vertical distance between the graphs of f and g for $x = w$ (see Figure 5.3). As in our discussion of Riemann sums in Chapter 4, let P denote a partition of $[a, b]$ determined by $a = x_0, x_1, \ldots, x_n = b$. For each k, let $\Delta x_k = x_k - x_{k-1}$, and let w_k be any number in the kth subinterval $[x_{k-1}, x_k]$ of P. By the definition of h,

$$h(w_k)\Delta x_k = [f(w_k) - g(w_k)]\Delta x_k,$$

which is the area of the rectangle of length $f(w_k) - g(w_k)$ and width Δx_k shown in Figure 5.4.

Figure 5.4



The Riemann sum

$$\sum_k h(w_k)\Delta x_k = \sum_k [f(w_k) - g(w_k)]\Delta x_k$$

is the sum of the areas of the rectangles in Figure 5.4 and is therefore an approximation to the area of the region between the graphs of f and g from a to b. By the definition of the definite integral,

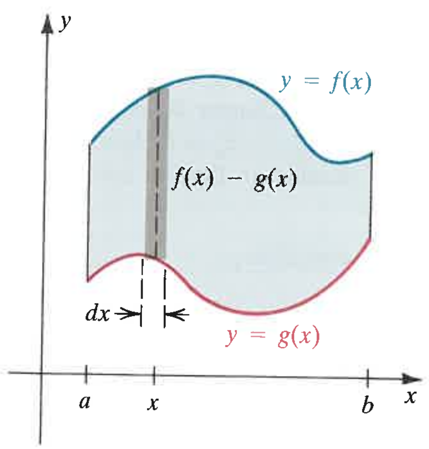
$$\lim_{\|P\|\to 0} \sum_k h(w_k)\Delta x_k = \int_a^b h(x)\,dx.$$

Since $h(x) = f(x) - g(x)$, we obtain the following corollary of Theorem (5.1).

Corollary 5.2

$$A = \lim_{\|P\|\to 0} \sum_k [f(w_k) - g(w_k)]\Delta x_k = \int_a^b [f(x) - g(x)]\,dx$$

Figure 5.5



We may use the following intuitive method for remembering this limit of sums formula (see Figure 5.5):

1. Use dx for the width Δx_k of a typical vertical rectangle.
2. Use $f(x) - g(x)$ for the length $f(w_k) - g(w_k)$ of the rectangle.
3. Regard the symbol $\int_a^b$ as an operator that takes a limit of sums of the rectangular areas $[f(x) - g(x)]\,dx$.

This method allows us to interpret the area formula in Theorem (5.1) as follows:

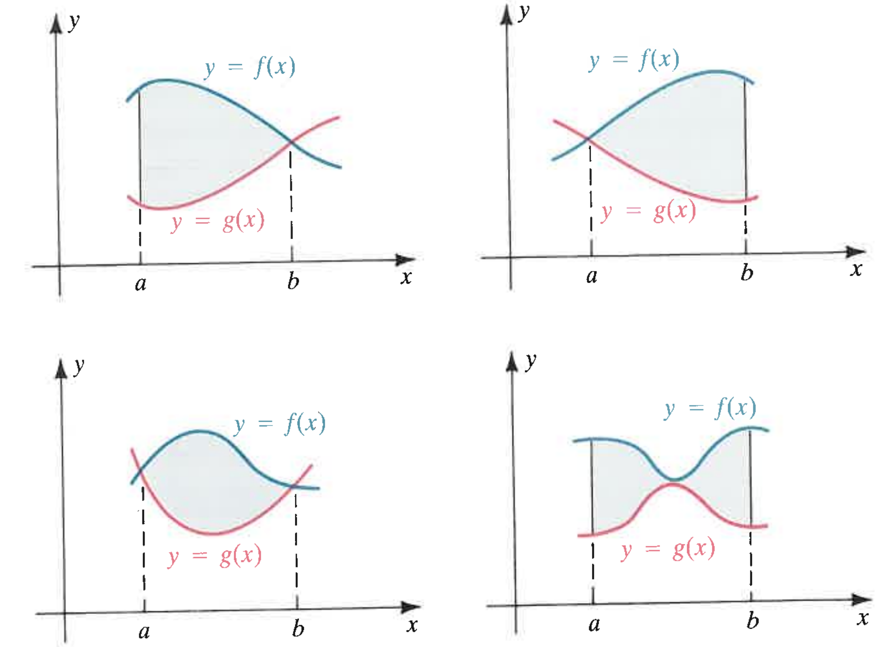
$$A = \int_a^b [f(x) - g(x)]\,dx$$

limit of sums — length of a rectangle — width of a rectangle

When using this technique, we visualize summing areas of vertical rectangles by moving through the region from left to right. Later in this section, we consider different types of regions, finding areas by using *horizontal* rectangles and integrating with respect to y.

Let us call a region an R_x **region** (for integration with respect to x) if it lies between the graphs of two equations $y = f(x)$ and $y = g(x)$, with f and g continuous, and $f(x) \geq g(x)$ for every x in $[a, b]$, where a and b are the smallest and largest x-coordinates, respectively, of the points (x, y) in the region. The regions in Figures 5.1–5.5 are R_x regions. Several others are sketched in Figure 5.6 on the following page. Note that the graphs of $y = f(x)$ and $y = g(x)$ may intersect one or more times; however, $f(x) \geq g(x)$ throughout the interval.

Figure 5.6 R_x regions



The following guidelines may be helpful when working problems.

Guidelines for Finding the Area of an R_x Region 5.3

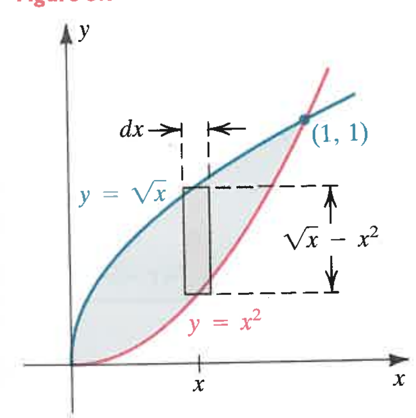
1. Sketch the region, labeling the upper boundary $y = f(x)$ and the lower boundary $y = g(x)$. Find the smallest value $x = a$ and the largest value $x = b$ for points (x, y) in the region.
2. Sketch a typical vertical rectangle and label its width dx.
3. Express the area of the rectangle in guideline (2) as

$$[f(x) - g(x)]\,dx.$$

4. Apply the limit of sums operator $\int_a^b$ to the expression in guideline (3) and evaluate the integral.

EXAMPLE ■ 1 Find the area of the region bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$.

Figure 5.7



SOLUTION Following guidelines (1)–(3), we sketch and label the region and show a typical vertical rectangle (see Figure 5.7). The points (0, 0) and (1, 1) at which the graphs intersect can be found by solving the equations $y = x^2$ and $y = \sqrt{x}$ simultaneously. Referring to the figure, we obtain the following facts:

upper boundary:	$y = \sqrt{x}$
lower boundary:	$y = x^2$
width of rectangle:	dx
length of rectangle:	$\sqrt{x} - x^2$
area of rectangle:	$(\sqrt{x} - x^2)\,dx$

Next, we follow guideline (4) with $a = 0$ and $b = 1$, remembering that applying $\int_0^1$ to the expression $(\sqrt{x} - x^2)\,dx$ represents taking a limit of sums of areas of vertical rectangles. We thus obtain

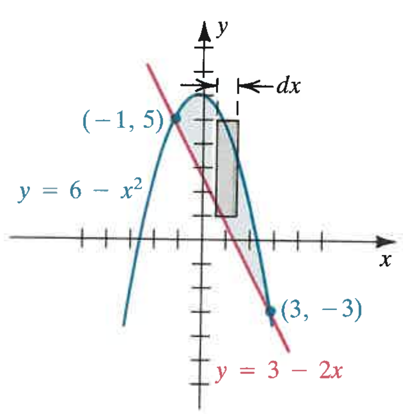
$$A = \int_0^1 (\sqrt{x} - x^2)\,dx = \int_0^1 (x^{1/2} - x^2)\,dx$$

$$= \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

EXAMPLE ■ 2 Find the area of the region bounded by the graphs of $y + x^2 = 6$ and $y + 2x - 3 = 0$.

SOLUTION The region and a typical rectangle are sketched in Figure 5.8. The points of intersection $(-1, 5)$ and $(3, -3)$ of the two graphs may be found by solving the two given equations simultaneously. To apply guideline (1), we must label the upper and lower boundaries $y = f(x)$ and $y = g(x)$, respectively, and hence we solve each of the given equations for y in terms of x, as shown in Figure 5.8. Here we obtain

Figure 5.8



upper boundary:	$y = 6 - x^2$
lower boundary:	$y = 3 - 2x$
width of rectangle:	dx
length of rectangle:	$(6 - x^2) - (3 - 2x)$
area of rectangle:	$[(6 - x^2) - (3 - 2x)]\,dx$

Next, we use guideline (4), with $a = -1$ and $b = 3$, regarding $\int_{-1}^3$ as an operator that takes a limit of sums of areas of rectangles. Thus,

$$A = \int_{-1}^3 [(6 - x^2) - (3 - 2x)]\,dx = \int_{-1}^3 (3 - x^2 + 2x)\,dx$$

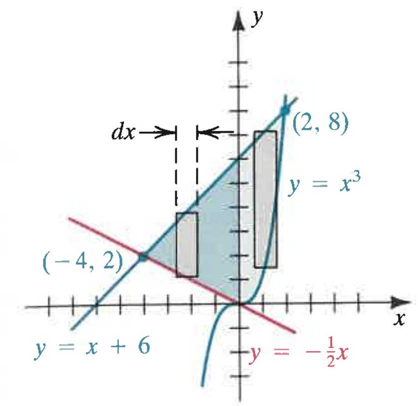
$$= \left[3x - \frac{x^3}{3} + x^2\right]_{-1}^3$$

$$= [9 - \tfrac{27}{3} + 9] - [-3 - (-\tfrac{1}{3}) + 1] = \tfrac{32}{3}.$$

The following example illustrates that it is sometimes necessary to subdivide a region into several R_x regions and then use more than one definite integral to find the area.

EXAMPLE ■ 3 Find the area of the region R bounded by the graphs of $y - x = 6$, $y - x^3 = 0$, and $2y + x = 0$.

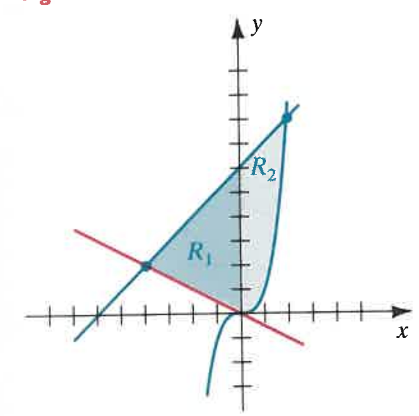
Figure 5.9



SOLUTION The graphs and the region are sketched in Figure 5.9. Each equation has been solved for y in terms of x, and the boundaries have been labeled as in guideline (1). Typical vertical rectangles are shown extending from the lower boundary to the upper boundary of R. Since the lower boundary consists of portions of two different graphs, the area cannot be found by using only one definite integral. However, if R is divided into two R_x regions, R_1 and R_2, as shown in Figure 5.10, then we can determine the area of each and add them together. Let us arrange our work as follows.

	Region R_1	Region R_2
upper boundary:	$y = x + 6$	$y = x + 6$
lower boundary:	$y = -\frac{1}{2}x$	$y = x^3$
width of rectangle:	dx	dx
length of rectangle:	$(x + 6) - (-\frac{1}{2}x)$	$(x + 6) - x^3$
area of rectangle:	$[(x + 6) - (-\frac{1}{2}x)]\,dx$	$[(x + 6) - x^3]\,dx$

Figure 5.10



Applying guideline (4), we find the areas A_1 and A_2 of R_1 and R_2:

$$A_1 = \int_{-4}^0 [(x + 6) - (-\tfrac{1}{2}x)]\,dx$$

$$= \int_{-4}^0 \left(\frac{3}{2}x + 6\right) dx = \left[\frac{3}{2}\left(\frac{x^2}{2}\right) + 6x\right]_{-4}^0$$

$$= 0 - (12 - 24) = 12$$

$$A_2 = \int_0^2 [(x + 6) - x^3]\,dx$$

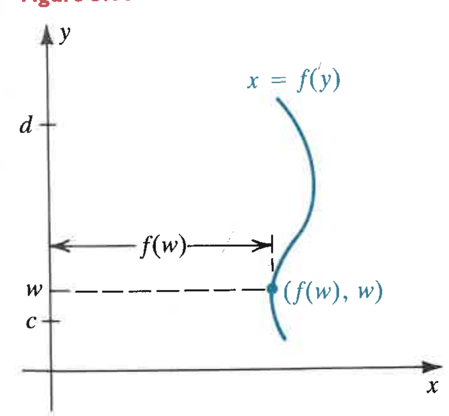
$$= \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_0^2$$

$$= (2 + 12 - 4) - 0 = 10$$

The area A of the entire region R is

$$A = A_1 + A_2 = 12 + 10 = 22.$$

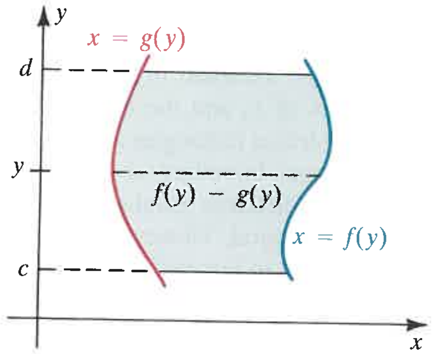
Figure 5.11



We have now evaluated many integrals similar to those in Example 3. For this reason, we sometimes merely *set up* an integral—that is, we express it in the proper form but do not find its numerical value.

If we consider an equation of the form $x = f(y)$, where f is continuous for $c \leq y \leq d$, then we *reverse the roles of x and y in the previous discussion, treating y as the independent variable and x as the dependent variable.* A typical graph of $x = f(y)$ is sketched in Figure 5.11. Note that if a value w is assigned to y, then *$f(w)$ is an x-coordinate* of the corresponding point on the graph.

Figure 5.12



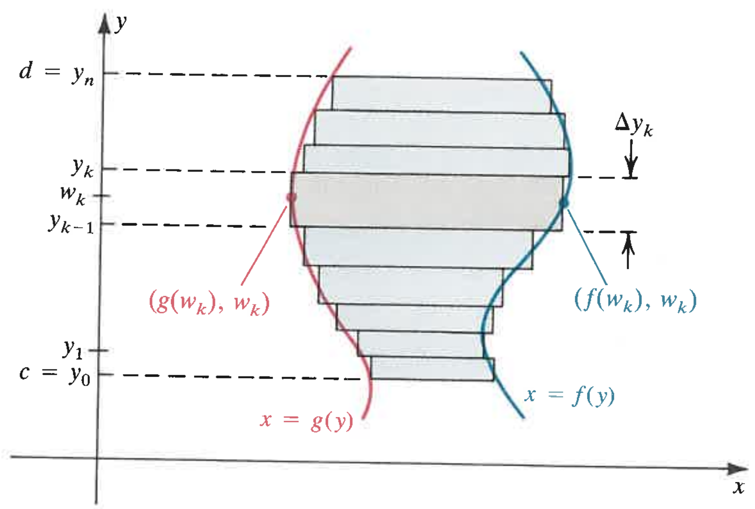
An R_y **region** is a region that lies between the graphs of two equations of the form $x = f(y)$ and $x = g(y)$, with f and g continuous, and with $f(y) \geq g(y)$ for every y in $[c, d]$, where c and d are the smallest and largest y-coordinates, respectively, of points in the region. One such region is illustrated in Figure 5.12. We call the graph of f the **right boundary** of the region and the graph of g the **left boundary**. For any y, the number $f(y) - g(y)$ is the horizontal distance between these boundaries, as shown in Figure 5.12.

We can use limits of sums to find the area A of an R_y region. We begin by selecting points on the y-axis with y-coordinates $c = y_0, y_1, \ldots, y_n = d$, obtaining a partition of the interval $[c, d]$ into subintervals of width $\Delta y_k = y_k - y_{k-1}$. For each k, we choose a number w_k in $[y_{k-1}, y_k]$ and consider horizontal rectangles that have areas $[f(w_k) - g(w_k)]\Delta y_k$, as illustrated in Figure 5.13. This procedure leads to

$$A = \lim_{\|P\|\to 0} \sum_k [f(w_k) - g(w_k)]\Delta y_k = \int_c^d [f(y) - g(y)]\,dy.$$

The last equality follows from the definition of the definite integral.

Figure 5.13



Using notation similar to that for R_x regions, we represent the width Δy_k of a horizontal rectangle by dy and the length $f(w_k) - g(w_k)$ of the rectangle by $f(y) - g(y)$ in the following guidelines.

Guidelines for Finding the Area of an R_y Region 5.4

1 Sketch the region, labeling the right boundary $x = f(y)$ and the left boundary $x = g(y)$. Find the smallest value $y = c$ and the largest value $y = d$ for points (x, y) in the region.

2 Sketch a typical horizontal rectangle and label its width dy.

3 Express the area of the rectangle in guideline (2) as

$$[f(y) - g(y)]\,dy.$$

4 Apply the limit of sums operator $\int_c^d$ to the expression in guideline (3) and evaluate the integral.

In guideline (4), we visualize summing areas of horizontal rectangles by moving from the lowest point of the region to the highest point.

Figure 5.14

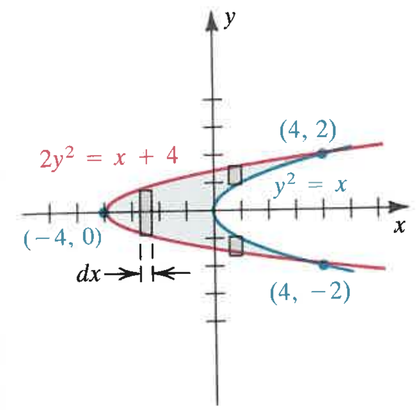
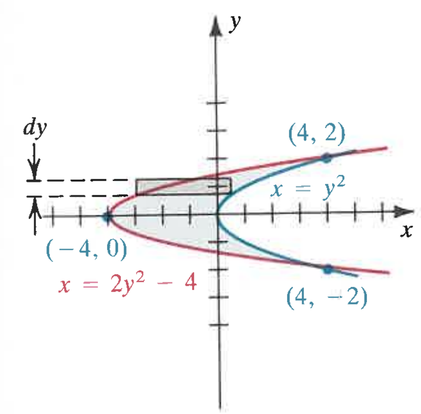


Figure 5.15



EXAMPLE 4 Find the area of the region bounded by the graphs of the equations $2y^2 = x + 4$ and $y^2 = x$.

SOLUTION The region is sketched in Figures 5.14 and 5.15. Figure 5.14 illustrates the use of vertical rectangles (integration with respect to x), and Figure 5.15 illustrates the use of horizontal rectangles (integration with respect to y). Referring to Figure 5.14, we see that several integrations with respect to x are required to find the area. However, for Figure 5.15, we need only one integration with respect to y. Thus we apply Guidelines (5.4), solving each equation for x in terms of y. Referring to Figure 5.15, we obtain the following:

right boundary:	$x = y^2$
left boundary:	$x = 2y^2 - 4$
width of rectangle:	dy
length of rectangle:	$y^2 - (2y^2 - 4)$
area of rectangle:	$[y^2 - (2y^2 - 4)]\,dy$

We could now use guideline (4) with $c = -2$ and $d = 2$, finding A by applying the operator $\int_{-2}^{2}$ to $[y^2 - (2y^2 - 4)]\,dy$. Another method is to use the symmetry of the region with respect to the x-axis and find A by doubling the area of the part that lies above the x-axis. Thus,

$$A = \int_{-2}^{2} [y^2 - (2y^2 - 4)]\,dy$$
$$= 2\int_0^2 (4 - y^2)\,dy$$
$$= 2\left[4y - \frac{y^3}{3}\right]_0^2 = 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}.$$

In following Guidelines (5.3) or (5.4) for finding the area of a region, we may need to use a graphing utility and numerical methods to obtain an accurate sketch of the region, find the smallest and largest x- or y-values in the region, and approximate the area. Our next example illustrates such a case.

EXAMPLE 5 For the region of the plane bounded by the curves $y = \cos(0.3x^2)$ and $y = x^2 + 0.6x - 2$,

(a) use a graphing utility to sketch the curves and determine the region

(b) find numerical approximations for the intersection points of the bounding curves

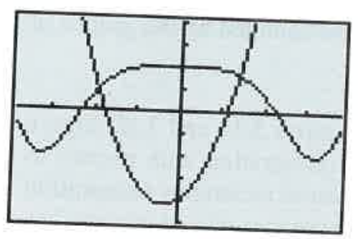
(c) set up a definite integral representing the area of the region

(d) approximate the area

SOLUTION

(a) After examining several different viewing windows, we obtain the view of the desired area shown in Figure 5.16.

Figure 5.16
$-3.8 \le x \le 3.8, \quad -2.5 \le y \le 2$

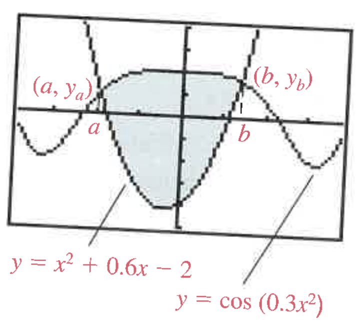


(b) By tracing on the graph, we obtain first approximations for the intersection points as $(-1.9, 0.5)$ and $(1.4, 0.8)$. At the intersection points, the bounding curves have equal y-values. Thus, $\cos(0.3x^2) = x^2 + 0.6x - 2$, so $\cos(0.3x^2) - [x^2 + 0.6x - 2] = 0$. We can use a solving routine or apply Newton's method to the function

$$f(x) = \cos(0.3x^2) - x^2 - 0.6x + 2$$

with starting values of $x = -1.9$ and then $x = 1.4$ to obtain values $a \approx -1.89968629228$ and $b \approx 1.40826496779$. Substituting these values into the equation for either bounding curve gives the other coordinates for the points of intersection, $y_a \approx 0.468996233702$ and $y_b \approx 0.828169200187$.

