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APPENDICES

I THEOREMS ON LIMITS, DERIVATIVES, AND INTEGRALS

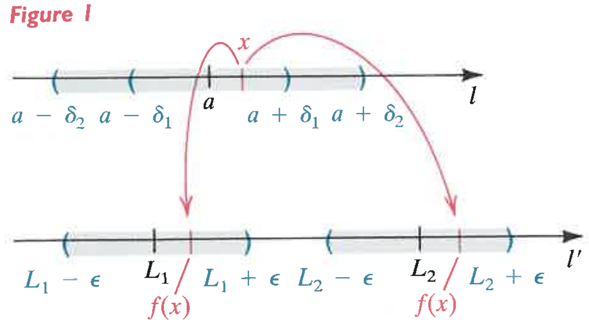
This appendix contains proofs for some theorems stated in the text. The numbering system corresponds to that given in previous chapters.

Uniqueness Theorem for Limits

If $f(x)$ has a limit as x approaches a, then the limit is unique.

PROOF Suppose $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ with $L_1 \neq L_2$. We may assume that $L_1 < L_2$. Choose $\epsilon > 0$ such that $\epsilon < \frac{1}{2}(L_2 - L_1)$ and consider the open intervals $(L_1 - \epsilon, L_1 + \epsilon)$ and $(L_2 - \epsilon, L_2 + \epsilon)$ on the coordinate line l' (see Figure 1). Since $\epsilon < \frac{1}{2}(L_2 - L_1)$, these two intervals do not intersect. By Definition (1.5), there is a $\delta_1 > 0$ such that whenever x is in $(a - \delta_1, a + \delta_1)$ and $x \neq a$, then $f(x)$ is in $(L_1 - \epsilon, L_1 + \epsilon)$. Similarly, there is a $\delta_2 > 0$ such that whenever x is in $(a - \delta_2, a + \delta_2)$ and $x \neq a$, then $f(x)$ is in $(L_2 - \epsilon, L_2 + \epsilon)$. This is illustrated in Figure 1, with $\delta_1 < \delta_2$. If an x is selected that is in *both* $(a - \delta_1, a + \delta_1)$ and $(a - \delta_2, a + \delta_2)$, then $f(x)$ is in $(L_1 - \epsilon, L_1 + \epsilon)$ and also in $(L_2 - \epsilon, L_2 + \epsilon)$, contrary to the fact that these two intervals do not intersect. Hence our original supposition is false, and consequently $L_1 = L_2$. ■

Figure 1



Theorem 1.8

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

(i) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$

PROOF Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

(i) According to Definition (1.4), we must show that for every $\epsilon > 0$ there is a $\delta > 0$ such that

(1) if $0 < |x - a| < \delta$, then $|f(x) + g(x) - (L + M)| < \epsilon$.

We begin by writing

(2) $|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|$.

Using the *triangle inequality*

$$|b + c| \leq |b| + |c|$$

for any real numbers b and c, we obtain

$$|(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M|.$$

Combining the last inequality with (2) gives us

(3) $|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$.

Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, the numbers $|f(x) - L|$ and $|g(x) - M|$ can be made arbitrarily small by choosing x sufficiently close to a. In particular, they can be made less than $\epsilon/2$. Thus, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

(4) if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon/2$, and
if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \epsilon/2$.

If δ denotes the *smaller* of δ_1 and δ_2, then whenever $0 < |x - a| < \delta$, the inequalities in (4) involving $f(x)$ and $g(x)$ are both true. Consequently, if $0 < |x - a| < \delta$, then, from (4) and (3),

$$|f(x) + g(x) - (L + M)| < \epsilon/2 + \epsilon/2 = \epsilon,$$

which is the desired statement (1).

(ii) We first show that if k is a function and

(5) if $\lim_{x\to a} k(x) = 0$, then $\lim_{x\to a} f(x)k(x) = 0$.

Since $\lim_{x\to a} f(x) = L$, it follows from Definition (1.4) (with $\epsilon = 1$) that there is a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < 1$ and hence also

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < 1 + |L|.$$

Consequently,

(6) if $0 < |x - a| < \delta_1$, then $|f(x)k(x)| < (1 + |L|)\,|k(x)|$.

Since $\lim_{x\to a} k(x) = 0$, for every $\epsilon > 0$ there is a $\delta_2 > 0$ such that

(7) if $0 < |x - a| < \delta_2$, then $|k(x) - 0| < \dfrac{\epsilon}{1 + |L|}$.