Figure 5.17



(c) In Figure 5.17, we have labeled the bounding curves and points of intersection and shaded the region. We see there that the graph of $y = \cos(0.3x^2)$ is above the graph of $y = x^2 + 0.6x - 2$ on the interval $[a, b]$. The integral representing the area of the region is

$$A = \int_a^b [\cos(0.3x^2) - (x^2 + 0.6x - 2)]\,dx$$

$$\approx \int_{-1.89968629228}^{1.40826496779} [\cos(0.3x^2) - x^2 - 0.6x + 2]\,dx.$$

(d) We compute numerical approximations for the definite integral in part (c) using Simpson's rule for several different values of n to approximate the area. For example, when $n = 64$, $A \approx 6.93542681443$ and when $n = 128$, $A \approx 6.93542681577$. Thus, we have confidence in the approximation that, to seven decimal places, $A \approx 6.9354268$.

As an application of the area between two curves, let us consider what economists call **capital formation**—that is, the process of increasing or decreasing a given holding of capital over time. If $K(t)$ is the amount of capital at time t, then dK/dt denotes the **rate of capital formation**. Economists consider the rate of capital formation to be identical to the **net investment flow**, which we will denote by $I(t)$. We can look at the

relationship between capital formation and net investment flow in two ways: in a derivative formulation,

$$\frac{dK}{dt} = I(t)$$

and in an integral form,

$$K(t) = \int I(t)\,dt.$$

Note that with $I(t) \ge 0$ for $a \le t \le b$, the amount of capital accumulation in this time interval is $\int_a^b I(t)\,dt$, the area under the graph of the function $I(t)$.

If we know the amount of capital $K(t)$ accumulated at time t, we may differentiate with respect to t to find the investment flow. Alternatively, if we are given the investment flow $I(t)$, we may integrate with respect to t to find the amount of capital—that is, $K(t)$ represents the total change in capital or the capital accumulation. As a derivative, the investment flow $I(t)$ is a rate of change of capital. That is, the value of $I(t)$ at a particular time t is the rate at which investment is flowing in or out of the given holding of capital, measured in units of capital per unit of time. For example, if $I(t) = 4 - t^2 + 2t$, where capital is measured in millions of dollars and time is measured in years, at time $t = 1$ year, we have $I(1) = 4 - 1 + 2 = 5$ million dollars per year. Hence, capital is increasing at an annual rate of \$5 million. At $t = 4$, $I(4) = 4 - 16 + 8 = -4$, so at time $t = 4$ years, capital is decreasing at an annual rate of \$4 million.

EXAMPLE 6 If the net investment flow is $I(t) = 4 - t^2 + 2t$ millions of dollars per time unit, find the capital formation during the time interval $[1, 2]$.

SOLUTION The capital formation is given by

$$\int_1^2 I(t)\,dt = \int_1^2 (4 - t^2 + 2t)\,dt.$$

We can evaluate the definite integral by finding an antiderivative for $I(t)$:

$$\int_1^2 (4 - t^2 + 2t)\,dt = \left[4t - \frac{t^3}{3} + t^2\right]_1^2$$

$$= \left[8 - \frac{8}{3} + 4\right] - \left[4 - \frac{1}{3} + 1\right] = 4\frac{2}{3}$$

Thus, the capital accumulation is about \$4.67 million.

EXAMPLE 7 Consider two different net investment flows given by $I_1(t) = 4 - t^2 + 2t$ and $I_2(t) = 4 - t$ (both in millions of dollars per year at year t).

(a) Find the time interval during which the first investment flow I_1 is at least as great as the second investment flow I_2.

(b) For the time interval found in part (a), determine how much more capital accumulates under the first investment flow than under the second investment flow.

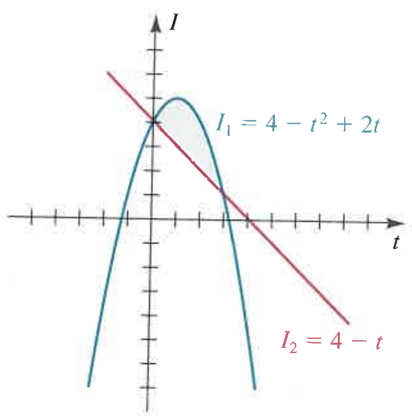


Figure 5.18

SOLUTION

(a) We need to find the interval $[a, b]$ during which $I_1(t) \geq I_2(t)$. We first sketch the graphs of the two functions (Figure 5.18) and then find the points of intersection by solving the equations $y = 4 - t^2 + 2t$ and $y = 4 - t$ simultaneously:

$$\begin{aligned} 4 - t^2 + 2t &= 4 - t \\ 3t - t^2 &= 0 \\ t(3 - t) &= 0 \\ t = 0 \quad \text{and} \quad t &= 3 \end{aligned}$$

Thus, we see from the graph that $I_1(t) \geq I_2(t)$ on the interval $[0, 3]$. If $t = 0$ corresponds to the present time, then the investment flow I_1 will exceed the investment flow I_2 for the next three years.

(b) The difference in capital accumulation between the two investment flows is the area of the region between the two curves I_1 and I_2 over the interval $[0, 3]$. That is,

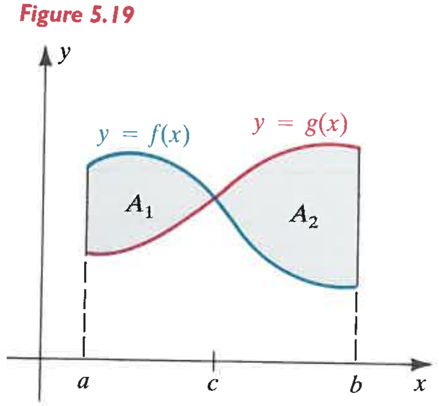
$$\begin{aligned} \int_0^3 [I_1(t) - I_2(t)]\,dt &= \int_0^3 [4 - t^2 + 2t - (4 - t)]\,dt \\ &= \int_0^3 [3t - t^2]\,dt \\ &= \left[\frac{3t^2}{2} - \frac{t^3}{3}\right]_0^3 = \left[\frac{27}{2} - \frac{27}{3}\right] - [0 - 0] = \frac{9}{2}. \end{aligned}$$

Thus, the first investment flow will generate \$4.5 million more in accumulated capital than the second investment flow during the next three years.

Throughout this section, we have assumed that the graphs of the functions (or equations) do not cross one another in the interval under discussion. If the graphs of f and g cross at one point $P(c, d)$, with $a < c < b$, and we wish to find the area bounded by the graphs from $x = a$ to $x = b$, then the methods developed in this section may still be used; however, *two* integrations are required, one corresponding to the interval $[a, c]$ and the other to $[c, b]$, as is illustrated in Figure 5.19, with $f(x) \geq g(x)$ on $[a, c]$ and $g(x) \geq f(x)$ on $[c, b]$. The area A is given by

$$A = A_1 + A_2 = \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx.$$

Figure 5.19

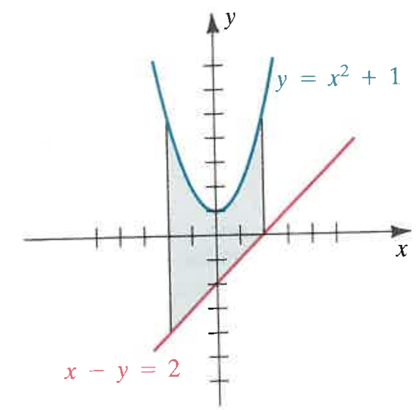


If the graphs cross *several* times, then several integrals may be necessary. Problems in which graphs cross one or more times appear in Exercises 31–36.

EXERCISES 5.1

Exer. 1–4: Set up an integral that can be used to find the area of the shaded region.

1



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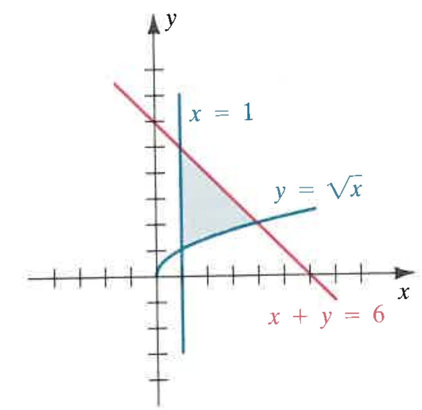

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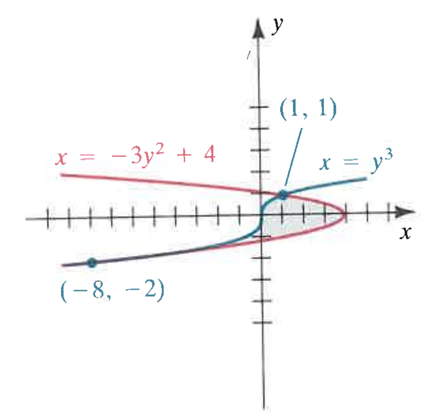

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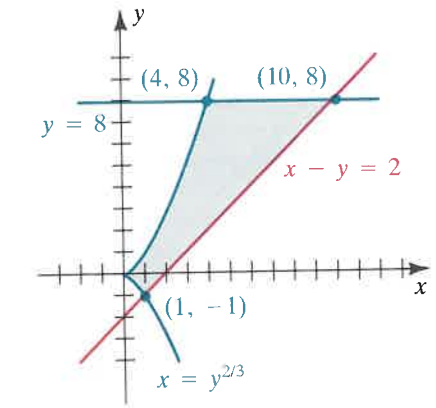


Exer. 5–22: Sketch the region bounded by the graphs of the equations and find its area.

5	$y = x^2$;	$y = 4x$		
6	$x + y = 3$;	$y + x^2 = 3$		
7	$y = x^2 + 1$;	$y = 5$		
8	$y = 4 - x^2$;	$y = -4$		
9	$y = 1/x^2$;	$y = -x^2$;	$x = 1$;	$x = 2$
10	$y = x^3$;	$y = x^2$		
11	$y^2 = -x$;	$x - y = 4$;	$y = -1$;	$y = 2$
12	$x = y^2$;	$y - x = 2$;	$y = -2$;	$y = 3$
13	$y^2 = 4 + x$;	$y^2 + x = 2$		
14	$x = y^2$;	$x - y = 2$		
15	$x = 4y - y^3$;	$x = 0$		
16	$x = y^{2/3}$;	$x = y^2$		
17	$y = x^3 - x$;	$y = 0$		
18	$y = x^3 - x^2 - 6x$;	$y = 0$		
19	$x = y^3 + 2y^2 - 3y$;	$x = 0$		
20	$x = 9y - y^3$;	$x = 0$		
21	$y = x\sqrt{4 - x^2}$;	$y = 0$		
22	$y = x\sqrt{x^2 - 9}$;	$y = 0$;	$x = 5$	

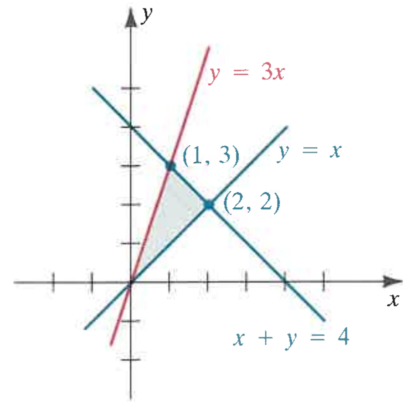
Exer. 23–24: Find the area of the region between the graphs of the two equations from $x = 0$ to $x = \pi$.

23 $y = \sin 4x$; $\quad y = 1 + \cos \frac{1}{3}x$

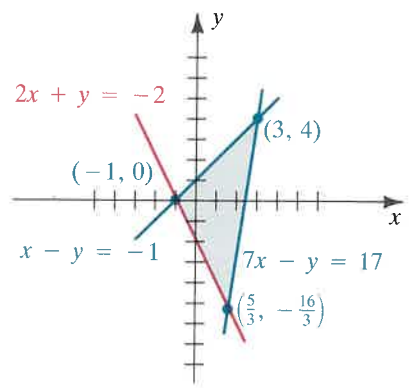
24 $y = 4 + \cos 2x$; $\quad y = 3 \sin \frac{1}{2}x$

Exer. 25–26: Set up sums of integrals that can be used to find the area of the shaded region by integrating with respect to (a) x and (b) y.

25



26



Exer. 27–30: Set up sums of integrals that can be used to find the area of the region bounded by the graphs of the equations by integrating with respect to (a) x and (b) y.

27 $y = \sqrt{x}$; $y = -x$; $x = 1$; $x = 4$

28 $y = 1 - x^2$; $y = x - 1$

29 $y = x + 3$; $x = -y^2 + 3$

30 $x = y^2$; $x = 2y^2 - 4$

Exer. 31–36: Find the area of the region between the graphs of f and g if x is restricted to the given interval.

31 $f(x) = 6 - 3x^2$; $g(x) = 3x$; $[0, 2]$

32 $f(x) = x^2 - 4$; $g(x) = x + 2$; $[1, 4]$

33 $f(x) = x^3 - 4x + 2$; $g(x) = 2$; $[-1, 3]$

34 $f(x) = x^2$; $g(x) = x^3$; $[-1, 2]$

35 $f(x) = \sin x$; $g(x) = \cos x$; $[0, 2\pi]$

36 $f(x) = \sin x$; $g(x) = \frac{1}{2}$; $[0, \pi/2]$

Exer. 37–38: Let R be the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. Set up a sum of integrals, not containing the absolute value symbol, that can be used to find the area of R.

37 $f(x) = |x^2 - 6x + 5|$; $a = 0$, $b = 7$

38 $f(x) = |-x^2 + 2x + 3|$; $a = -3$, $b = 4$

39 Show that the area of the region bounded by an ellipse whose major and minor axes have lengths $2a$ and $2b$, respectively, is πab. (*Hint:* Use an equation of the ellipse to show first that the area is given by $2(b/a)\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx$, and then interpret the definite integral as the area of a semicircle of radius a.)

40 Suppose that the function values of f and g in the following table were obtained empirically. Assuming that f and g are continuous, approximate the area between their graphs from $x = 1$ to $x = 5$ using **(a)** the trapezoidal rule, with $n = 8$, and **(b)** Simpson's rule, with $n = 4$.

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	3.5	2.5	3	4	3.5	2.5	2	2	3
$g(x)$	1.5	2	2	1.5	1	0.5	1	1.5	1

c **41** Graph $f(x) = |x^3 - 0.7x^2 - 0.8x + 1.3|$ on $[-1.5, 1.5]$. Set up a sum of integrals, not containing the absolute value symbol, that can be used to approximate the area of the region bounded by the graph of f, the x-axis, and the lines $x = -1.5$ and $x = 1.5$.

c **42** Graph, on the same coordinate axes, $f(x) = \sin x$ and $g(x) = x^3 - x + 0.2$ for $-2 \le x \le 2$. Set up a sum of integrals that can be used to approximate the area of the region bounded by the graphs.

c **Exer. 43–46: Plot the graphs of the equations. (a) Find numerical approximations for the intersection points of the different bounding curves. (b) Set up a definite integral representing the area of the bounded region. (c) Approximate this area to four-decimal-place accuracy using Simpson's rule.**

43 $y = x^3 - 2x^2 - x + 1$; $y = \sqrt{10x}$

44 $y = 4x^4 - 8x^2 + x - 1$; $y = -2x^2 - x + 4$

45 $y = 50\cos(0.5x)$; $y = x^2 - 20$

46 $y = 0.2x^4 - x^3 + 0.4x^2 - 2$; $y = \cos(0.7x)$

c **Exer. 47–50: Plot the graphs of the equations. (a) Set up a definite integral representing the area of the bounded region. (b) Approximate this area to four-decimal-place accuracy using Simpson's rule.**

47 $y = \sqrt{25 - x^2}$; $y = \sqrt{29 - x^2} - 2$

48 $y = \sin[\pi(x^2 - 1)]$; $y = 1 - x^2$

49 $y = \sin x$; $y = \sin(\sin x)$; $x = 0$, $x = \pi$

50 $y = 1 + 1.6x - 0.3x^2$; $y = \sqrt{1 + x^3}$

Exer 51–54: For each pair of net investment flows $I_1(t)$ and $I_2(t)$, (a) find the time interval during which I_1 is at least as great as I_2, and (b) for the time interval found in part (a), determine how much more capital accumulates under the first investment flow than the second investment flow.

51 $I_1(t) = t$; $I_2(t) = t^2$

52 $I_1(t) = 4(1 - t^2)$; $I_2(t) = 1 - t^2$

53 $I_1(t) = 2(1 - t^2)$; $I_2(t) = t^2 - 1$

54 $I_1(t) = -t^2 + 4t$; $I_2(t) = 3t/2$

c **Exer. 55–56: Graph, on the same coordinate axes, the given ellipses. (a) Estimate their points of intersection. (b) Set up an integral that can be used to approximate the area of the region bounded by and inside both ellipses.**

55 $\dfrac{x^2}{2.9} + \dfrac{y^2}{2.1} = 1$; $\dfrac{x^2}{4.3} + \dfrac{(y - 2.1)^2}{4.9} = 1$

56 $\dfrac{x^2}{3.9} + \dfrac{y^2}{2.4} = 1$; $\dfrac{(x + 1.9)^2}{4.1} + \dfrac{y^2}{2.5} = 1$

c **Exer. 57–58: Graph, on the same coordinate axes, the given hyperbolas. (a) Estimate their first-quadrant point of intersection. (b) Set up an integral that can be used to approximate the area of the region in the first quadrant bounded by the hyperbolas and a coordinate axis.**

57 $\dfrac{(y - 0.1)^2}{1.6} - \dfrac{(x + 0.2)^2}{0.5} = 1$; $\dfrac{(y - 0.5)^2}{2.7} - \dfrac{(x - 0.1)^2}{5.3} = 1$

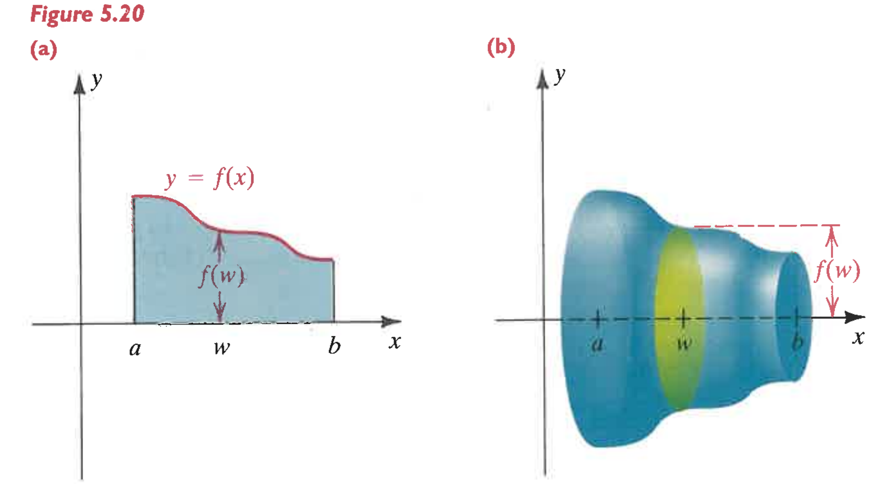
58 $\dfrac{(x - 0.1)^2}{0.12} - \dfrac{y^2}{0.1} = 1$; $\dfrac{x^2}{0.9} - \dfrac{(y - 0.3)^2}{2.1} = 1$

5.2 SOLIDS OF REVOLUTION

The volume of an object plays an important role in many problems in the physical sciences. In this section and the next two sections, we consider several methods for computing volumes. Since it is difficult to determine the volume of an irregularly shaped object, we begin with objects that have simple shapes, including the solids of revolution.

If a region in a plane is revolved about a line in the plane, the resulting solid is a **solid of revolution**, and we say that the solid is **generated** by the region. The line is an **axis of revolution**. In particular, if the R_x region shown in Figure 5.20(a) is revolved about the x-axis, we obtain the solid illustrated in Figure 5.20(b). As a special case, if f is a constant function,

Figure 5.20

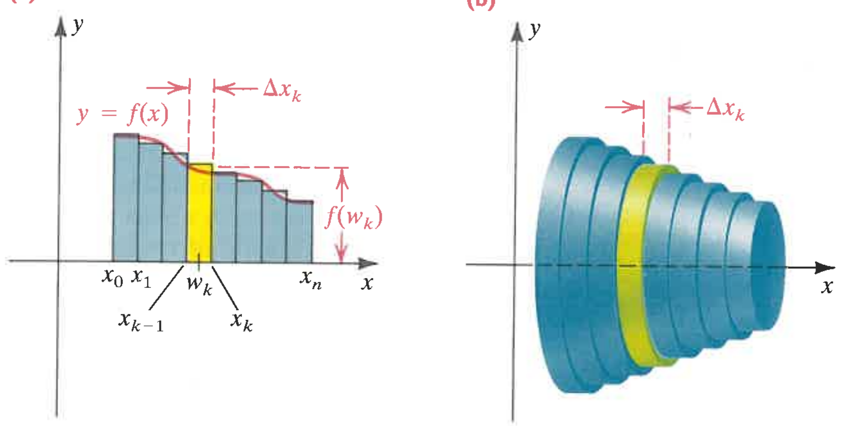


say $f(x) = k$, then the region is rectangular and the solid generated is a right circular cylinder. If the graph of f is a semicircle with endpoints of a diameter at the points $(a, 0)$ and $(b, 0)$, then the solid of revolution is a sphere. If the region is a right triangle with base on the x-axis and two vertices at the points $(a, 0)$ and $(b, 0)$ with the right angle at one of these points, then the solid generated is a right circular cone.

If a plane perpendicular to the x-axis intersects the solid shown in Figure 5.20(b), a circular cross section is obtained. If, as indicated in the figure, the plane passes through the point on the axis with x-coordinate w, then the radius of the circle is $f(w)$, and hence its area is $\pi[f(w)]^2$. We shall arrive at a definition for the volume of such a solid of revolution by using Riemann sums.

Let us partition the interval $[a, b]$, as we did for areas in Section 5.1, and consider the rectangles in Figure 5.21(a). The solid of revolution generated by these rectangles has the shape shown in Figure 5.21(b). Beginning with Figure 5.25, we shall remove, or cut out, parts of solids of revolution to help us visualize portions generated by typical rectangles. When referring to such figures, remember that the entire solid is obtained by one *complete* revolution about an axis, not a partial one.

Figure 5.21
(a) (b)



Observe that the kth rectangle generates a **circular disk** (a flat right circular cylinder) of base radius $f(w_k)$ and altitude (thickness) $\Delta x_k = x_k - x_{k-1}$. The volume of this disk is the area of the base times the altitude—that is, $\pi[f(w_k)]^2\Delta x_k$. The volume of the solid shown in Figure 5.21(b) is the sum of the volumes of all such disks:

$$\sum_k \pi[f(w_k)]^2 \Delta x_k$$

This sum may be regarded as a Riemann sum for $\pi[f(x)]^2$. If the norm $\|P\|$ of the partition is close to zero, then the sum should be close to the

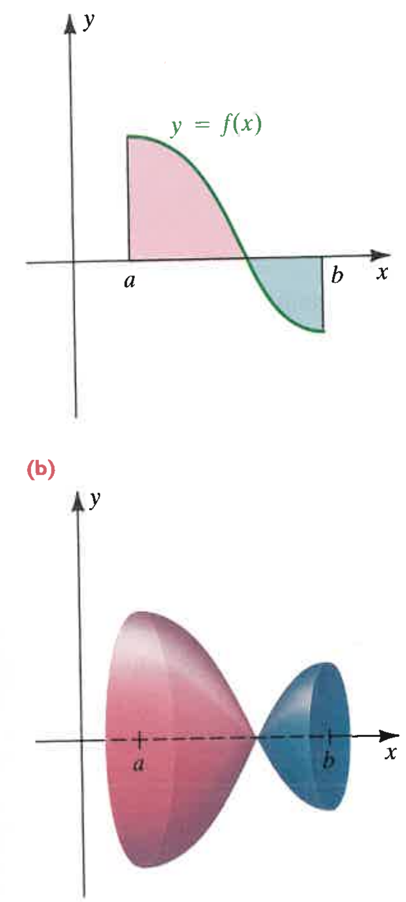
volume of the solid. Hence we define the volume of the solid of revolution as a limit of these sums.

Definition 5.5

Let f be continuous on $[a, b]$, and let R be the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$. The **volume** V of the solid of revolution generated by revolving R about the x-axis is

$$V = \lim_{\|P\|\to 0} \sum_k \pi[f(w_k)]^2 \Delta x_k = \int_a^b \pi[f(x)]^2\, dx.$$

Figure 5.22
(a)

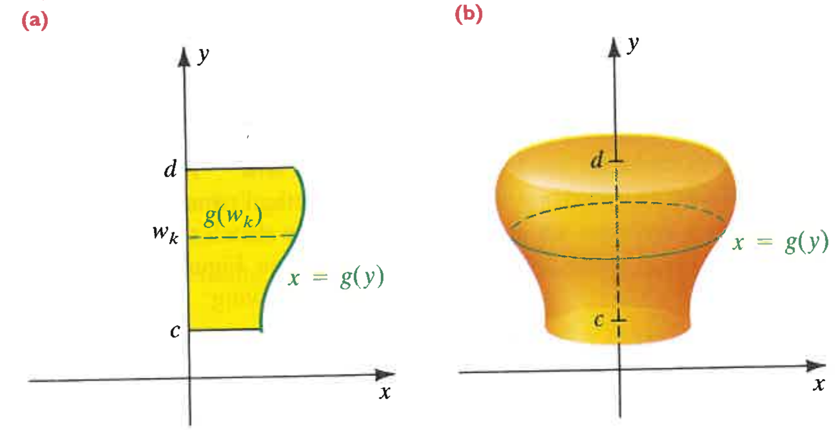


The fact that the limit of sums in this definition equals $\int_a^b \pi[f(x)]^2\, dx$ follows from the definition of the definite integral. We shall not ordinarily specify the units of measure for volume. If the linear measurement is inches, the volume is in cubic inches (in^3). If x is measured in centimeters, then V is in cubic centimeters (cm^3), and so on.

The requirement that $f(x) \geq 0$ was omitted intentionally in Definition (5.5). If f is negative for some x, as in Figure 5.22(a), and if the region bounded by the graphs of f, $x = a$, $x = b$, and the x-axis is revolved about the x-axis, we obtain the solid shown in Figure 5.22(b). This solid is the same as that generated by revolving the region under the graph of $y = |f(x)|$ from a to b about the x-axis. Since $|f(x)|^2 = [f(x)]^2$, the limit in Definition (5.5) gives us the volume.

Let us interchange the roles of x and y and revolve the R_y region in Figure 5.23(a) about the y-axis, obtaining the solid illustrated in Figure 5.23(b). If we partition the y-interval $[c, d]$ and use *horizontal* rectangles of width Δy_k and length $g(w_k)$, the same type of reasoning that gave us (5.5) leads to Definition (5.6) on the following page.

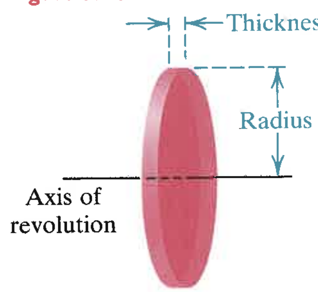
Figure 5.23
(a) (b)



Definition 5.6

$$V = \lim_{\|P\|\to 0} \sum_k \pi[g(w_k)]^2 \Delta y_k = \int_c^d \pi[g(y)]^2\, dy$$

Figure 5.24



Since we may revolve a region about the x-axis, the y-axis, or some other line, *it is not advisable to merely memorize the formulas in* (5.5) *and* (5.6). It is better to remember the following general rule for finding the volume of a circular disk (see Figure 5.24).

Volume V of a Circular Disk 5.7

$$V = \pi(\text{radius})^2 \cdot (\text{thickness})$$

When working problems, we shall use the intuitive method developed in Section 5.1, replacing Δx_k or Δy_k by dx or dy, and so on. The following guidelines may be helpful.

Guidelines for Finding the Volume of a Solid of Revolution Using Disks 5.8

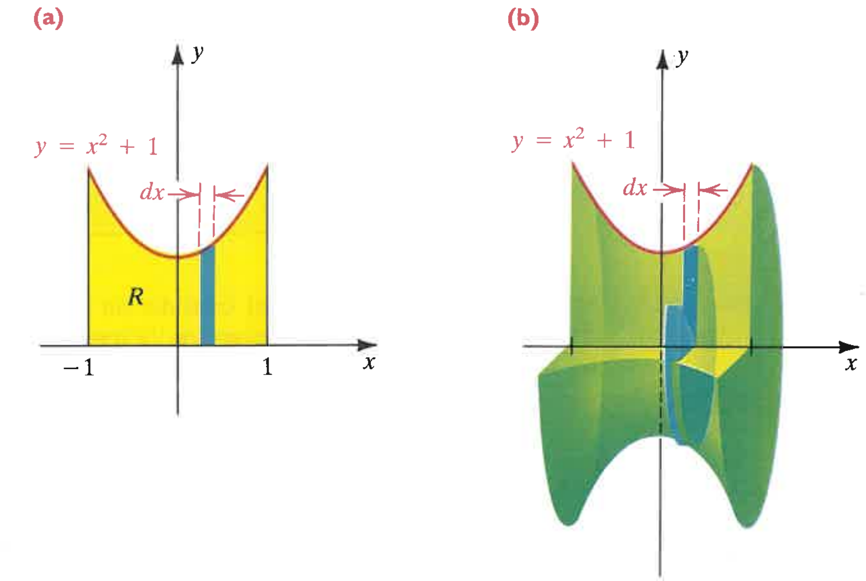
1. Sketch the region R to be revolved, and label the boundaries. Show a typical vertical rectangle of width dx or a horizontal rectangle of width dy.
2. Sketch the solid generated by R and the disk generated by the rectangle in guideline (1).
3. Express the radius of the disk in terms of x or y, depending on whether its thickness is dx or dy.
4. Use (5.7) to find a formula for the volume of the disk.
5. Apply the limit of sums operator $\int_a^b$ or $\int_c^d$ to the expression in guideline (4) and evaluate the integral.

EXAMPLE 1 The region bounded by the x-axis, the graph of the equation $y = x^2 + 1$, and the lines $x = -1$ and $x = 1$ is revolved about the x-axis. Find the volume of the resulting solid.

SOLUTION As specified in guideline (1), we sketch the region and show a vertical rectangle of width dx (see Figure 5.25a). Following guideline (2), we sketch the solid generated by R and the disk generated by the rectangle (see Figure 5.25b). As specified in guidelines (3) and (4), we note the following:

thickness of disk: dx
radius of disk: $x^2 + 1$
volume of disk: $\pi(x^2+1)^2\, dx$

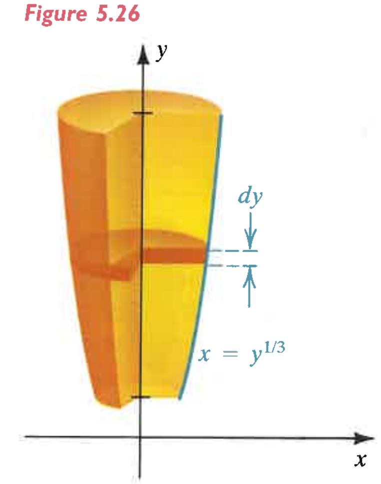
Figure 5.25



We could next apply guideline (5) with $a = -1$ and $b = 1$, finding the volume V by regarding $\int_{-1}^1$ as an operator that takes a limit of sums of volumes of disks. Another method is to use the symmetry of the region with respect to the y-axis and find V by applying $\int_0^1$ to $\pi(x^2+1)^2\, dx$ and doubling the result. Thus,

$$\begin{aligned} V &= \int_{-1}^{1} \pi(x^2+1)^2\, dx \\ &= 2\int_0^1 \pi(x^4 + 2x^2 + 1)\, dx \\ &= 2\pi\left[\frac{x^5}{5} + 2\left(\frac{x^3}{3}\right) + x\right]_0^1 \\ &= 2\pi\left(\tfrac{1}{5} + \tfrac{2}{3} + 1\right) = \tfrac{56}{15}\pi \approx 11.7. \end{aligned}$$

Figure 5.26



EXAMPLE 2 The region bounded by the y-axis and the graphs of $y = x^3$, $y = 1$, and $y = 8$ is revolved about the y-axis. Find the volume of the resulting solid.

SOLUTION The region and the solid are sketched in Figure 5.26, together with a disk generated by a typical horizontal rectangle. Since we plan to integrate with respect to y, we solve the equation $y = x^3$ for x in terms of y, obtaining $x = y^{1/3}$. We note the following facts (see guidelines 3 and 4):

thickness of disk: dy
radius of disk: $y^{1/3}$
volume of disk: $\pi(y^{1/3})^2\, dy$

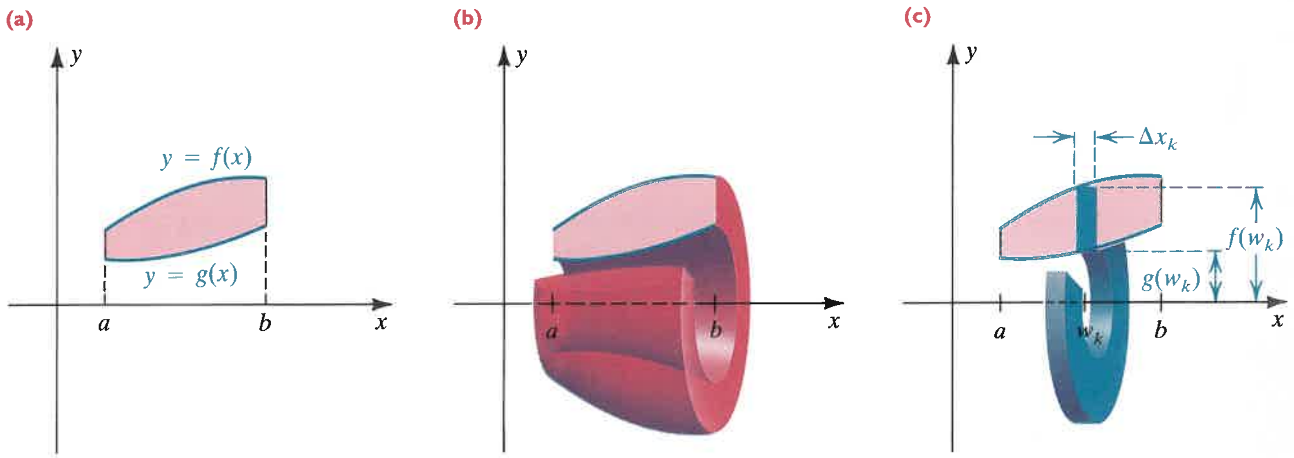
Finally, we apply guideline (5), with $c = 1$ and $d = 8$, regarding $\int_1^8$ as an operator that takes a limit of sums of disks:

$$V = \int_1^8 \pi (y^{1/3})^2\, dy = \pi \int_1^8 y^{2/3}\, dy = \pi \left[\frac{y^{5/3}}{\frac{5}{3}} \right]_1^8$$

$$= \tfrac{3}{5}\pi \left[y^{5/3} \right]_1^8 = \tfrac{3}{5}\pi\, [32 - 1] = \tfrac{93}{5}\pi \approx 58.4$$

Let us next consider an R_x region of the type illustrated in Figure 5.27(a). If this region is revolved about the x-axis, we obtain the solid illustrated in Figure 5.27(b). Note that if $g(x) > 0$ for every x in $[a, b]$, there is a hole through the solid.

Figure 5.27



The volume V of the solid may be found by subtracting the volume of the solid generated by the smaller region from the volume of the solid generated by the larger region. Using Definition (5.5) gives us

$$V = \int_a^b \pi [f(x)]^2\, dx - \int_a^b \pi [g(x)]^2\, dx$$

$$= \int_a^b \pi \{[f(x)]^2 - [g(x)]^2\}\, dx.$$

The last integral has an interesting interpretation as a limit of sums. As illustrated in Figure 5.27(c), a vertical rectangle extending from the graph of g to the graph of f, through the points with x-coordinate w_k, generates a washer-shaped solid whose volume is

$$\pi [f(w_k)]^2 \Delta x_k - \pi [g(w_k)]^2 \Delta x_k = \pi \{[f(w_k)]^2 - [g(w_k)]^2\} \Delta x_k.$$

Summing the volumes of all such washers and taking the limit gives us the desired definite integral. When working problems of this type, it is convenient to use the following general rule.

Volume V of a Washer 5.9

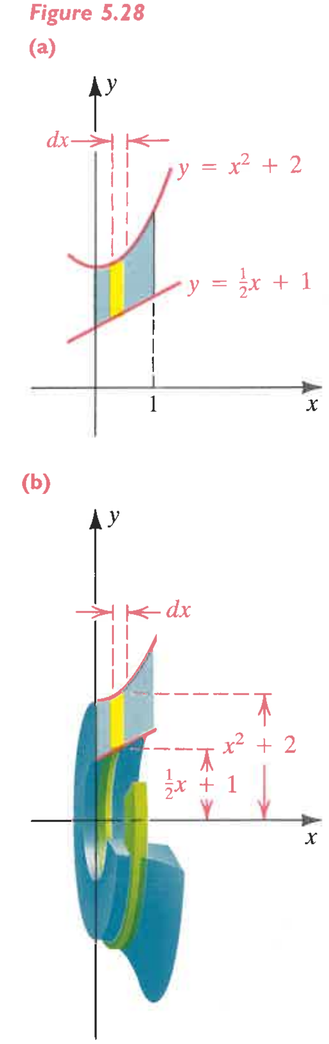
$$V = \pi [(\text{outer radius})^2 - (\text{inner radius})^2] \cdot (\text{thickness})$$

In applying (5.9), a common error is to use the square of the difference of the radii instead of the difference of the squares. Note that

$$\text{volume of a washer} \neq \pi [(\text{outer radius}) - (\text{inner radius})]^2 \cdot (\text{thickness}).$$

Guidelines similar to (5.8) can be stated for problems involving washers. The principal differences are that in guideline (3), we find expressions for the outer radius and inner radius of a typical washer, and in guideline (4), we use (5.9) to find a formula for the volume of the washer.

Figure 5.28



EXAMPLE 3 The region bounded by the graphs of the equations $x^2 = y - 2$ and $2y - x - 2 = 0$ and the vertical lines $x = 0$ and $x = 1$ is revolved about the x-axis. Find the volume of the resulting solid.

SOLUTION The region and a typical vertical rectangle are sketched in Figure 5.28(a). Since we wish to integrate with respect to x, we solve the first two equations for y in terms of x, obtaining $y = x^2 + 2$ and $y = \frac{1}{2}x + 1$. The solid and a washer generated by the rectangle are illustrated in Figure 5.28(b). Using (5.9) we obtain the following:

thickness of washer: dx
outer radius: $x^2 + 2$
inner radius: $\frac{1}{2}x + 1$
volume: $\pi [(x^2 + 2)^2 - (\frac{1}{2}x + 1)^2]\, dx$

We take a limit of sums of volumes of washers by applying $\int_0^1$:

$$V = \int_0^1 \pi [(x^2 + 2)^2 - (\tfrac{1}{2}x + 1)^2]\, dx$$

$$= \pi \int_0^1 (x^4 + \tfrac{15}{4}x^2 - x + 3)\, dx$$

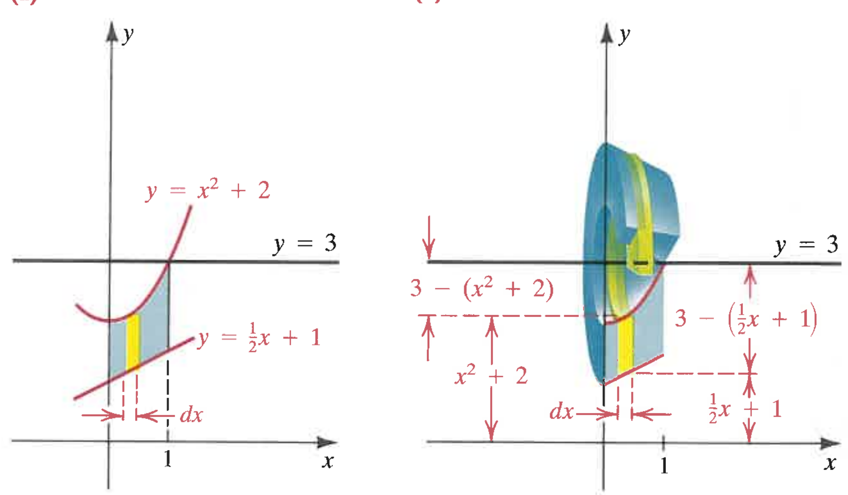
$$= \pi \left[\frac{x^5}{5} + \frac{15}{4}\left(\frac{x^3}{3}\right) - \frac{x^2}{2} + 3x \right]_0^1 = \frac{79\pi}{20} \approx 12.4$$

EXAMPLE 4 Find the volume of the solid generated by revolving the region described in Example 3 about the line $y = 3$.

SOLUTION The region and a typical vertical rectangle are resketched in Figure 5.29(a), together with the axis of revolution $y = 3$. The solid and a washer generated by the rectangle are illustrated in Figure 5.29(b). We note the following:

thickness of washer: dx
outer radius: $3 - (\frac{1}{2}x + 1) = 2 - \frac{1}{2}x$
inner radius: $3 - (x^2 + 2) = 1 - x^2$
volume: $\pi[(2 - \frac{1}{2}x)^2 - (1 - x^2)^2]\,dx$

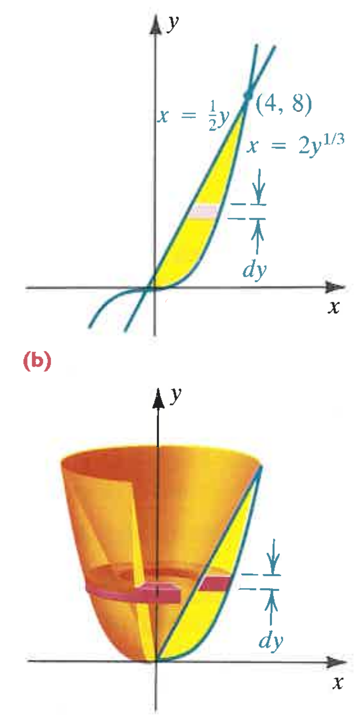
Figure 5.29



Applying the limit of sums operator $\int_0^1$ gives us the volume:

$$V = \int_0^1 \pi[(2 - \tfrac{1}{2}x)^2 - (1 - x^2)^2]\,dx$$
$$= \pi \int_0^1 (3 - 2x + \tfrac{9}{4}x^2 - x^4)\,dx$$
$$= \pi \left[3x - x^2 + \frac{9}{4}\left(\frac{x^3}{3}\right) - \frac{x^5}{5}\right]_0^1$$
$$= \frac{51\pi}{20} \approx 8.01$$

Figure 5.30



EXAMPLE 5 The region in the first quadrant bounded by the graphs of $y = \frac{1}{8}x^3$ and $y = 2x$ is revolved about the y-axis. Find the volume of the resulting solid.

SOLUTION The region and a typical horizontal rectangle are shown in Figure 5.30(a). We wish to integrate with respect to y, so we solve the given equations for x in terms of y, obtaining

$$x = \tfrac{1}{2}y \quad \text{and} \quad x = 2y^{1/3}.$$

Figure 5.30(b) illustrates the volume generated by the region and the washer generated by the rectangle. We note the following:

thickness of washer: dy
outer radius: $2y^{1/3}$
inner radius: $\frac{1}{2}y$
volume: $\pi[(2y^{1/3})^2 - (\frac{1}{2}y)^2]\,dy = \pi(4y^{2/3} - \frac{1}{4}y^2)\,dy$

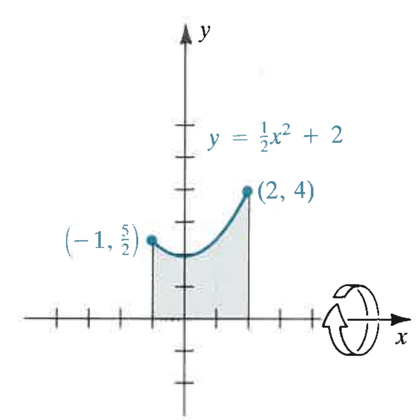
Applying the limit of sums operator $\int_0^8$ gives us the volume:

$$V = \int_0^8 \pi(4y^{2/3} - \tfrac{1}{4}y^2)\,dy$$
$$= \pi\left[\tfrac{12}{5}y^{5/3} - \tfrac{1}{12}y^3\right]_0^8 = \tfrac{512}{15}\pi \approx 107.2$$

EXERCISES 5.2

Exer. 1–4: Set up an integral that can be used to find the volume of the solid obtained by revolving the shaded region about the indicated axis.