If δ denotes the smaller of δ_1 and δ_2, then whenever $0 < |x - a| < \delta$, both inequalities (6) and (7) are true and, consequently,

$$|f(x)k(x)| < (1 + |L|) \cdot \frac{\epsilon}{1 + |L|}.$$

Therefore,

$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |f(x)k(x) - 0| < \epsilon,$$

which proves (5).

Next consider the identity

(8) $f(x)g(x) - LM = f(x)[g(x) - M] + M[f(x) - L]$.

Since $\lim_{x\to a}[g(x) - M] = 0$, it follows from (5), with $k(x) = g(x) - M$, that $\lim_{x\to a} f(x)[g(x) - M] = 0$. In addition, $\lim_{x\to a} M[f(x) - L] = 0$ and hence, from (8), $\lim_{x\to a}[f(x)g(x) - LM] = 0$. The last statement is equivalent to $\lim_{x\to a} f(x)g(x) = LM$.

(iii) It is sufficient to show that $\lim_{x\to a} 1/g(x) = 1/M$, for once this is done, the desired result may be obtained by applying (ii) to the product $f(x) \cdot 1/g(x)$. Consider

(9) $$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \left|\frac{M - g(x)}{g(x)M}\right| = \frac{1}{|M|\,|g(x)|}\,|g(x) - M|.$$

Since $\lim_{x\to a} g(x) = M$, there is a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|g(x) - M| < |M|/2$. Consequently, for every such x,

$$\begin{aligned} |M| &= |g(x) + (M - g(x))| \\ &\le |g(x)| + |M - g(x)| \\ &< |g(x)| + |M|/2 \end{aligned}$$

and therefore,

$$\frac{|M|}{2} < |g(x)|, \quad \text{or} \quad \frac{1}{|g(x)|} < \frac{2}{|M|}.$$

Substitution in (9) leads to

(10) $$\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \frac{2}{|M|^2}\,|g(x) - M|, \quad \text{provided } 0 < |x - a| < \delta_1.$$

Again, since $\lim_{x\to a} g(x) = M$, it follows that for every $\epsilon > 0$ there is a $\delta_2 > 0$ such that

(11) if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \dfrac{|M|^2}{2}\epsilon$.

If δ denotes the smaller of δ_1 and δ_2, then both inequalities (10) and (11) are true. Thus,

$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad \left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon,$$

which means that $\lim_{x\to a} 1/g(x) = 1/M$. ■

Theorem I.13

> If $a > 0$ and n is a positive integer, or if $a \le 0$ and n is an odd positive integer, then
>
> $$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}.$$

PROOF Suppose $a > 0$ and n is any positive integer. We must show that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon,$$

or, equivalently,

(1) if $-\delta < x - a < \delta$ and $x \ne a$, then $-\epsilon < \sqrt[n]{x} - \sqrt[n]{a} < \epsilon$.

It is sufficient to prove (1) if $\epsilon < \sqrt[n]{a}$, for if a δ exists under this condition, then the same δ can be used for any *larger* value of ϵ. Thus, in the remainder of the proof, $\sqrt[n]{a} - \epsilon$ is considered to be a positive number less than $\sqrt[n]{a}$. The inequalities in the following list are all equivalent:

$$\begin{gathered} -\epsilon < \sqrt[n]{x} - \sqrt[n]{a} < \epsilon \\ \sqrt[n]{a} - \epsilon < \sqrt[n]{x} < \sqrt[n]{a} + \epsilon \\ (\sqrt[n]{a} - \epsilon)^n < x < (\sqrt[n]{a} + \epsilon)^n \\ (\sqrt[n]{a} - \epsilon)^n - a < x - a < (\sqrt[n]{a} + \epsilon)^n - a \\ -[a - (\sqrt[n]{a} - \epsilon)^n] < x - a < (\sqrt[n]{a} + \epsilon)^n - a \end{gathered}$$

If δ denotes the smaller of the two positive numbers $a - (\sqrt[n]{a} - \epsilon)^n$ and $(\sqrt[n]{a} + \epsilon)^n - a$, then whenever $-\delta < x - a < \delta$, the last inequality in the list is true and hence so is the first. This gives us (1).

Next suppose $a < 0$ and n is an odd positive integer. In this case, $-a$ and $\sqrt[n]{-a}$ are positive and, by the first part of the proof, we may write

$$\lim_{-x\to -a} \sqrt[n]{-x} = \sqrt[n]{-a}.$$

Thus, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{if} \quad 0 < |-x - (-a)| < \delta, \quad \text{then} \quad |\sqrt[n]{-x} - \sqrt[n]{-a}| < \epsilon,$$

or equivalently,

$$\text{if} \quad 0 < |x - a| < \delta, \quad \text{then} \quad |\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon.$$

The last inequalities imply that $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$. ■

Sandwich Theorem 1.15

> Suppose $f(x) \le h(x) \le g(x)$ for every x in an open interval containing a, except possibly at a.
>
> If $\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x)$, then $\lim_{x \to a} h(x) = L$.