1



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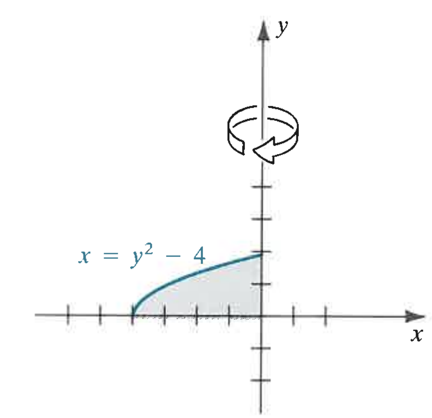

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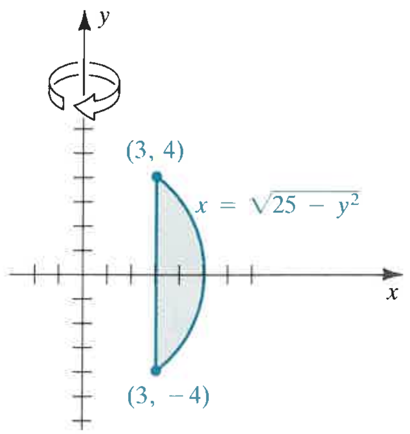

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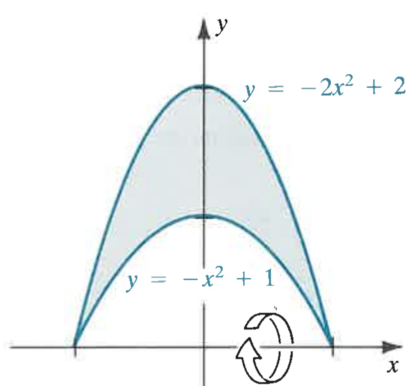


Exer. 5–24: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated if R is revolved about the indicated axis.

5 $y = 1/x$, $x = 1$, $x = 3$, $y = 0$; x-axis

6 $y = \sqrt{x}$, $x = 4$, $y = 0$; x-axis

7 $y = x^2 - 4x$, $y = 0$; x-axis

8 $y = x^3$, $x = -2$, $y = 0$; x-axis

9 $y = x^2$, $y = 2$; y-axis

10 $y = 1/x$, $y = 1$, $y = 3$, $x = 0$; y-axis

11 $x = 4y - y^2$, $x = 0$; y-axis

12 $y = x$, $y = 3$, $x = 0$; y-axis

13 $y = x^2$, $y = 4 - x^2$; x-axis

14 $x = y^3$, $x^2 + y = 0$; x-axis

15 $y = x$, $x + y = 4$, $x = 0$; x-axis

16 $y = (x-1)^2 + 1$, $y = -(x-1)^2 + 3$; x-axis

17 $y^2 = x$, $2y = x$; y-axis

18 $y = 2x$, $y = 4x^2$; y-axis

19 $x = y^2$, $x - y = 2$; y-axis

20 $x + y = 1$, $x - y = -1$, $x = 2$; y-axis

21 $y = \sin 2x$, $x = 0$, $x = \pi$, $y = 0$; x-axis
(*Hint:* Use a half-angle formula.)

22 $y = 1 + \cos 3x$, $x = 0$, $x = 2\pi$, $y = 0$; x-axis
(*Hint:* Use a half-angle formula.)

23 $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$; x-axis
(*Hint:* Use a double angle formula.)

24 $y = \sec x$, $y = \sin x$, $x = 0$, $x = \pi/4$; x-axis

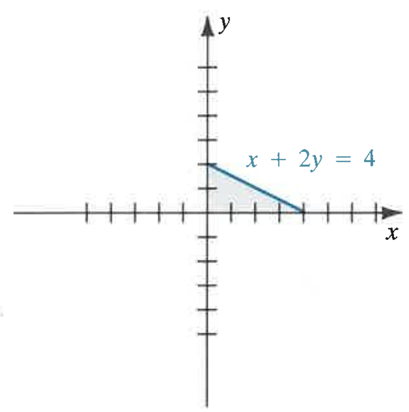
Exer. 25–26: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated if R is revolved about the given line.

25 $y = x^2$, $y = 4$
(a) $y = 4$ (b) $y = 5$
(c) $x = 2$ (d) $x = 3$

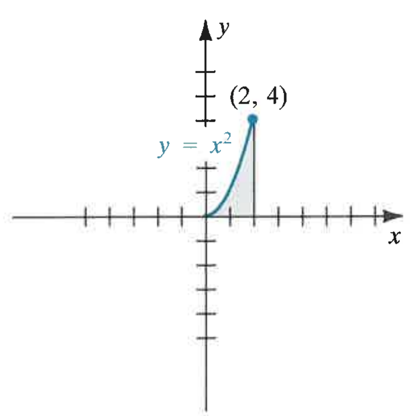
26 $y = \sqrt{x}$, $y = 0$, $x = 4$
(a) $x = 4$ (b) $x = 6$
(c) $y = 2$ (d) $y = 4$

Exer. 27–28: Set up an integral that can be used to find the volume of the solid generated by revolving the shaded region about the line (a) $y = -2$, (b) $y = 5$, (c) $x = 7$, and (d) $x = -4$.

27



28



Exer. 29–34: Sketch the region R bounded by the graphs of the equations, and set up integrals that can be used to find the volume of the solid generated if R is revolved about the given line.

29 $y = x^3$, $y = 4x$; $y = 8$

30 $y = x^3$, $y = 4x$; $x = 4$

31 $x + y = 3$, $y + x^2 = 3$; $x = 2$

32 $y = 1 - x^2$, $x - y = 1$; $y = 3$

33 $x^2 + y^2 = 1$; $x = 5$

34 $y = x^{2/3}$, $y = x^2$; $y = -1$

Exer. 35–40: Use a definite integral to derive a formula for the volume of the indicated solid.

35 A right circular cylinder of altitude h and radius r

36 A cylindrical shell of altitude h, outer radius R, and inner radius r

37 A right circular cone of altitude h and base radius r

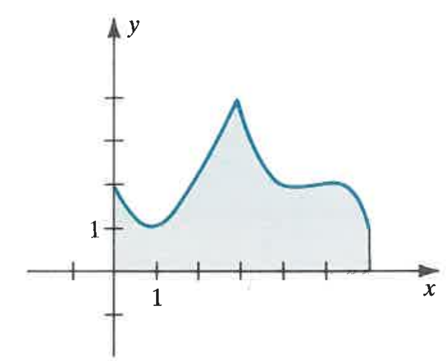
38 A sphere of radius r

39 A frustum of a right circular cone of altitude h, lower base radius R, and upper base radius r

40 A spherical segment of altitude h in a sphere of radius r

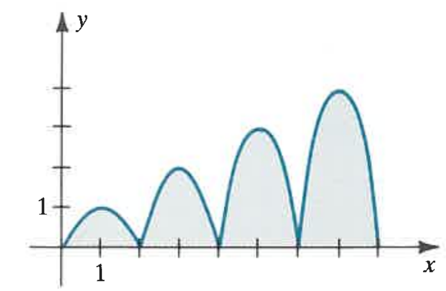
41 If the region shown in the figure is revolved about the x-axis, use the trapezoidal rule with $n = 6$ to approximate the volume of the resulting solid.

Exercise 41



42 If the region shown in the figure is revolved about the x-axis, use Simpson's rule with $n = 4$ to approximate the volume of the resulting solid.

Exercise 42



c **Exer. 43–44: Graph f and g on the same coordinate axes for $0 \le x \le \pi$. (a) Estimate the x-coordinates of the points of intersection of the graphs. (b) If the region bounded by the graphs of f and g is revolved about the x-axis, use Simpson's rule with $n = 2$ to approximate the volume of the resulting solid.**

43 $f(x) = \dfrac{\sin x}{1 + x}$; $g(x) = 0.3$

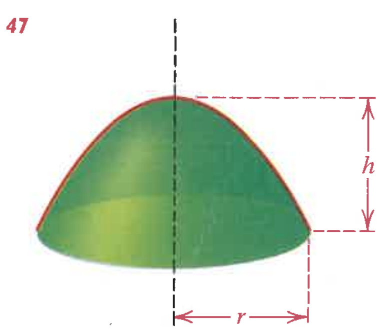
44 $f(x) = \sqrt[4]{|\sin x|}$; $g(x) = 0.2x + 0.7$

45 Find the volume of the solid obtained by revolving the region bounded by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x-axis.

46 Work Exercise 45 with the region revolved about the y-axis.

47 A *paraboloid of revolution* is formed by revolving a parabola about its axis. Paraboloids are the basic shape for a wide variety of collectors and reflectors. Shown in the figure is a (finite) paraboloid of altitude h and radius of base r.
(a) The *focal length* of the paraboloid is the distance p between the vertex and the focus of the parabola. Express p in terms of r and h.
(b) Find the volume of the paraboloid.

Exercise 47



Mathematicians and Their Times

JOHN BERNOULLI

WE OFTEN HEAR OF SCIENTIFIC or artistic genius emerging in an individual of modest circumstances whose forebears gave little evidence of greatness. Such was the case with Newton. There have been notable instances, however, of enormous talent displayed by several generations of the same family. In music, the Bach family included a score of eminent artists. In mathematics, the premier example is the Bernoulli family, with eight mathematicians in three generations and dozens of other distinguished descendants who played a leading part in developing calculus and making it accessible to a wider audience.

The first generation of Bernoulli mathematicians were the three sons of a druggist who fled from Antwerp to Switzerland to escape religious persecution. Two of these brothers, James (1654–1705) and John (1667–1748), are the most eminent mathematicians among the Bernoullis. Their brother Nicholas (1662–1716), also a gifted mathematician, first earned a degree in philosophy at age 16 and then turned to law before joining the mathematics faculty at St. Petersburg Academy.

James Bernoulli, a mathematics professor at the University of Basel, introduced the term *integral* into the field and developed the calculus significantly beyond the state in which Leibniz and Newton left it. He also made important contributions to probability and the calculus of variations.

John Bernoulli began his career as a physician and studied mathematics under his older brother James, eventually replacing him as mathematics professor at the University of Basel. He became deeply interested in calculus and was indirectly responsible for the first calculus textbook (1696), published by the French marquis G. F. A. de l'Hôpital. Bernoulli tutored l'Hôpital and sold the marquis the rights to a number of his own mathematical discoveries. Later Bernoulli virtually accused l'Hôpital of plagiarism.

John also became locked in a bitter quarrel over mathematics with his brother James, exchanging words that later writers characterized as more in keeping with horse thieves or street brawlers than well-known scientists. When the French Academy of Sciences awarded his son Daniel a prize, John was so jealous that he expelled Daniel from his home.

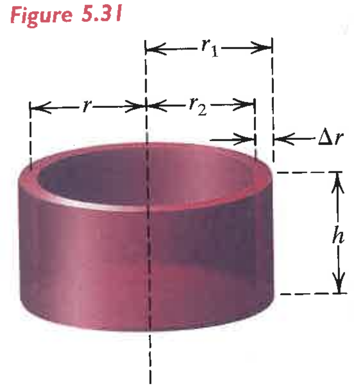
Most notable of the second generation of Bernoullis were John's sons: Nicholas, Daniel, and John II. Daniel's discoveries in science were so extensive that he is considered the founder of mathematical physics. John II began his career in law, later became a professor of eloquence, and eventually succeeded his father as Basel's mathematics professor.

In the third generation, the sons of John II, John III and Jacob, also did significant work in the sciences. John III became the royal astronomer at Berlin, but his brother's promising career was tragically cut short by drowning at age 30.

5.3 VOLUMES BY CYLINDRICAL SHELLS

In the preceding section, we found volumes of solids of revolution by using circular disks or washers. In this section, we shall see that for certain types of solids, it is convenient to use hollow circular cylinders—that is, thin **cylindrical shells** of the type illustrated in Figure 5.31, where r_1 is the *outer radius*, r_2 is the *inner radius*, h is the *altitude*, and $\Delta r = r_1 - r_2$ is the *thickness* of the shell. The **average radius** of the shell is $r = \frac{1}{2}(r_1 + r_2)$. We can find the volume of the shell by subtracting the volume $\pi r_2^2 h$ of the inner cylinder from the volume $\pi r_1^2 h$ of the outer cylinder. If we do so and change the form of the resulting expression, we obtain

Figure 5.31



$$\begin{aligned}\pi r_1^2 h - \pi r_2^2 h &= \pi(r_1^2 - r_2^2)h \\ &= \pi(r_1 + r_2)(r_1 - r_2)h \\ &= 2\pi \cdot \tfrac{1}{2}(r_1 + r_2)h(r_1 - r_2) \\ &= 2\pi r h \Delta r,\end{aligned}$$

which gives us the following general rule.

Volume V of a Cylindrical Shell 5.10

$$V = 2\pi(\text{average radius})(\text{altitude})(\text{thickness})$$

If the R_x region in Figure 5.32(a) is revolved about the y-axis, we obtain the solid illustrated in Figure 5.32(b).

Let P be a partition of $[a, b]$, and consider the typical vertical rectangle in Figure 5.32(c), where w_k is the midpoint of $[x_{k-1}, x_k]$. If we revolve this rectangle about the y-axis, we obtain a cylindrical shell of average radius w_k, altitude $f(w_k)$, and thickness Δx_k. Hence, by (5.10), the volume of the shell is

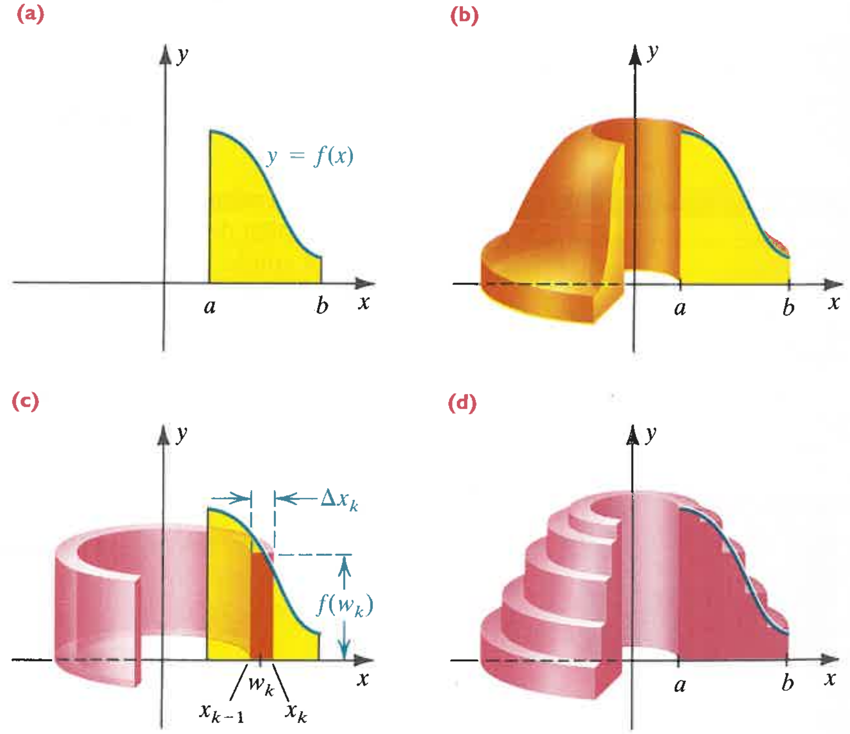
$$2\pi w_k f(w_k)\Delta x_k.$$

Revolving the rectangular polygon formed by *all* the rectangles determined by P gives us the solid illustrated in Figure 5.32(d). The volume of this solid is a Reimann sum:

$$\sum_k 2\pi w_k f(w_k)\Delta x_k$$

The smaller the norm $\|P\|$ of the partition, the better the sum approximates the volume V of the solid shown in Figure 5.32(b). This discussion provides the motivation for Definition (5.11) on the following page.

Figure 5.32



Definition 5.11

Let f be continuous and suppose $f(x) \geq 0$ on the interval $[a, b]$, where $0 \leq a \leq b$. Let R be the region under the graph of f from a to b. The volume V of the solid of revolution generated by revolving R about the y-axis is

$$V = \lim_{\|P\|\to 0} \sum_k 2\pi w_k f(w_k)\Delta x_k = \int_a^b 2\pi x f(x)\,dx.$$

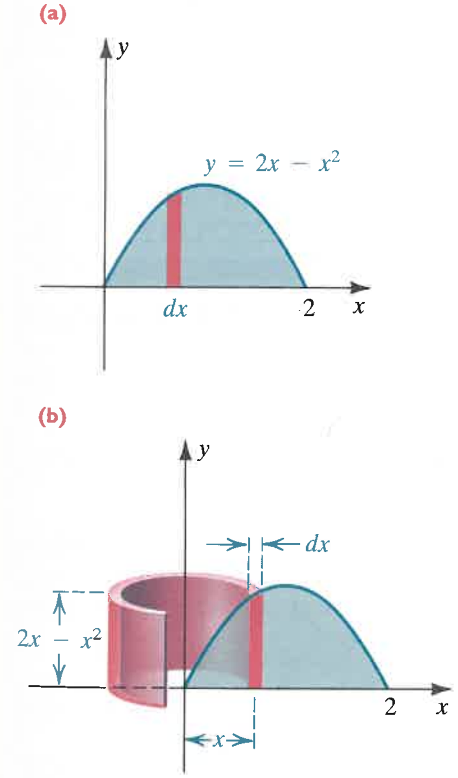
It can be proved that if the methods of Section 5.2 are also applicable, then both methods lead to the same answer.

We may also consider solids obtained by revolving a region about the x-axis or some other line. The following guidelines may be useful.

Guidelines for Finding the Volume of a Solid of Revolution Using Cylindrical Shells 5.12

1. Sketch the region R to be revolved, and label the boundaries. Show a typical vertical rectangle of width dx or a horizontal rectangle of width dy.
2. Sketch the cylindrical shell generated by the rectangle in guideline (1).
3. Express the average radius of the shell in terms of x or y, depending on whether its thickness is dx or dy. *Remember that x represents a distance from the y-axis to a vertical rectangle, and y represents a distance from the x-axis to a horizontal rectangle.*
4. Express the altitude of the shell in terms of x or y, depending on whether its thickness is dx or dy.
5. Use (5.10) to find a formula for the volume of the shell.
6. Apply the limit of sums operator $\int_a^b$ or $\int_c^d$ to the expression in guideline (5) and evaluate the integral.

Figure 5.33



EXAMPLE ■ 1 The region bounded by the graph of $y = 2x - x^2$ and the x-axis is revolved about the y-axis. Find the volume of the resulting solid.

SOLUTION The region to be revolved is sketched in Figure 5.33(a), together with a typical vertical rectangle of width dx. Figure 5.33(b) shows the cylindrical shell generated by revolving the rectangle about the y-axis. Note that x represents the distance from the y-axis to the midpoint of the rectangle (the average radius of the shell). Referring to the figure and using (5.10) gives us the following:

thickness of shell: dx

average radius: x

altitude: $2x - x^2$

volume: $2\pi x(2x - x^2)\,dx$

To sum all such shells, we move from left to right through the region from $a = 0$ to $b = 2$ (do *not* sum from -2 to 2). Hence, the limit of sums is

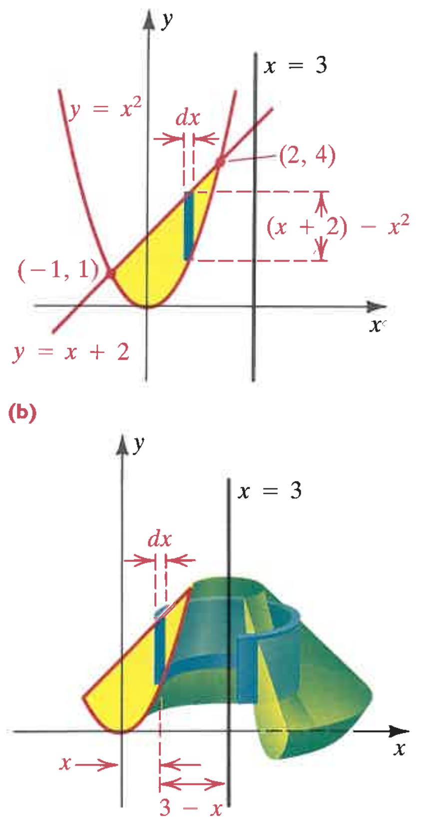
$$V = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi \int_0^2 (2x^2 - x^3)\,dx$$

$$= 2\pi \left[2\left(\frac{x^3}{3}\right) - \frac{x^4}{4} \right]_0^2 = \frac{8\pi}{3} \approx 8.4.$$

The volume V can also be found using washers; however, the calculations would be much more involved, since the equation $y = 2x - x^2$ would have to be solved for x in terms of y.

Figure 5.34

(a)



EXAMPLE 2 The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved about the line $x = 3$. Set up the integral for the volume of the resulting solid.

SOLUTION The region is sketched in Figure 5.34(a), together with a typical vertical rectangle extending from the lower boundary $y = x^2$ to the upper boundary $y = x + 2$. Also shown is the axis of revolution $x = 3$. In Figure 5.34(b), we have illustrated both the cylindrical shell and the solid that are generated by revolving the rectangle and the region about the line $x = 3$. It is important to note that since x is the distance from the y-axis to the rectangle, the radius of the shell is $3 - x$. Referring to Figure 5.34 and using (5.10) gives us the following:

thickness of shell:	dx
average radius:	$3 - x$
altitude:	$(x + 2) - x^2$
volume:	$2\pi(3 - x)(x + 2 - x^2)\,dx$

To sum all such shells, we move from left to right through the region from $a = -1$ to $b = 2$. Hence, the limit of sums is

$$V = \int_{-1}^2 2\pi(3 - x)(x + 2 - x^2)\,dx.$$

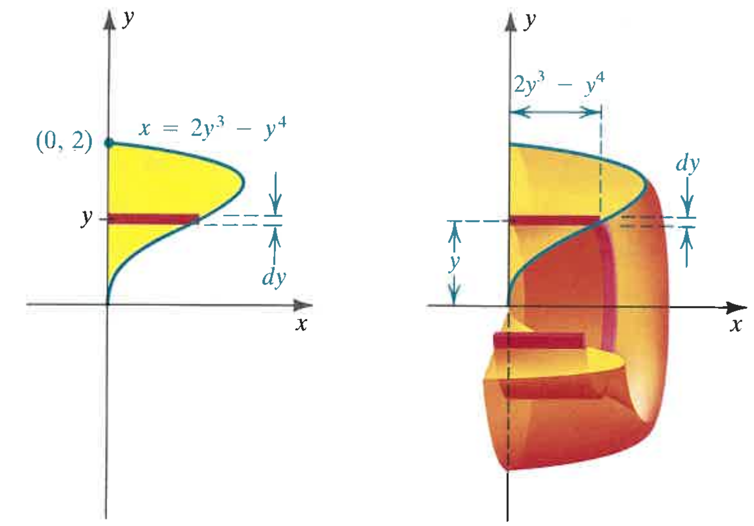
EXAMPLE 3 The region in the first quadrant bounded by the graph of the equation $x = 2y^3 - y^4$ and the y-axis is revolved about the x-axis. Set up the integral for the volume of the resulting solid.

SOLUTION The region is sketched in Figure 5.35(a), together with a typical horizontal rectangle. Figure 5.35(b) shows the cylindrical shell and the solid that are generated by the revolution about the x-axis. Referring to the figure and using (5.10) gives the following:

thickness of shell:	dy
average radius:	y
altitude:	$2y^3 - y^4$
volume:	$2\pi y(2y^3 - y^4)\,dy$

Figure 5.35

(a) (b)



To sum all such shells, we move upward through the region from $c = 0$ to $d = 2$. Hence, the limit of sums is

$$V = \int_0^2 2\pi y(2y^3 - y^4)\,dy.$$

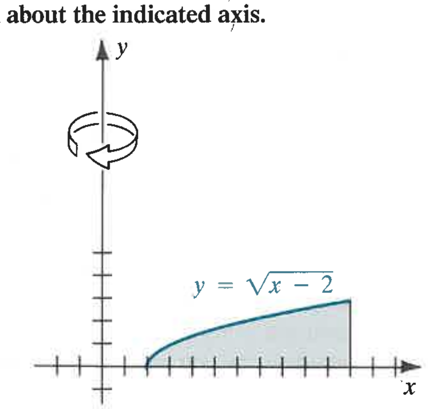
It is worth noting that in the preceding example we were forced to use shells and to integrate with respect to y, since the use of washers and integration with respect to x would require that we solve the equation $x = 2y^3 - y^4$ for y in terms of x, a rather formidable task.

EXERCISES 5.3

Use cylindrical shells for each exercise.

Exer. 1–4: Set up an integral that can be used to find the volume of the solid obtained by revolving the shaded region about the indicated axis.

1



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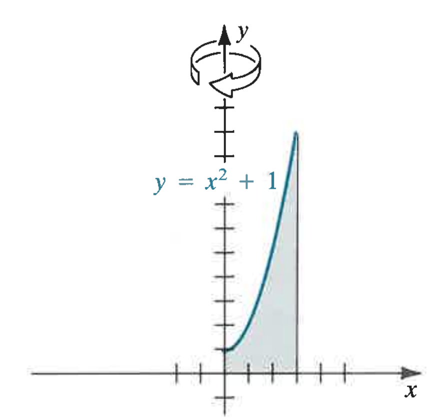

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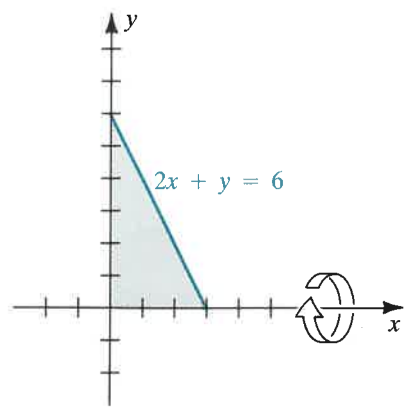

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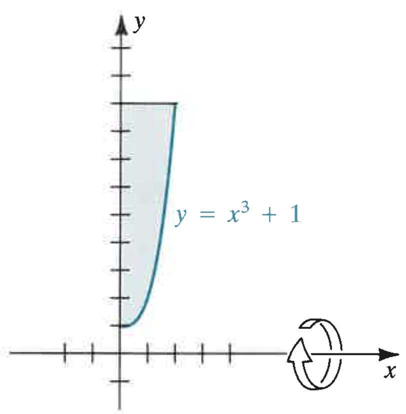


Exer. 5–18: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated if R is revolved about the indicated axis.

5 $y=\sqrt{x}$, $x=4$, $y=0$; y-axis

6 $y=1/x$; $x=1$, $x=2$, $y=0$; y-axis

7 $y=x^2$, $y^2=8x$; y-axis

8 $16y=x^2$, $y^2=2x$; y-axis

9 $2x-y=12$, $x-2y=3$, $x=4$; y-axis

10 $y=x^3+1$, $x+2y=2$, $x=1$; y-axis

11 $2x-y=4$, $x=0$, $y=0$; y-axis

12 $y=x^2-5x$, $y=0$; y-axis

13 $x^2=4y$, $y=4$; x-axis

14 $y^3=x$, $y=3$, $x=0$; x-axis

15 $y=2x$, $y=6$, $x=0$; x-axis

16 $2y=x$, $y=4$, $x=1$; x-axis

17 $y=\sqrt{x+4}$, $y=0$, $x=0$; x-axis

18 $y=-x$, $x-y=-4$, $y=0$; x-axis

Exer. 19–26: Let R be the region bounded by the graphs of the equations. Set up an integral that can be used to find the volume of the solid generated if R is revolved about the given line.

19 $y=x^2+1$, $x=0$, $x=2$, $y=0$
(a) $x=3$ (b) $x=-1$

20 $y=4-x^2$, $y=0$
(a) $x=2$ (b) $x=-3$

21 $y=x^2$, $y=4$
(a) $y=4$ (b) $y=5$ (c) $x=2$ (d) $x=-3$

22 $y=\sqrt{x}$, $y=0$, $x=4$
(a) $x=4$ (b) $x=6$ (c) $y=2$ (d) $y=-4$

23 $x+y=3$, $y+x^2=3$; $x=2$

24 $y=1-x^2$, $x-y=1$; $y=3$

25 $x^2+y^2=1$; $x=5$

26 $y=x^{2/3}$, $y=x^2$; $y=-1$

Exer. 27–30: Let R be the region bounded by the graphs of the equations. Set up integrals that can be used to find the volume of the solid generated if R is revolved about the given axis using (a) cylindrical shells and (b) disks or washers.

27 $y=1/\sqrt{x}$, $x=1$, $x=4$, $y=0$; x-axis

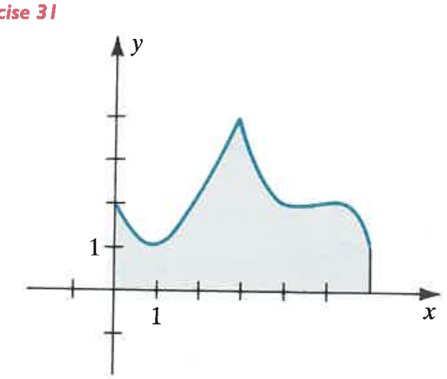
28 $y=9-x^2$, $x=0$, $x=2$, $y=0$; x-axis

29 $y=x^2+2$, $x=0$, $x=1$, $y=0$; y-axis

30 $y=x+1$, $x=0$, $x=1$, $y=0$; y-axis

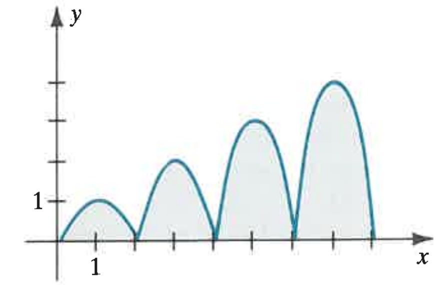
31 If the region shown in the figure is revolved about the y-axis, use the trapezoidal rule, with $n=6$, to approximate the volume of the resulting solid.

Exercise 31



32 If the region shown in the figure on the following page is revolved about the y-axis, use Simpson's rule, with $n=4$, to approximate the volume of the resulting solid.

Exercise 32



c 33 Graph $f(x)=-x^4+2.21x^3-3.21x^2+4.42x-2$.
(a) Estimate the x-intercepts of the graph.
(b) If the region bounded by the graph of f and the x-axis is revolved about the y-axis, set up an integral that can be used to approximate the volume of the resulting solid.

c 34 Graph, on the same coordinate axes, $f(x)=\csc x$ and $g(x)=x+1$ for $0<x<\pi$.
(a) Use Newton's method to approximate, to four decimal places, the x-coordinates of the points of intersection of the graphs.
(b) If the region bounded by the graphs is revolved about the y-axis, use the trapezoidal rule, with $n=6$, to approximate the volume of the resulting solid.

35 Let R be the region bounded by the parabola $x^2=4y$ and the line l through the focus that is perpendicular to the axis of the parabola.
(a) Find the area of R.
(b) If R is revolved about the y-axis, find the volume of the resulting solid.
(c) If R is revolved about the x-axis, find the volume of the resulting solid.

36 Work (a)–(c) of Exercise 35 if R is the region bounded by the graphs of $y^2=2x-6$ and $x=5$.

Exer. 37–38: Let R be the region bounded by the hyperbola with equation $b^2x^2-a^2y^2=a^2b^2$ and a vertical line through a focus.

37 Show that the area of the region R is given by

$$\frac{2b}{a}\int_a^c \sqrt{x^2-a^2}\,dx, \text{ where } c=\sqrt{a^2+b^2}.$$

38 Find the volume of the solid obtained by revolving R about the y-axis.

5.4 VOLUMES BY CROSS SECTIONS

If a plane intersects a solid, then the region common to the plane and the solid is a **cross section** of the solid. In Section 5.2, we used circular and washer-shaped cross sections to find volumes of solids of revolution. In this section, we shall study solids that have the following property (see Figure 5.36): For every x in $[a,b]$, the plane perpendicular to the x-axis at x intersects the solid in a cross section whose area is $A(x)$, where A is a continuous function on $[a,b]$.

Figure 5.36

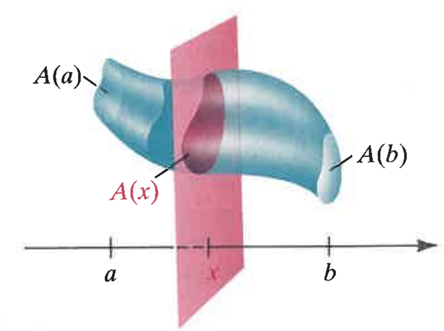


Figure 5.37

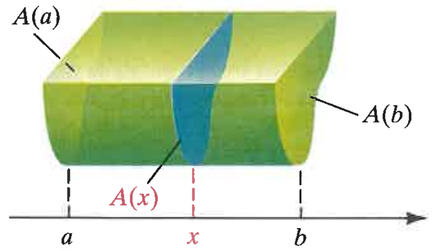
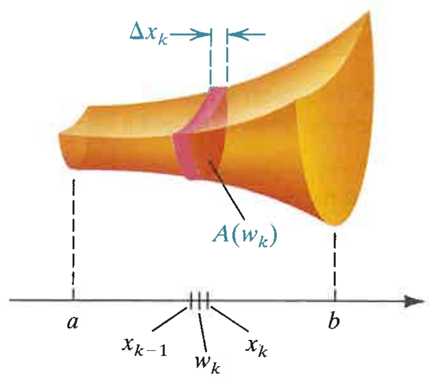


Figure 5.38



The solid is called a **cylinder** if, as illustrated in Figure 5.37 a line parallel to the x-axis that traces the boundary of the cross section corresponding to a also traces the boundary of the cross section corresponding to every x in $[a, b]$. The cross sections determined by the planes through $x = a$ and $x = b$ are the **bases** of the cylinder. The distance between the bases is the **altitude** of the cylinder. By definition, the volume of the cylinder is the area of a base multiplied by the altitude. Thus, the volume of the solid in Figure 5.37 is $A(a) \cdot (b - a)$.

To find the volume of a noncylindrical solid of the type illustrated in Figure 5.38, we begin with a partition P of $[a, b]$. Planes perpendicular to the x-axis at each x_k in the partition slice the solid into smaller pieces. If we choose any number w_k in $[x_{k-1}, x_k]$, the volume of a typical slice can be approximated by the volume $A(w_k)\Delta x_k$ of the red cylinder shown in Figure 5.38. If V is the volume of the solid and if the norm $\|P\|$ is small, then

$$V \approx \sum_k A(w_k)\Delta x_k.$$

Since this approximation improves as $\|P\|$ gets smaller, we define the volume of the solid by

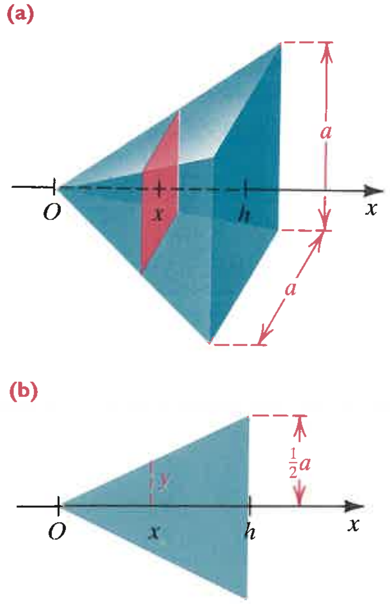
$$V = \lim_{\|P\|\to 0} \sum_k A(w_k)\Delta x_k = \int_a^b A(x)\, dx,$$

where the last equality follows from the definition of the definite integral. We may summarize our discussion as follows.

Volumes by Cross Sections 5.13

> Let S be a solid bounded by planes that are perpendicular to the x-axis at a and b. If, for every x in $[a, b]$, the cross-sectional area of S is given by $A(x)$, where A is continuous on $[a, b]$, then the volume of S is
>
> $$V = \int_a^b A(x)\, dx.$$

Figure 5.39



An analogous result can be stated for a y-interval $[c, d]$ and a cross-sectional area $A(y)$.

EXAMPLE 1 Find the volume of a right pyramid with a square base of side a and altitude h.

SOLUTION As in Figure 5.39(a), let us take the vertex of the pyramid at the origin, with the x-axis passing through the center of the square base, a distance h from O. Cross sections by planes perpendicular to the x-axis are squares. Figure 5.39(b) is a side view of the pyramid. Since $2y$ is the length of the side of the square cross section corresponding to x, the cross-sectional area $A(x)$ is

$$A(x) = (2y)^2 = 4y^2.$$

Using similar triangles in Figure 5.39(b), we have

$$\frac{y}{x} = \frac{\frac{1}{2}a}{h}, \quad \text{or} \quad y = \frac{ax}{2h}.$$

Hence,

$$A(x) = 4y^2 = \frac{4a^2x^2}{4h^2} = \frac{a^2}{h^2}x^2.$$

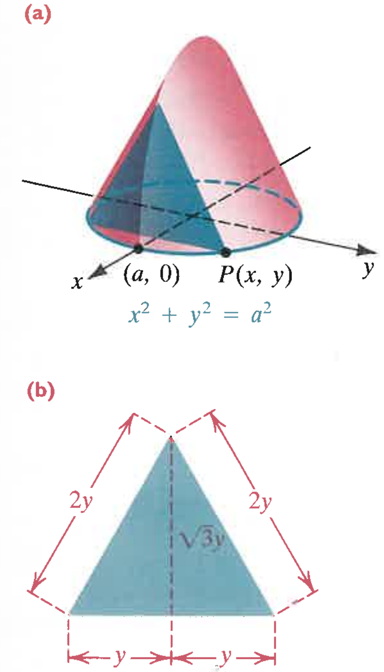
Applying (5.13) yields

$$V = \int_0^h A(x)\, dx = \int_0^h \left(\frac{a^2}{h^2}\right) x^2\, dx$$

$$= \left(\frac{a^2}{h^2}\right)\left[\frac{x^3}{3}\right]_0^h = \frac{a^2}{h^2}\frac{h^3}{3} = \frac{1}{3}a^2h.$$

EXAMPLE 2 A solid has, as its base, the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an equilateral triangle with one side in the base.

Figure 5.40



SOLUTION A triangular cross section by a plane x units from the origin is illustrated in Figure 5.40(a). If the point $P(x, y)$ is on the circle and $y > 0$, then the lengths of the sides of this equilateral triangle are $2y$. Referring to Figure 5.40(b), we see, by the Pythagorean theorem, that the altitude of the triangle is

$$\sqrt{(2y)^2 - y^2} = \sqrt{3y^2} = \sqrt{3}y.$$

Hence, the area $A(x)$ of the cross section is

$$A(x) = \tfrac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2 = \sqrt{3}(a^2 - x^2).$$

Applying (5.13) gives us

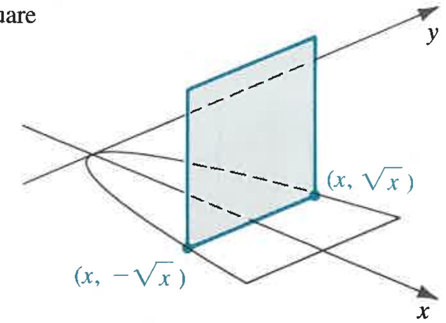
$$V = \int_{-a}^a A(x)\, dx = \int_{-a}^a \sqrt{3}(a^2 - x^2)\, dx$$

$$= \sqrt{3}\left[a^2x - \frac{x^3}{3}\right]_{-a}^a = \frac{4\sqrt{3}}{3}a^3.$$

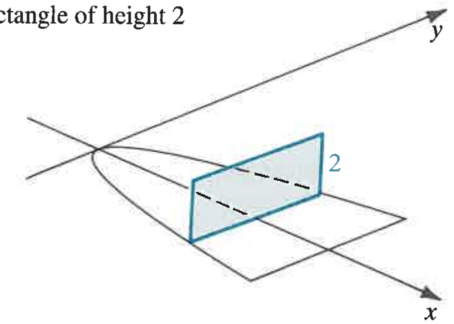
EXERCISES 5.4

Exer. 1–8: Let R be the region bounded by the graphs of $x = y^2$ and $x = 9$. Find the volume of the solid that has R as its base if every cross section by a plane perpendicular to the x-axis has the given shape.

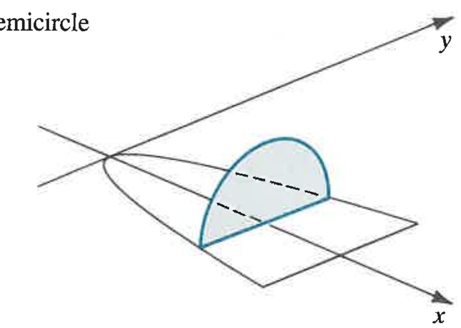
1 A square



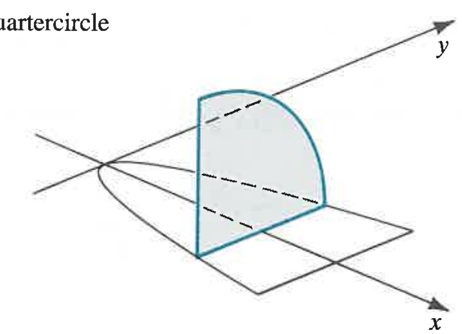
2 A rectangle of height 2



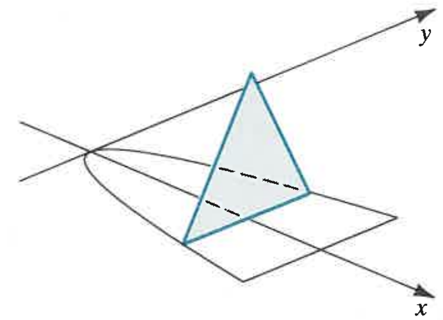
3 A semicircle



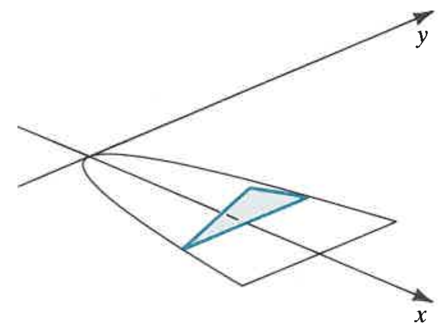
4 A quartercircle



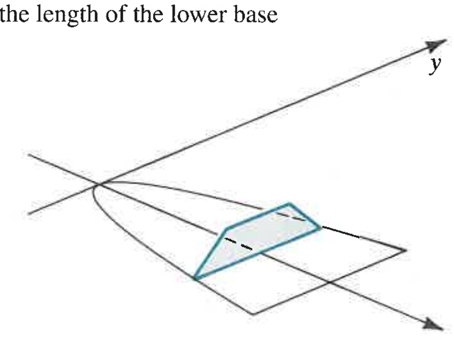
5 An equilateral triangle



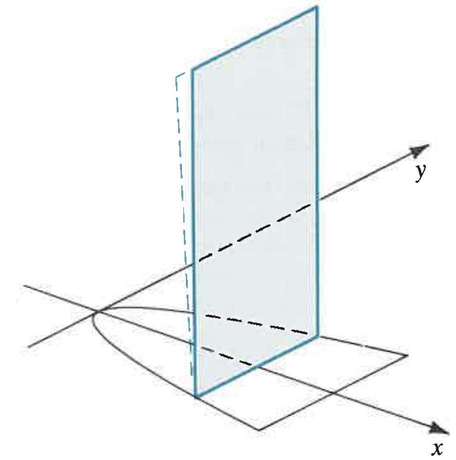
6 A triangle with height equal to $\frac{1}{4}$ the length of the base



7 A trapezoid with lower base in the xy-plane, upper base equal to $\frac{1}{2}$ the length of the lower base, and height equal to $\frac{1}{4}$ the length of the lower base



8 A parallelogram with base in the xy-plane and height equal to twice the length of the base



9 A solid has as its base the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a square.

10 Work Exercise 9 if every cross section is an isosceles triangle with base on the xy-plane and altitude equal to the length of the base.

11 A solid has as its base the region in the xy-plane bounded by the graphs of $y = 4$ and $y = x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles right triangle with hypotenuse on the xy-plane.

12 Work Exercise 11 if every cross section is a square.

13 Find the volume of a pyramid of the type illustrated in Figure 5.39 if the altitude is h and the base is a rectangle of dimensions a and $2a$.

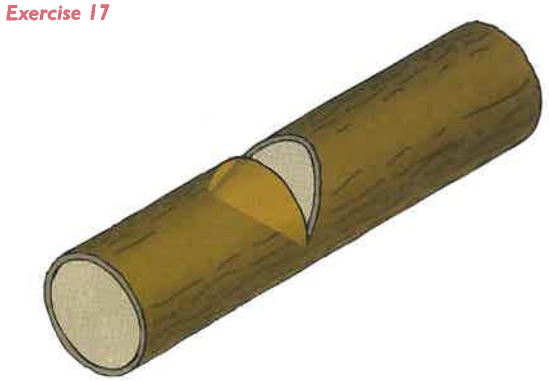
14 A solid has as its base the region in the xy-plane bounded by the graphs of $y = x$ and $y^2 = x$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a semicircle with diameter in the xy-plane.

15 A solid has as its base the region in the xy-plane bounded by the graphs of $y^2 = 4x$ and $x = 4$. If every cross section by a plane perpendicular to the y-axis is a semicircle, find the volume of the solid.

16 A solid has as its base the region in the xy-plane bounded by the graphs of $x^2 = 16y$ and $y = 2$. Every cross section by a plane perpendicular to the y-axis is a rectangle whose height is twice that of the side in the xy-plane. Find the volume of the solid.

17 A log having the shape of a right circular cylinder of radius a is lying on its side. A wedge is removed from the log by making a vertical cut and another cut at an angle of 45°, both cuts intersecting at the center of the log (see figure). Find the volume of the wedge.

Exercise 17



18 The axes of two right circular cylinders of radius a intersect at right angles. Find the volume of the solid bounded by the cylinders.

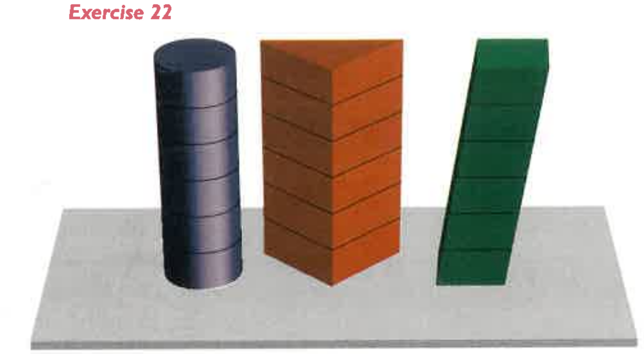
19 The base of a solid is the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles triangle of constant altitude h. (*Hint:* Interpret $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$ as an area.)

20 Cross sections of a horn-shaped solid by planes perpendicular to its axis are circles. If a cross section that is s inches from the smaller end of the solid has diameter $6 + \frac{1}{36}s^2$ inches and if the length of the solid is 2 ft, find its volume.

21 A tetrahedron has three mutually perpendicular faces and three mutually perpendicular edges of lengths 2, 3, and 4 cm, respectively. Find its volume.

22 *Cavalieri's theorem* states that if two solids have equal altitudes and if all cross sections by planes parallel to their bases and at the same distances from their bases have equal areas, then the solids have the same volume (see figure). Prove Cavalieri's theorem.

Exercise 22



23 The base of a solid is an isosceles right triangle whose equal sides have length a. Find the volume if cross sections that are perpendicular to the base and to one of the equal sides are semicircular.

24 Work Exercise 23 if the cross sections are regular hexagons with one side in the base.

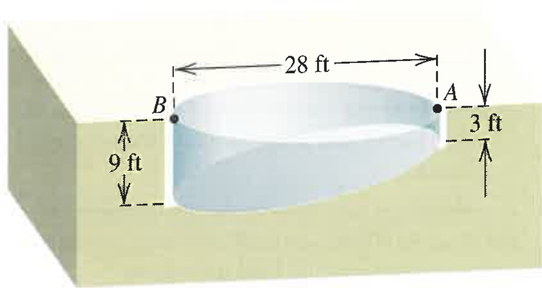
25 Show that the disk and washer methods discussed in Section 5.2 are special cases of (5.13).

c **26** A circular swimming pool has diameter 28 ft. The depth of the water changes slowly from 3 ft at a point A on one side of the pool to 9 ft at a point B diametrically opposite A (see figure). Depth readings $h(x)$ (in feet) taken along the diameter AB are given in the following table, where x is the distance (in feet) from A.

x	0	4	8	12	16	20	24	28
$h(x)$	3	3.5	4	5	6.5	8	8.5	9

Use the trapezoidal rule, with $n = 7$, to estimate the volume of water in the pool. Approximate the number of gallons of water contained in the pool (1 gal $\approx$ 0.134 ft^3).

Exercise 26

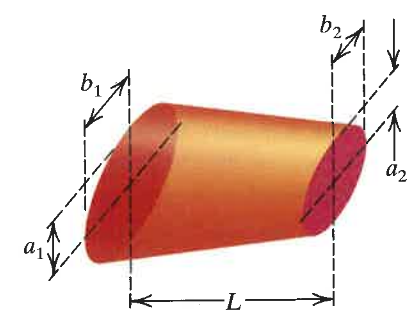


27 The base of a solid is a region bounded by an ellipse with major and minor axes of lengths 16 and 9, respectively. Find the volume of the solid if every cross section by a plane perpendicular to the major axis has the shape of a square.

28 Work Exercise 27 with the cross section having the shape of an equilateral triangle.

29 A common model for human limbs is the *elliptical frustum* shown in the figure, where cross sections perpendicular to the axis of the frustum are elliptical and have the same eccentricity. For human limbs, the eccentricity typically varies from 0.6 to values near 1. If $k = a_1/b_1 = a_2/b_2$ and if L is the length of the limb, show that the volume V is given by the equation $V = (\frac{1}{3}\pi L/k)(a_1^2 + a_1a_2 + a_2^2)$. (*Hint*: Use Exercise 39 in Section 5.1.)

Exercise 29



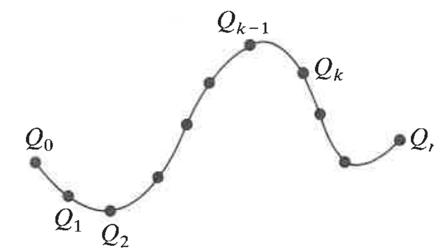
30 The base of a right elliptic cone has major and minor axes of lengths $2a$ and $2b$, respectively. Find the volume if the altitude of the cone is h. (*Hint:* Use Exercise 39 in Section 5.1.)

5.5 ARC LENGTH AND SURFACES OF REVOLUTION

In earlier sections, we considered the volume of the solid created when the graph of a function is revolved about an axis. In this section, we will determine the *length* of the graph and the *surface area* of the solid.

For some applications, we must determine the *length* of the graph of a function. To obtain a suitable formula, we shall employ a process similar to one that could be used to approximate the length of a bent wire. Let us imagine dividing the wire into many small pieces by placing dots at

Figure 5.41
Bent wire



$Q_0, Q_1, Q_2, \ldots, Q_n$, as illustrated in Figure 5.41. We may then approximate the length of the wire between Q_{k-1} and Q_k (for each k) by measuring the distance $d(Q_{k-1}, Q_k)$ with a ruler. The sum of all these distances is an approximation for the total length of the wire. Evidently, the closer together we place the dots, the better the approximation. The process we shall use for the graph of a function is similar; however, we shall find the *exact* length by taking a *limit of sums* of lengths of line segments. This process leads to a definite integral. To guarantee that the integral exists, we must place restrictions on the function, as indicated in the following discussion.

A function f is **smooth** on an interval if it has a derivative f' that is continuous throughout the interval. Intuitively, this means that a small change in x produces a small change in the slope $f'(x)$ of the tangent line to the graph of f. Thus, the graph has no corners or cusps. We shall define the **length of arc** between two points A and B on the graph of a smooth function.

If f is smooth on a closed interval $[a, b]$, the points $A(a, f(a))$ and $B(b, f(b))$ are called the **endpoints** of the graph of f. Let P be the partition of $[a, b]$ determined by $a = x_0, x_1, x_2, \ldots, x_n = b$, and let Q_k denote the point with coordinates $(x_k, f(x_k))$ on the graph of f, as illustrated in Figure 5.42. If we connect each Q_{k-1} to Q_k by a line segment of length $d(Q_{k-1}, Q_k)$, the length L_P of the resulting broken line is

$$L_P = \sum_{k=1}^{n} d(Q_{k-1}, Q_k).$$

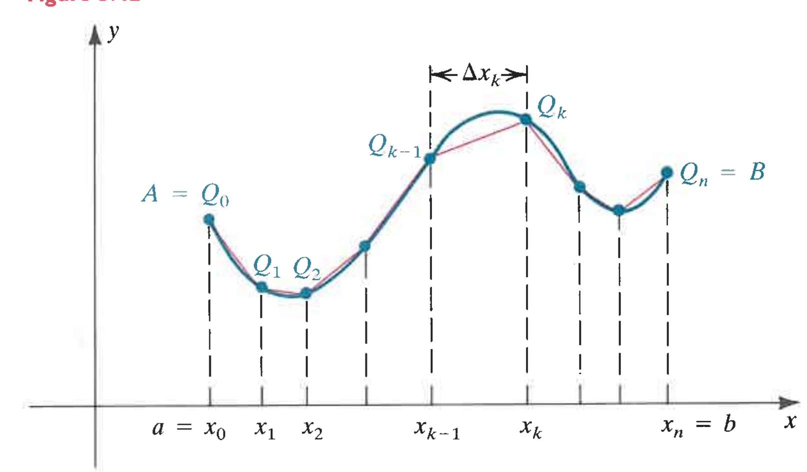
Using the distance formula, we get

$$d(Q_{k-1}, Q_k) = \sqrt{(x_k - x_{k-1})^2 + [f(x_k) - f(x_{k-1})]^2}.$$

By the mean value theorem (3.12),

$$f(x_k) - f(x_{k-1}) = f'(w_k)(x_k - x_{k-1})$$

Figure 5.42



for some number w_k in the open interval (x_{k-1}, x_k). Substituting for $f(x_k) - f(x_{k-1})$ in the preceding formula and letting $\Delta x_k = x_k - x_{k-1}$, we obtain

$$d(Q_{k-1}, Q_k) = \sqrt{(\Delta x_k)^2 + [f'(w_k)\Delta x_k]^2}$$
$$= \sqrt{1 + [f'(w_k)]^2}\,\Delta x_k.$$

Consequently,

$$L_P = \sum_{k=1}^{n} \sqrt{1 + [f'(w_k)]^2}\,\Delta x_k.$$

Observe that L_P is a Riemann sum for $g(x) = \sqrt{1 + [f'(x)]^2}$. Moreover, g is continuous on $[a, b]$, since f' is continuous. If the norm $\|P\|$ is small, then the length L_P of the broken line approximates the length of the graph of f from A to B. This approximation should improve as $\|P\|$ decreases, so we define the *length* (also called the *arc length*) of the graph of f from A to B as the limit of sums L_P. Since $g = \sqrt{1 + (f')^2}$ is a continuous function, the limit exists and equals the definite integral $\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$. This arc length will be denoted by L_a^b.

Definition 5.14

Let f be smooth on $[a, b]$. The **arc length of the graph** of f from $A(a, f(a))$ to $B(b, f(b))$ is

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Definition (5.14) will be extended to more general graphs in Chapter 9. If a function f is defined implicitly by an equation in x and y, then we shall also refer to the *arc length of the graph of the equation.*

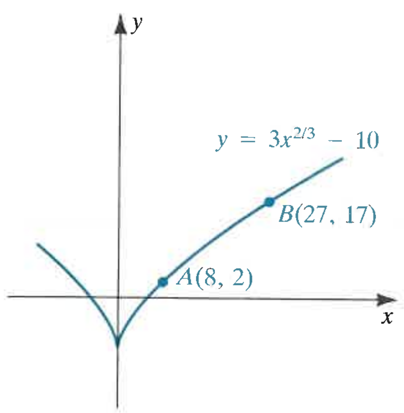


Figure 5.43

EXAMPLE 1 If $f(x) = 3x^{2/3} - 10$, find the arc length of the graph of f from the point $A(8, 2)$ to $B(27, 17)$.

SOLUTION The graph of f is sketched in Figure 5.43. Since

$$f'(x) = 2x^{-1/3} = \frac{2}{x^{1/3}},$$

we have, by Definition (5.14),

$$L_8^{27} = \int_8^{27} \sqrt{1 + \left(\frac{2}{x^{1/3}}\right)^2}\,dx = \int_8^{27} \sqrt{1 + \frac{4}{x^{2/3}}}\,dx$$
$$= \int_8^{27} \sqrt{\frac{x^{2/3} + 4}{x^{2/3}}}\,dx = \int_8^{27} \sqrt{x^{2/3} + 4}\,\frac{1}{x^{1/3}}\,dx.$$

To evaluate this integral, we make the substitution

$$u = x^{2/3} + 4, \qquad du = \frac{2}{3}x^{-1/3}\,dx = \frac{2}{3}\frac{1}{x^{1/3}}\,dx.$$

The integral can be expressed in a suitable form for integration by introducing the factor $\frac{2}{3}$ in the integrand and compensating by multiplying the integral by $\frac{3}{2}$:

$$L_8^{27} = \frac{3}{2}\int_8^{27} \sqrt{x^{2/3} + 4}\left(\frac{2}{3}\frac{1}{x^{1/3}}\right)dx$$

We next calculate the values of $u = x^{2/3} + 4$ that correspond to the limits of integration $x = 8$ and $x = 27$:

(i) If $x = 8$, then $u = 8^{2/3} + 4 = 8$.

(ii) If $x = 27$, then $u = 27^{2/3} + 4 = 13$.

Substituting in the integrand and changing the limits of integration gives us the arc length:

$$L_8^{27} = \tfrac{3}{2}\int_8^{13} \sqrt{u}\,du = \left[u^{3/2}\right]_8^{13} = 13^{3/2} - 8^{3/2} \approx 24.2$$

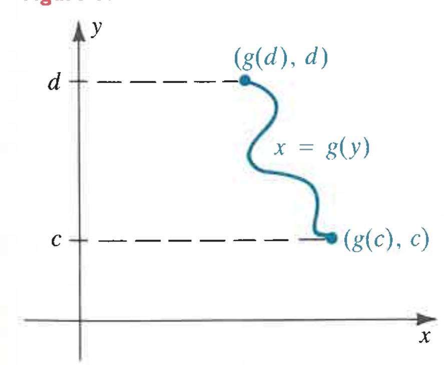
Interchanging the roles of x and y in Definition (5.14) gives us the following formula for integration with respect to y.

Definition 5.15

Let $x = g(y)$ with g smooth on the interval $[c, d]$. The arc length of the graph of g from $(g(c), c)$ to $(g(d), d)$ (see Figure 5.44) is

$$L_c^d = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy.$$

Figure 5.44



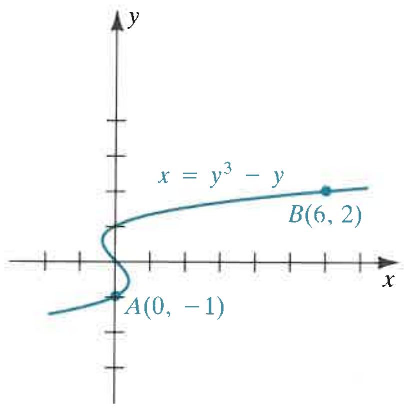
The integrands $\sqrt{1 + [f'(x)]^2}$ and $\sqrt{1 + [g'(y)]^2}$ in formulas (5.14) and (5.15) often result in expressions that have no obvious antiderivatives. In such cases, numerical integration may be used to approximate arc length, as illustrated in the next example.

EXAMPLE 2

(a) Set up an integral for finding the arc length of the graph of the equation $y^3 - y - x = 0$ from $A(0, -1)$ to $B(6, 2)$.

(b) Approximate the integral in part (a) to at least four-decimal-place accuracy by using Simpson's rule (4.38).

Figure 5.45



SOLUTION

(a) Since the equation is not of the form $y = f(x)$, we cannot apply Definition (5.14) directly. However, if we write $x = y^3 - y$, then we can use (5.15) with $g(y) = y^3 - y$. The graph of the equation is sketched in Figure 5.45. Using (5.15) with $c = -1$ and $d = 2$ yields

$$L_{-1}^{2} = \int_{-1}^{2} \sqrt{1 + (3y^2 - 1)^2}\, dy$$
$$= \int_{-1}^{2} \sqrt{9y^4 - 6y^2 + 2}\, dy.$$

(b) We compute Simpson's rule repeatedly, beginning with $n = 1$ and then successively doubling the number of subintervals—that is, $n = 1, 2, 4, 8, \ldots$. The following table shows the results of our calculations.

n	S_n
1	8.70226731015
2	8.94490388877
4	8.70891806925
8	8.72498046484
16	8.72499726385
32	8.72500017224

From the table, we see that to four decimal places we obtain the approximation 8.7250 for $n = 8$, 16, and 32. We know, furthermore, from Theorem (4.39) that another doubling of n would divide the error by 16. Thus, subsequent changes in the approximation based on larger values of n would not affect the first four decimal places. Our conclusion is

$$\int_{-1}^{2} \sqrt{9y^4 - 6y^2 + 2}\, dy \approx 8.7250.$$

A function f is **piecewise smooth** on its domain if the graph of f can be decomposed into a finite number of parts, each of which is the graph of a smooth function. We define the arc length of the graph as the sum of the arc lengths of the individual graphs.

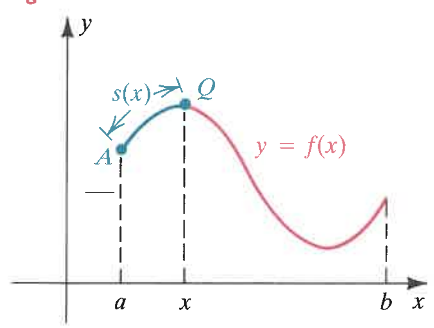
To avoid any misunderstanding in the following discussion, we shall denote the variable of integration by t. In this case, the arc length formula in Definition (5.14) is written

$$L_a^b = \int_a^b \sqrt{1 + [f'(t)]^2}\, dt.$$

If f is smooth on $[a, b]$, then f is smooth on $[a, x]$ for every number x in $[a, b]$, and the length of the graph from the point $A(a, f(a))$ to the point $Q(x, f(x))$ is

$$L_a^x = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

Figure 5.46



If we change the notation and use the symbol $s(x)$ in place of L_a^x, then s may be regarded as a function with domain $[a, b]$, since to each x in $[a, b]$ there corresponds a unique number $s(x)$. As shown in Figure 5.46, $s(x)$ is the length of arc of the graph of f from $A(a, f(a))$ to $Q(x, f(x))$. We shall call s the *arc length function* for the graph of f, as in the next definition.

Definition 5.16

Let f be smooth on $[a, b]$. The **arc length function** s for the graph of f on $[a, b]$ is defined by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

for $a \le x \le b$.

If s is the arc length function, the differential ds is called the **differential of arc length**. The next theorem specifies formulas for finding ds.

Theorem 5.17

Let f be smooth on $[a, b]$, and let s be the arc length function for the graph of $y = f(x)$ on $[a, b]$. If Δx is an increment in the variable x, then

(i) $\dfrac{ds}{dx} = \sqrt{1 + [f'(x)]^2}$

(ii) $ds = \sqrt{1 + [f'(x)]^2}\,\Delta x$

PROOF By Definition (5.16) and Theorem (4.35),

$$\frac{ds}{dx} = \frac{d}{dx}\left[\int_a^x \sqrt{1 + [f'(t)]^2}\, dt\right] = \sqrt{1 + [f'(x)]^2}.$$

Applying Definition (2.34) yields Theorem (5.17)(ii). ■

EXAMPLE 3 Approximate the arc length of the graph of the equation $y = x^3 + 2x$ from $A(1, 3)$ to $B(1.2, 4.128)$ by

(a) differentials at $x_0 = 1$

(b) the line segment from A to B

(c) the trapezoidal rule applied to the definite integral from Definition (5.15) with $n = 10$ and $n = 20$

SOLUTION

(a) If we let $g(x) = x^3 + 2x$, then $g'(x) = 3x^2 + 2$, so by Theorem (5.17)(ii),

$$ds = \sqrt{1 + (3x^2 + 2)^2}\,\Delta x.$$

We obtain an approximation by letting $x = 1$ and $\Delta x = 0.2$:

$$\Delta s \approx ds = \sqrt{1 + 5^2}(0.2) = \sqrt{26}(0.2) \approx 1.01980$$

(b) By the distance formula (4) on page 10, the length of the line segment from $A(1, 3)$ to $B(1.2, 4.128)$ is

$$d(A, B) = \sqrt{(1.2 - 1)^2 + (4.128 - 3)^2} = \sqrt{(0.2)^2 + (1.128)^2} \approx 1.14559.$$

(c) For the trapezoidal rule, with $n = 10$, we have

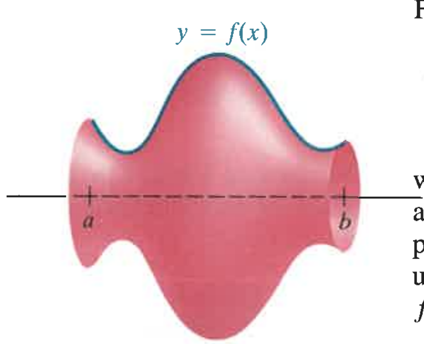
$$\Delta x = \frac{b - a}{n} = \frac{1.2 - 1}{10} = 0.02 \quad \text{so that} \quad x_k = 1 + 0.02k.$$

From (4.37), we have

$$T_{10} = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_9) + f(x_{10})]$$
$$= (0.01)[f(1) + 2f(1.02) + 2f(1.04) + \cdots + 2f(1.18) + f(1.2)]$$

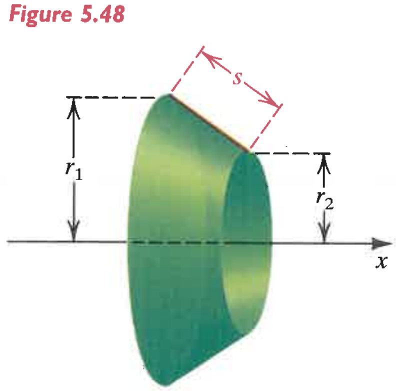
with $f(x) = \sqrt{1 + (3x^2 + 2)^2}$. Calculating each term in this sum and then adding them together is a task best left to a computing device with a built-in program for the trapezoidal rule. Using such a program, which allows the user to enter the values for a, b, and n and an expression for the function f, we obtain $T_{10} \approx 1.456709$ and $T_{20} \approx 1.456811$.

Figure 5.47



Let f be a function that is nonnegative throughout a closed interval $[a, b]$. If the graph of f is revolved about the x-axis, a **surface of revolution** is generated (see Figure 5.47). For example, if $f(x) = \sqrt{r^2 - x^2}$ for a positive constant r, then the graph of f on $[-r, r]$ is the upper half of the circle $x^2 + y^2 = r^2$, and a revolution about the x-axis produces a sphere of radius r having surface area $4\pi r^2$.

Figure 5.48



If the graph of f is the line segment shown in Figure 5.48, then the surface generated is a frustum of a cone having base radii r_1 and r_2 and slant height s. It can be shown that the surface area is

$$\pi(r_1 + r_2)s = 2\pi\left(\frac{r_1 + r_2}{2}\right)s.$$

You may remember this formula as follows.

Surface Area S of a Frustum of a Cone 5.18

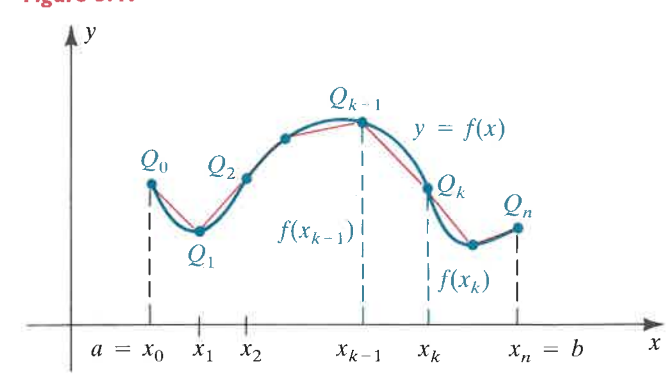
$$S = 2\pi\,(\text{average radius})(\text{slant height})$$

We shall use this fact in the following discussion.

Let f be a smooth function that is nonnegative on $[a, b]$, and consider the surface generated by revolving the graph of f about the x-axis (see Figure 5.47). We wish to find a formula for the area S of this surface. Let P be a partition of $[a, b]$ determined by $a = x_0, x_1, \ldots, x_n = b$, and for each k, let Q_k denote the point $(x_k, f(x_k))$ on the graph of f (see Figure 5.49). If the norm $\|P\|$ is close to zero, then the broken line l_P obtained by connecting Q_{k-1} to Q_k for each k is an approximation to the graph of f, and hence the area of the surface generated by revolving l_P about the x-axis should approximate S. The line segment $Q_{k-1}Q_k$ generates a frustum of a cone having base radii $f(x_{k-1})$ and $f(x_k)$ and slant height $d(Q_{k-1}, Q_k)$. By (5.18), its surface area is

$$2\pi\frac{f(x_{k-1}) + f(x_k)}{2}d(Q_{k-1}, Q_k).$$

Figure 5.49



Summing terms of this form from $k = 1$ to $k = n$ gives us the area S_P of the surface generated by the broken line l_P. If we use the expression for $d(Q_{k-1}, Q_k)$ on page 474, then

$$S_P = \sum_{k=1}^{n} 2\pi\frac{f(x_{k-1}) + f(x_k)}{2}\sqrt{1 + [f'(w_k)]^2}\,\Delta x_k,$$

where $x_{k-1} < w_k < x_k$. We define the area S of the surface of revolution as

$$S = \lim_{\|P\| \to 0} S_P.$$

From the form of S_P, it is reasonable to expect that the limit is given by

$$\int_a^b 2\pi\frac{f(x) + f(x)}{2}\sqrt{1 + [f'(x)]^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

The proof of this fact requires results from advanced calculus and is omitted. The following definition summarizes our discussion.

Definition 5.19

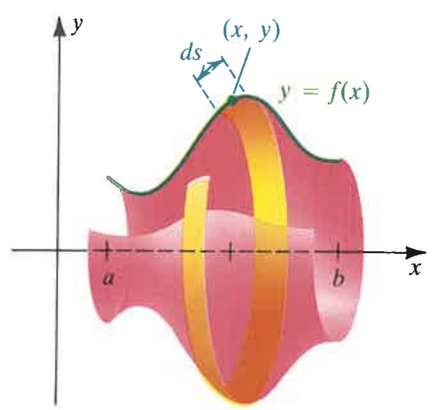
If f is smooth and $f(x) \geq 0$ on $[a, b]$, then the **area** S of the surface generated by revolving the graph of f about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

If f is negative for some x in $[a, b]$, then the following extension of Definition (5.19) can be used to find the surface area S:

$$S = \int_a^b 2\pi\,|f(x)|\,\sqrt{1 + [f'(x)]^2}\,dx$$

Figure 5.50



We can use (5.18) to remember the formula for S in Definition (5.19). As in Figure 5.50, let (x, y) denote an arbitrary point on the graph of f and, as in Theorem (5.17)(ii), consider the differential of arc length

$$ds = \sqrt{1 + [f'(x)]^2}\,\Delta x.$$

Next, regard ds as the slant height of the frustum of a cone that has average radius $y = f(x)$ (see Figure 5.50). Applying (5.18), the surface area of this frustum is given by

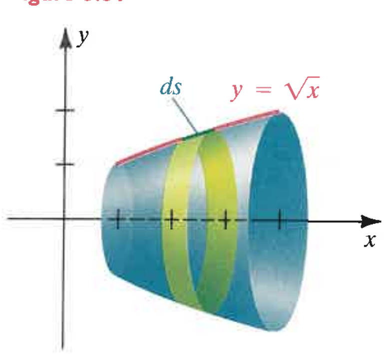
$$2\pi f(x)\,ds = 2\pi y\,ds.$$

As with our work in Sections 5.1–5.3, applying $\int_b^a$ may be regarded as taking a limit of sums of these areas of frustums. Thus,

$$S = \int_a^b 2\pi f(x)\,ds = \int_a^b 2\pi y\,ds.$$

EXAMPLE 4 The graph of $y = \sqrt{x}$ from (1, 1) to (4, 2) is revolved about the x-axis. Find the area of the resulting surface.

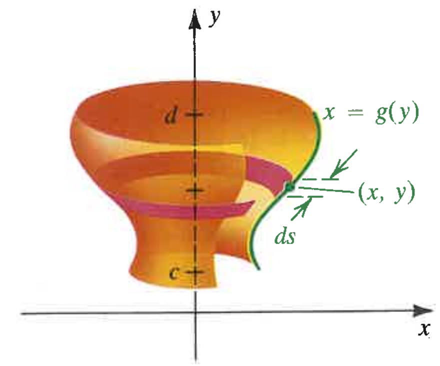
Figure 5.51



SOLUTION The surface is illustrated in Figure 5.51. Using Definition (5.19) or the previous discussion, we have

$$\begin{aligned} S &= \int_1^4 2\pi y\,ds \\ &= \int_1^4 2\pi x^{1/2}\sqrt{1 + \left(\frac{1}{2x^{1/2}}\right)^2}\,dx \\ &= \int_1^4 2\pi x^{1/2}\sqrt{\frac{4x+1}{4x}}\,dx = \pi\int_1^4 \sqrt{4x+1}\,dx \\ &= \frac{\pi}{6}\left[(4x+1)^{3/2}\right]_1^4 = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85 \text{ square units.} \end{aligned}$$

Figure 5.52



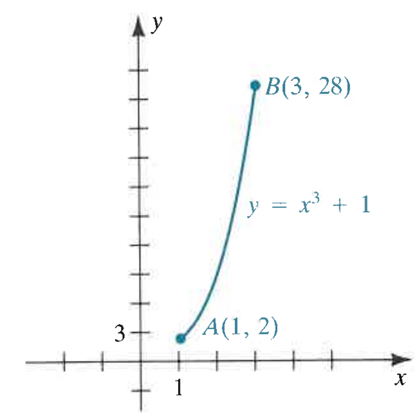
If we interchange the roles of x and y in the preceding discussion, then a formula analogous to (5.19) can be obtained for integration with respect to y. Thus, if $x = g(y)$ and g is smooth and nonnegative on $[c, d]$, then the area S of the surface generated by revolving the graph of g about the y-axis (see Figure 5.52) is

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\,dy.$$

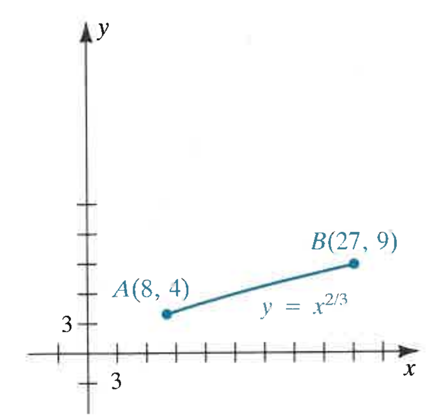
EXERCISES 5.5

Exer. 1–4: Set up an integral that can be used to find the arc length of the graph from A to B by integrating with respect to (a) x and (b) y.

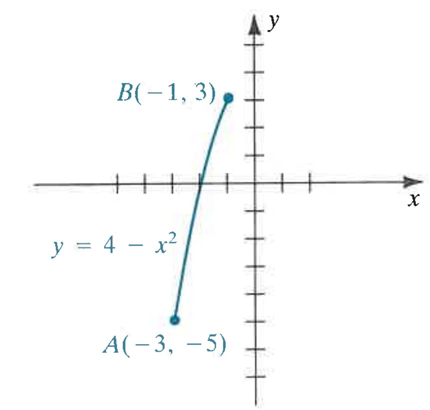
1



2



3



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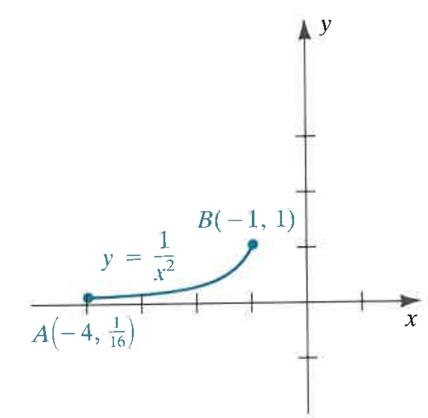


Exer. 5–12: Find the arc length of the graph of the equation from A to B.

5 $y = \frac{2}{3}x^{2/3}$; $A(1, \frac{2}{3})$, $B(8, \frac{8}{3})$

6 $(y+1)^2 = (x-4)^3$; $A(5, 0)$, $B(8, 7)$

$7\ \ y = 5 - \sqrt{x^3}$; $\quad A(1,\ 4)$, $\quad B(4,\ -3)$

$8\ \ y = 6\sqrt[3]{x^2} + 1$; $\quad A(-1,\ 7)$, $\quad B(-8,\ 25)$

$9\ \ y = \dfrac{x^3}{12} + \dfrac{1}{x}$; $\quad A(1,\ \frac{13}{12})$, $\quad B(2,\ \frac{7}{6})$

$10\ \ y + \dfrac{1}{4x} + \dfrac{x^3}{3} = 0$; $\quad A(2,\ \frac{67}{24})$, $\quad B(3,\ \frac{109}{12})$

$11\ \ 30xy^3 - y^8 = 15$; $\quad A(\frac{8}{15},\ 1)$, $\quad B(\frac{271}{240},\ 2)$

$12\ \ x = \dfrac{y^4}{16} + \dfrac{1}{2y^2}$; $\quad A(\frac{9}{8},\ -2)$, $\quad B(\frac{9}{16},\ -1)$

Exer. 13–14: Set up an integral for finding the arc length of the graph of the equation from A to B.

13 $2y^3 - 7y + 2x = 8$; $\quad A(3, 2)$, $\quad B(4, 0)$

14 $11x - 4x^3 - 7y = -7$; $\quad A(1, 2)$, $\quad B(0, 1)$

15 Find the arc length of the graph of $x^{2/3} + y^{2/3} = 1$. (*Hint:* Use symmetry with respect to the line $y = x$.)

16 Find the arc length of the graph of $y = \dfrac{3x^8 + 5}{30x^3}$ from $(1, \frac{4}{5})$ to $(2, \frac{773}{240})$.

c **Exer. 17–22: (a) Set up an integral for finding the arc length of the graph of f between A and B, and approximate the integral by using Simpson's rule with $n = 4$. (b) Approximate the arc length from A to B by differentials. (c) Approximate the arc length from A to B by computing the length of the line segment from A to B.**

17 $f(x) = \sqrt[3]{x^2}$; $\quad A(1,\ 1)$, $\quad B\left(1.1,\ \sqrt[3]{1.1^2}\right)$

18 $f(x) = \sqrt{x^3}$; $\quad A(1,\ 1)$, $\quad B\left(1.1,\ \sqrt{1.1^3}\right)$

19 $f(x) = x^2$; $\quad A(2,\ 4)$, $\quad B(2.1,\ 4.41)$

20 $f(x) = -x^3$; $\quad A(1,\ -1)$, $\quad B(1.1,\ -1.331)$

21 $f(x) = \cos x$; $\quad A\left(\dfrac{\pi}{6},\ \dfrac{\sqrt{3}}{2}\right)$, $\quad B\left(\dfrac{31\pi}{180},\ \cos\dfrac{31\pi}{180}\right)$

22 $f(x) = \sin x$; $\quad A(0,\ 0)$, $\quad B\left(\dfrac{\pi}{90},\ \sin\dfrac{\pi}{90}\right)$

c **Exer. 23–26: Use Simpson's rule, with $n = 8$ or larger, or use the numerical integration provided by a calculator or computer application to approximate the arc length of the graph of the equation from A to B.**

23 $y = x^2 + x + 3$; $\quad A(-2,\ 5)$, $\quad B(2,\ 9)$

24 $y = x^3$; $\quad A(0,\ 0)$, $\quad B(2,\ 8)$

25 $y = \csc x$; $\quad A\left(\dfrac{\pi}{4},\ \sqrt{2}\right)$, $\quad B\left(\dfrac{3\pi}{4},\ \sqrt{2}\right)$

26 $y = \tan x$; $\quad A(0,\ 0)$, $\quad B\left(\dfrac{\pi}{4},\ 1\right)$

c **Exer. 27–28: Consider the arc length L_a^b of the graph of f from the point $A(a, f(a))$ to the point $B(b, f(b))$. Let $a = x_0, x_1, \ldots, x_n = b$ be a regular partition of $[a, b]$ with $\Delta x = (b - a)/n$. (a) Approximate L_a^b by $\sum_{k=1}^{n} d(Q_{k-1}, Q_k)$, where Q_k is the point $(x_k, f(x_k))$ on the graph, for $n = 4$ and $n = 8$. In general, for any n, how does this approximation compare to L_a^b? (b) Set up a definite integral for L_a^b and approximate this integral using the trapezoidal rule for $n = 4$ and $n = 8$.**

27 $f(x) = \sin x$; $\quad A(0,\ 0)$, $\quad B(\pi,\ 0)$

28 $f(x) = \sin(\sin x)$; $\quad A(0,\ 0)$, $\quad B(\pi,\ 0)$

Exer. 29–32: The graph of the equation from A to B is revolved about the x-axis. Find the area of the resulting surface.

29 $4x = y^2$; $\quad A(0,\ 0)$, $\quad B(1,\ 2)$

30 $y = x^3$; $\quad A(1,\ 1)$, $\quad B(2,\ 8)$

31 $8y = 2x^4 + x^{-2}$; $\quad A(1,\ \frac{3}{8})$, $\quad B(2,\ \frac{129}{32})$

32 $y = 2\sqrt{x + 1}$; $\quad A(0,\ 2)$, $\quad B(3,\ 4)$

Exer. 33–34: The graph of the equation from A to B is revolved about the y-axis. Find the area of the resulting surface.

33 $y = 2\sqrt[3]{x}$; $\quad A(1,\ 2)$, $\quad B(8,\ 4)$

34 $x = 4\sqrt{y}$; $\quad A(4,\ 1)$, $\quad B(12,\ 9)$

Exer. 35–36: If the smaller arc of the circle $x^2 + y^2 = 25$ between the points $(-3, 4)$ and $(3, 4)$ is revolved about the given axis, find the area of the resulting surface.

35 The y-axis

36 The x-axis

Exer. 37–39: Use a definite integral to derive a formula for the surface area of the indicated solid.

37 A right circular cone of altitude h and base radius r

38 A spherical segment of altitude h in a sphere of radius r

39 A sphere of radius r

40 Show that the area of the surface of a sphere of radius a between two parallel planes depends only on the distance between the planes. (*Hint:* Use Exercise 38.)

41 If the graph in Figure 5.50 is revolved about the y-axis, show that the area of the resulting surface is given by

$$\int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\,dx.$$

42 Use Exercise 41 to find the area of the surface generated by revolving the graph of $y = 3\sqrt[3]{x}$ from $A(1, 3)$ to $B(8, 6)$ about the y-axis.

c **Exer. 43–44: Let S be the area of the surface generated by revolving the graph of f from $A(a, f(a))$ to $B(b, f(b))$ about the x-axis. Let $a = x_0, x_1, \ldots, x_n = b$ be a regular partition of $[a, b]$ with $\Delta x = (b - a)/n$. (a) Approximate S by**

$$\sum_{k=1}^{n} 2\pi\left[\frac{f(x_{k-1}) + f(x_k)}{2}\right] d(Q_{k-1}, Q_k),$$

where Q_k is the point $(x_k, f(x_k))$ on the graph for $n = 4$ and $n = 8$. In general, for any n, how does this approximation compare to S? (b) Set up a definite integral for S and approximate this integral using the trapezoidal rule for $n = 4$ and $n = 8$.

43 $f(x) = \sin x$; $\quad A(0,\ 0)$, $\quad B(\pi,\ 0)$

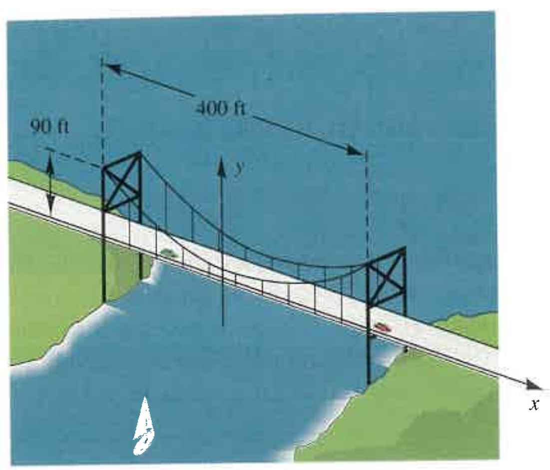
44 $f(x) = 1 - x^3$; $\quad A(0,\ 1)$, $\quad B(1,\ 0)$

c 45 An American football has the approximate shape of the solid generated by revolving the arc of a circle, $x^2 + (y + k)^2 = r^2$, where $y \geq 0$ and $0 \leq k < r$. For a full-sized football, the arc from point to point measures about 14 in. along a seam. Around the widest part, the circumference measures about 22 in. Approximate the surface area of the football.

c 46 For a junior-sized football, the arc from point to point measures about 13 in. along a seam. (Assume the same model for a football used in Exercise 45.) Around the widest part, the circumference measures about 18 in. Approximate the surface area for a junior-sized football.

47 One section of a suspension bridge has its weight uniformly distributed between twin towers that are 400 ft apart and that rise 90 ft above the horizontal roadway. A cable strung between the tops of the towers has the shape of a parabola, with center point 10 ft above the roadway. Suppose coordinate axes are introduced, as shown in the figure.

Exercise 47



(a) Find an equation for the parabola.

(b) Set up an integral whose value is the length of the cable.

(c) If nine equispaced vertical cables are used to support the parabolic cable, find the total length of these supports.

48 Let R be the region bounded by the parabola $x = ay^2$ and the line l through the focus that is perpendicular to the axis of the parabola. Find the area of the curved surface obtained by revolving R about the x-axis.

49 A radio telescope has the shape of a paraboloid of revolution (see Exercise 47 of Section 5.2) with focal length p and diameter of base $2a$.

(a) Show that the surface area S available for collecting radio waves is

$$S = \frac{8\pi p^2}{3}\left[\left(1 + \frac{a^2}{4p^2}\right)^{3/2} - 1\right].$$

c (b) One of the largest radio telescopes, located in Jodrell Bank, Cheshire, England, has diameter 250 ft and focal length 50 ft. Approximate S to the nearest square foot.

50 (a) Show that the circumference C of the ellipse with the equation

$$(x^2/a^2) + (y^2/b^2) = 1$$

is given by

$$C = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta,$$

where e is the eccentricity. (This is called an *elliptic integral*, and it cannot be evaluated exactly using methods we have presented to this point.)

c (b) The planet Mercury travels in an elliptical orbit with $e = 0.206$ and $a = 0.387$ AU. Use part (a) and Simpson's rule, with $n = 10$, to approximate the length of the orbit.

(c) Find the maximum and minimum distances between Mercury and the sun.

5.6 WORK

We may consider the concept of *force* as a push or pull on an object. For example, we need a force to push or pull furniture across a floor, to lift an object off the ground, to stretch or compress a spring, or to move a charged particle through an electromagnetic field. In this section, we discuss the *work* done by a continuously varying force.

If an object weighs 10 lb, then by definition the force required to lift it (or hold it off the ground) is 10 lb. A force of this type is a **constant force,** since its magnitude does not change while it is being applied to the object.

The concept of *work* is used when a force acts through a distance. The following definition covers the simplest case, in which the object moves along a line in the same direction as the applied force.

Definition 5.20

If a constant force F acts on an object, moving it a distance d in the direction of the force, the **work** W done is

$$W = Fd.$$

The following table lists units of force and work in the British system and the International System (abbreviated SI, for the French *Système International*). In SI units, 1 Newton is the force required to impart an acceleration of 1 m/sec^2 to a mass of 1 kilogram.

System	Unit of force	Unit of distance	Unit of work
British	pound (lb)	foot (ft)	foot-pound (ft-lb)
		inch (in.)	inch-pound (in.-lb)
International (SI)	Newton (N)	meter (m)	Newton-meter (N-m)

A Newton-meter is also called a *joule* (J). It can be shown that

$$1 \text{ N} \approx 0.225 \text{ lb} \quad \text{and} \quad 1 \text{ N-m} \approx 0.74 \text{ ft-lb}.$$

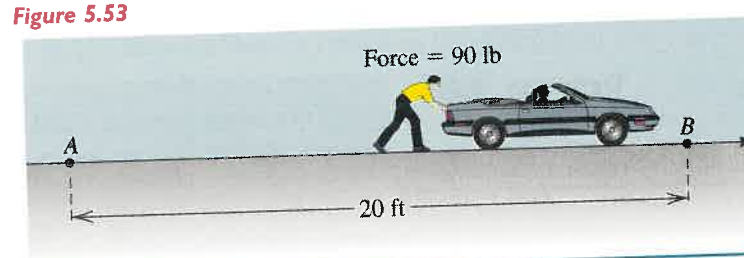
For simplicity, in examples and most exercises we will use the British system, in which the magnitude of the force is the same as the weight, in pounds, of the object. In using SI units, it is often necessary to consider a gravitational constant a (9.81 m/sec^2) and use Newton's second law of motion, $F = ma$, to change a mass m (in kilograms) to a force F (in Newtons).

EXAMPLE 1 Find the work done in pushing an automobile a distance of 20 ft along a level road while exerting a constant force of 90 lb.

SOLUTION The problem is illustrated in Figure 5.53. Since the constant force is $F = 90$ lb and the distance that the automobile moves is $d = 20$ ft, it follows from Definition (5.20) that the work done is

$$W = (90)(20) = 1800 \text{ ft-lb}.$$

Figure 5.53

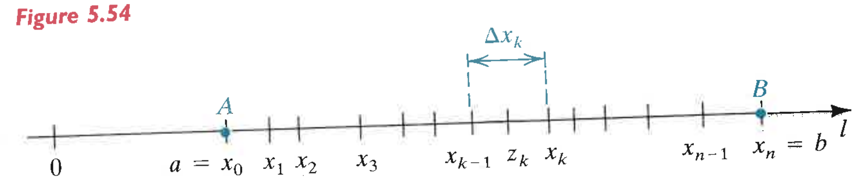


Anyone who has pushed an automobile (or some other object) is aware that the force applied is seldom constant. Thus, if an automobile is stalled, a larger force may be required to get it moving than to keep it in motion. The force may also vary because of friction, since part of the road may be smooth and another part rough. A force that is not constant is a **variable force.** We next develop a method for determining the work done by a variable force in moving an object rectilinearly in the same direction as the force.

Suppose that a force moves an object along the x-axis from $x = a$ to $x = b$ and that the force at x is given by $f(x)$, where f is continuous on $[a, b]$. (The phrase *force at x* means the force acting at the point with coordinate x.) As shown in Figure 5.54, we begin by considering a partition P of $[a, b]$ determined by

$$a = x_0, x_1, x_2, \ldots, x_n = b \quad \text{with} \quad \Delta x_k = x_k - x_{k-1}.$$

Figure 5.54



If ΔW_k is the **increment of work**—that is, the amount of work done from x_{k-1} to x_k—then the work W done from a to b is the sum

$$W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_n = \sum_{k=1}^{n} \Delta W_k.$$

To approximate ΔW_k, we choose any number z_k in $[x_{k-1}, x_k]$ and consider the force $f(z_k)$ at z_k. If the norm $\|P\|$ is small, then intuitively we know that the function values change very little on $[x_{k-1}, x_k]$—that is, f is *almost constant* on this interval. Applying Definition (5.20) gives us

$$\Delta W_k \approx f(z_k)\Delta x_k$$

and hence

$$W = \sum_{k=1}^{n} \Delta W_k \approx \sum_{k=1}^{n} f(z_k)\Delta x_k.$$

Since this approximation should improve as $\|P\| \to 0$, we define W as a limit of such sums. This limit leads to a definite integral.

Definition 5.21

If $f(x)$ is the force at x and if f is continuous on $[a, b]$, then the **work** W done in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \lim_{\|P\|\to 0} \sum_k f(z_k)\Delta x_k$$
$$= \int_a^b f(x)\,dx.$$

An analogous definition can be stated for an interval on the y-axis by replacing x with y throughout our discussion.

Definition (5.21) can be used to find the work done in stretching or compressing a spring. To solve problems of this type, it is necessary to use the following law from physics.

Hooke's Law: The force $f(x)$ required to stretch a spring x units beyond its natural length is given by $f(x) = kx$, where k is a constant called the **spring constant.**

EXAMPLE 2 A force of 9 lb is required to stretch a spring from its natural length of 6 in. to a length of 8 in. Find the work done in stretching the spring

(a) from its natural length to a length of 10 in.

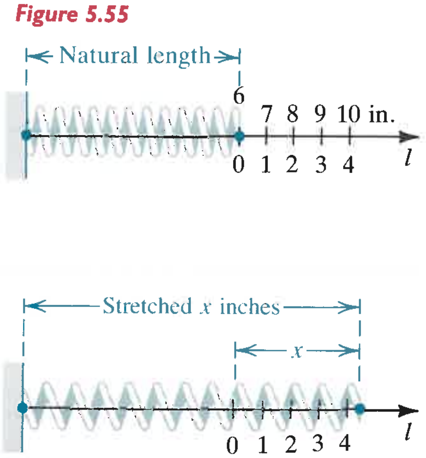
(b) from a length of 7 in. to a length of 9 in.

SOLUTION

(a) Let us introduce an x-axis as shown in Figure 5.55, with one end of the spring attached to a point to the left of the origin and the end to be pulled located at the origin. According to Hooke's law, the force $f(x)$ required to stretch the spring x units beyond its natural length is $f(x) = kx$ for some constant k. Since a 9-lb force is required to stretch the spring 2 in. beyond its natural length, we have $f(2) = 9$. We let $x = 2$ in $f(x) = kx$:

$$9 = k \cdot 2, \quad \text{or} \quad k = \tfrac{9}{2}$$

Figure 5.55



Consequently, for this spring, Hooke's law has the form

$$f(x) = \tfrac{9}{2}x.$$

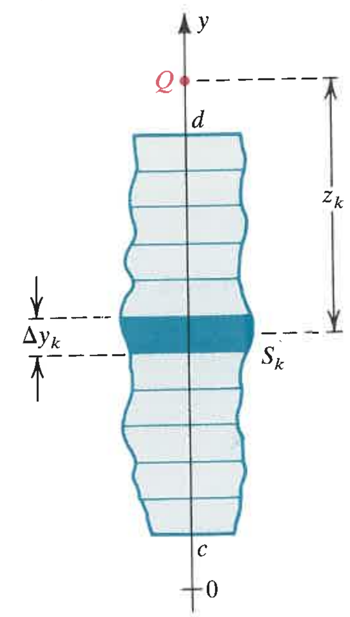
Applying Definition (5.21) with $a = 0$ and $b = 4$, we can determine the work done in stretching the spring 4 in.:

$$W = \int_0^4 \frac{9}{2}x\,dx = \frac{9}{2}\left[\frac{x^2}{2}\right]_0^4 = 36 \text{ in.-lb}$$

(b) We again use the force $f(x) = \frac{9}{2}x$ obtained in part (a). By Definition (5.21), the work done in stretching the spring from $x = 1$ to $x = 3$ is

$$W = \int_1^3 \frac{9}{2}x\,dx = \frac{9}{2}\left[\frac{x^2}{2}\right]_1^3 = 18 \text{ in.-lb.}$$

Figure 5.56



In some applications, we wish to determine the work done in pumping out a tank containing a fluid or in lifting an object, such as a chain or a cable, that extends vertically between two points. A general situation is illustrated in Figure 5.56, which shows a solid that extends along the y-axis from $y = c$ to $y = d$. We wish to vertically lift all particles contained in the solid to the level of point Q. Let us consider a partition P of $[c, d]$ and imagine slicing the solid by means of planes perpendicular to the y-axis at each number y_k in the partition. As shown in the figure, $\Delta y_k = y_k - y_{k-1}$, and S_k represents the kth slice. We next introduce the following notation:

$$z_k = \text{the (approximate) distance } S_k \text{ is lifted}$$
$$\Delta F_k = \text{the (approximate) force required to lift } S_k$$

If ΔW_k is the work done in lifting S_k, then, by Definition (5.20),

$$\Delta W_k \approx \Delta F_k \cdot z_k = z_k \cdot \Delta F_k.$$

We define the work W done in lifting the entire solid as a limit of sums.

Definition 5.22

$$W = \lim_{\|P\|\to 0} \sum_k z_k \cdot \Delta F_k$$

The limit leads to a definite integral. Note the difference between this type of problem and that in our earlier discussion. To obtain (5.21), we considered *distance increments* Δx_k and the force $f(z_k)$ that acts through Δx_k. In the present situation, we consider *force increments* ΔF_k and the distance z_k through which ΔF_k acts. The next two examples illustrate this technique. As in preceding sections, we shall use dy to represent a typical increment Δy_k and y to denote a number in $[c, d]$.

Figure 5.57

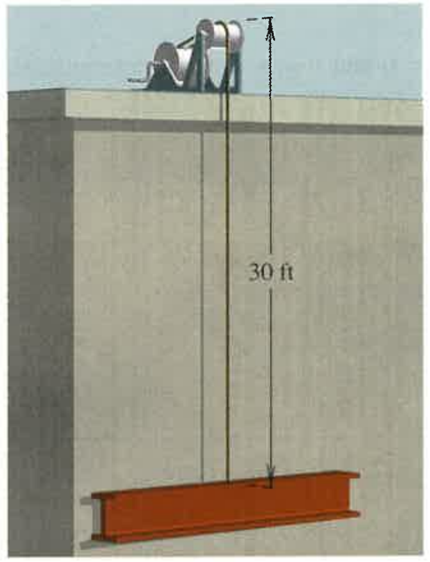
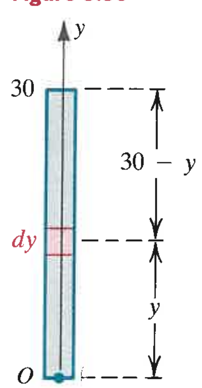


Figure 5.58



EXAMPLE ■ 3 A uniform cable 30 ft long and weighing 60 lb hangs vertically from a pulley system at the top of a building, as shown in Figure 5.57. A steel beam weighing 500 lb is attached to the end of the cable. Find the work required to pull it to the top.

SOLUTION Let W_B denote the work required to pull the beam to the top, and let W_C denote the work required for the cable. Since the beam weighs 500 lb and must move through a distance of 30 ft, we have, by Definition (5.20),

$$W_B = 500 \cdot 30 = 15{,}000 \text{ ft-lb.}$$

The work required to pull the cable to the top may be found by the method used to obtain (5.22). Consider a y-axis with the lower end of the cable at the origin and the upper end at $y = 30$, as in Figure 5.58. Let dy denote an increment of length of the cable. Since each foot of cable weighs $60/30 = 2$ lb, the weight of the increment (and hence the force required to lift it) is $2\,dy$. If y denotes the distance from O to a point in the increment, then we have the following:

$$\begin{aligned} \text{increment of force:}&\quad 2\,dy \\ \text{distance lifted:}&\quad 30 - y \\ \text{increment of work:}&\quad (30 - y)2\,dy \end{aligned}$$

Applying $\int_0^{30}$ takes a limit of sums of the increments of work. Hence,

$$W_C = \int_0^{30} (30 - y)2\,dy$$

$$= 2\left[30y - \tfrac{1}{2}y^2\right]_0^{30} = 900 \text{ ft-lb.}$$

The total work required is

$$W = W_B + W_C = 15{,}000 + 900 = 15{,}900 \text{ ft-lb.}$$

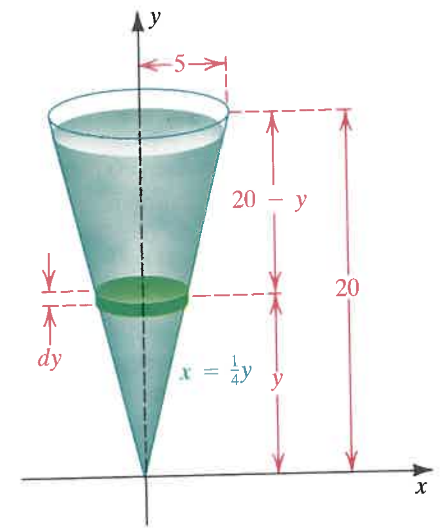
EXAMPLE ■ 4 A right circular conical tank of altitude 20 ft and radius of base 5 ft has its vertex at ground level and axis vertical. If the tank is full of water weighing 62.5 lb/ft^3, find the work done in pumping all the water over the top of the tank.

SOLUTION We begin by introducing a coordinate system, as shown in Figure 5.59. The cone intersects the xy-plane along lines of slope -4 and 4 through the origin. An equation for the line with slope 4 is

$$y = 4x, \quad \text{or} \quad x = \tfrac{1}{4}y.$$

Let us imagine subdividing the water into slices, using planes perpendicular to the y-axis, from $y = 0$ to $y = 20$. If dy represents the width of a typical slice, then its volume may be approximated by the circular disk

Figure 5.59



shown in Figure 5.59. As we did in our work with volumes of revolution in Section 5.2, we obtain

$$\text{volume of disk } = \pi x^2\,dy = \pi\left(\tfrac{1}{4}y\right)^2 dy.$$

Since water weighs 62.5 lb/ft^3, the weight of the disk, and hence the force required to lift it, is $62.5\pi\left(\frac{1}{4}y\right)^2 dy$. Thus, we have

$$\begin{aligned} \text{increment of force:}&\quad 62.5\pi\left(\tfrac{1}{16}y^2\right) dy \\ \text{distance lifted:}&\quad 20 - y \\ \text{increment of work:}&\quad (20 - y)62.5\pi\left(\tfrac{1}{16}y^2\right) dy \end{aligned}$$

Applying $\int_0^{20}$ takes a limit of sums of the increments of work. Hence,

$$W = \int_0^{20} (20 - y)62.5\pi\left(\tfrac{1}{16}y^2\right) dy = \frac{62.5}{16}\pi \int_0^{20} (20y^2 - y^3)\,dy$$

$$= \frac{62.5}{16}\pi \left[20\left(\frac{y^3}{3}\right) - \frac{y^4}{4}\right]_0^{20} = \frac{62.5}{16}\pi\left(\frac{40{,}000}{3}\right) \approx 163{,}625 \text{ ft-lb.}$$

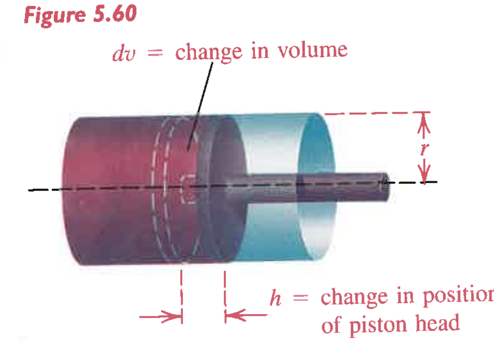
The next example is another illustration of how work may be calculated by means of a limit of sums—that is, by a definite integral.

EXAMPLE ■ 5 A confined gas has pressure p (lb/in^2) and volume v (in^3). If the gas expands from $v = a$ to $v = b$, show that the work done (in.-lb) is given by

$$W = \int_a^b p\,dv.$$

SOLUTION Since the work done is independent of the shape of the container, we may assume that the gas is enclosed in a right circular cylinder of radius r and that the expansion takes place against a piston head, as illustrated in Figure 5.60. As in the figure, let dv denote the

Figure 5.60



change in volume that corresponds to a change of h inches in the position of the piston head. Thus,

$$dv = \pi r^2 h, \quad \text{or} \quad h = \frac{1}{\pi r^2}\, dv.$$

If p denotes the pressure at some point in the volume increment shown in Figure 5.60, then the force against the piston head is the product of p and the area πr^2 of the piston head. Thus, we have the following for the indicated volume increment:

force against piston head: $p(\pi r^2)$

distance piston head moves: h

increment of work: $(p\pi r^2)h = (p\pi r^2)\dfrac{1}{\pi r^2}\, dv = p\, dv$

Applying $\int_a^b$ to the increments of work gives us the work done as the gas expands from $v = a$ to $v = b$:

$$W = \int_a^b p\, dv$$

EXERCISES 5.6

1 A 400-lb gorilla climbs a vertical tree 15 ft high. Find the work done if the gorilla reaches the top in

(a) 10 sec **(b)** 5 sec

2 Find the work done in lifting an 80-lb sandbag a height of 4 ft.

3 A spring of natural length 10 in. stretches 1.5 in. under a weight of 8 lb. Find the work done in stretching the spring

(a) from its natural length to a length of 14 in.

(b) from a length of 11 in. to a length of 13 in.

4 A force of 25 lb is required to compress a spring of natural length 0.80 ft to a length of 0.75 ft. Find the work done in compressing the spring from its natural length to a length of 0.70 ft.

5 If a spring is 12 in. long, compare the work W_1 done in stretching it from 12 in. to 13 in. with the work W_2 done in stretching it from 13 in. to 14 in.

6 It requires 60 in.-lb of work to stretch a certain spring from a length of 6 in. to 7 in. and another 120 in.-lb of work to stretch it from 7 in. to 8 in. Find the spring constant and the natural length of the spring.

7 A freight elevator weighing 3000 lb is supported by a 12-ft-long cable that weighs 14 lb per linear foot. Approximate the work required to lift the elevator 9 ft by winding the cable onto a winch.

8 A construction worker pulls a 50-lb motor from ground level to the top of a 60-ft-high building using a rope that weighs $\frac{1}{4}$ lb/ft. Find the work done.

9 A bucket containing water is lifted vertically at a constant rate of 1.5 ft/sec by means of a rope of negligible weight. As the bucket rises, water leaks out at a rate of 0.25 lb/sec. If the bucket weighs 4 lb when empty and if it contained 20 lb of water at the instant that the lifting began, determine the work done in raising the bucket 12 ft.

10 In Exercise 9, find the work required to raise the bucket until half the water has leaked out.

11 A fishtank has a rectangular base of width 2 ft and length 4 ft, and rectangular sides of height 3 ft. If the tank is filled with water weighing 62.5 lb/ft^3, find the work required to pump all the water over the top of the tank.

12 Generalize Example 4 to the case of a conical tank of altitude h feet and radius of base a feet that is filled with a liquid weighing ρ lb/ft^3.

13 A vertical cylindrical tank of diameter 3 ft and height 6 ft is full of water. Find the work required to pump all the water

(a) over the top of the tank

(b) through a pipe that rises to a height of 4 ft above the top of the tank

14 Work Exercise 13 if the tank is only half full of water.

15 The ends of an 8-ft-long water trough are equilateral triangles having sides of length 2 ft. If the trough is full of water, find the work required to pump all of it over the top.

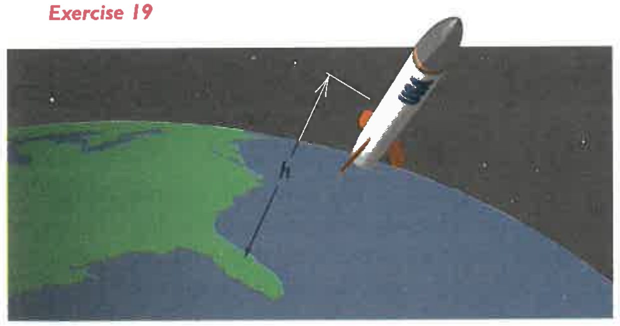
16 A cistern has the shape of the lower half of a sphere of radius 5 ft. If the cistern is full of water, find the work required to pump all the water to a point 4 ft above the top of the cistern.

17 Refer to Example 5. The volume and the pressure of a certain gas vary in accordance with the law $pv^{1.2} = 115$, where the units of measurement are inches and pounds. Find the work done if the gas expands from 32 in^3 to 40 in^3.

18 Refer to Example 5. The pressure and the volume of a quantity of enclosed steam are related by the formula $pv^{1.14} = c$, where c is a constant. If the initial pressure and volume are p_0 and v_0, respectively, find a formula for the work done if the steam expands to twice its volume.

19 Newton's law of gravitation states that the force F of attraction between two particles having masses m_1 and m_2 is given by $F = Gm_1m_2/s^2$, where G is a gravitational constant and s is the distance between the particles. If the mass m_1 of the earth is regarded as concentrated at the center of the earth and a rocket of mass m_2 is on the surface (a distance 4000 mi from the center), find a general formula for the work done in firing the rocket vertically upward to an altitude h (see figure).

Exercise 19



20 In the study of electricity, the formula $F = kq/r^2$, where k is a constant, is used to find the force (in Newtons) with which a positive charge Q of strength q units repels a unit positive charge located r meters from Q. Find the work done in moving a unit charge from a point d centimeters from Q to a point $\frac{1}{2}d$ centimeters from Q.

c **Exer. 21–22: Suppose the table was obtained experimentally for a force $f(x)$ acting at the point with coordinate x on a coordinate line. Use the trapezoidal rule to approximate the work done on the interval $[a, b]$, where a and b are the smallest and largest values of x, respectively.**

21

x (ft)	0	0.5	1.0	1.5	2.0	2.5
$f(x)$ (lb)	7.4	8.1	8.4	7.8	6.3	7.1

x (ft)	3.0	3.5	4.0	4.5	5.0
$f(x)$ (lb)	5.9	6.8	7.0	8.0	9.2

22

x (m)	1	2	3	4	5
$f(x)$ (N)	125	120	130	146	165

x (m)	6	7	8	9
$f(x)$ (N)	157	150	143	140

23 The force (in Newtons) with which two electrons repel each other is inversely proportional to the square of the distance (in meters) between them.

(a) If one electron is held fixed at the point (5, 0), find the work done in moving a second electron along the x-axis from the origin to the point (3, 0).

(b) If two electrons are held fixed at the points (5, 0) and (−5, 0), respectively, find the work done in moving a third electron from the origin to (3, 0).

24 If the force function is constant, show that Definition (5.21) reduces to Definition (5.20).