PROOF For every $\epsilon > 0$, there is a $\delta_1 > 0$ and a $\delta_2 > 0$ such that

$$\text{(1)} \quad \begin{aligned} &\text{if } 0 < |x - a| < \delta_1, \text{ then } |f(x) - L| < \epsilon, \text{ and} \\ &\text{if } 0 < |x - a| < \delta_2, \text{ then } |g(x) - L| < \epsilon. \end{aligned}$$

If δ denotes the smaller of δ_1 and δ_2, then whenever $0 < |x - a| < \delta$, both inequalities in (1) that involve ϵ are true—that is,

$$-\epsilon < f(x) - L < \epsilon \quad \text{and} \quad -\epsilon < g(x) - L < \epsilon.$$

Thus, if $0 < |x - a| < \delta$, then $L - \epsilon < f(x)$ and $g(x) < L + \epsilon$. Since $f(x) \le h(x) \le g(x)$, if $0 < |x - a| < \delta$, then $L - \epsilon < h(x) < L + \epsilon$, or, equivalently, $|h(x) - L| < \epsilon$, which is what we wished to prove. ■

Theorem 1.18

> If k is a positive rational number and c is any real number, then
>
> $$\lim_{x \to \infty} \frac{c}{x^k} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{c}{x^k} = 0,$$
>
> provided x^k is always defined.

PROOF To use Definition (1.16) to prove that $\lim_{x \to \infty} (c/x^k) = 0$, we must show that for every $\epsilon > 0$ there is a positive number N such that

$$\left| \frac{c}{x^k} - 0 \right| < \epsilon \quad \text{whenever} \quad x > N.$$

If $c = 0$, any $N > 0$ will suffice. If $c \ne 0$, the following four inequalities are equivalent for $x > 0$:

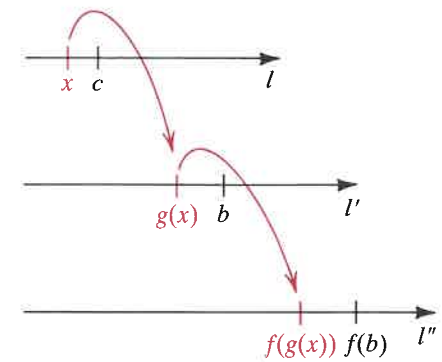
$$\left| \frac{c}{x^k} - 0 \right| < \epsilon, \qquad \frac{|x|^k}{|c|} > \frac{1}{\epsilon}, \qquad |x|^k > \frac{|c|}{\epsilon}, \qquad x > \left(\frac{|c|}{\epsilon} \right)^{1/k}$$

The last inequality gives us a clue to a choice for N. Letting $N = (|c|/\epsilon)^{1/k}$, we see that whenever $x > N$, the fourth, and hence the first, inequality is true, which is what we wished to show. The second part of the theorem may be proved in similar fashion. ■

Theorem 1.24

> If $\lim_{x \to c} g(x) = b$ and if f is continuous at b, then
>
> $$\lim_{x \to c} f(g(x)) = f(b) = f\left(\lim_{x \to c} g(x) \right).$$

Figure 2



PROOF The composite function $f(g(x))$ may be represented geometrically by means of three real lines l, l', and l'', as shown in Figure 2. To each coordinate x on l, there corresponds the coordinate $g(x)$ on l' and then, in turn, $f(g(x))$ on l''. We wish to prove that $f(g(x))$ has the limit $f(b)$ as x approaches c. In terms of Definition (1.4), we must show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{(1)} \quad \text{if } 0 < |x - c| < \delta, \text{ then } |f(g(x)) - f(b)| < \epsilon.$$

Let us begin by considering the interval $(f(b) - \epsilon, f(b) + \epsilon)$ on l'', shown in Figure 3. Since f is continuous at b, $\lim_{z \to b} f(z) = f(b)$ and hence, as illustrated in the figure, there exists a number $\delta_1 > 0$ such that

$$\text{(2)} \quad \text{if } |z - b| < \delta_1, \text{ then } |f(z) - f(b)| < \epsilon.$$

In particular, if we let $z = g(x)$ in (2), it follows that

$$\text{(3)} \quad \text{if } |g(x) - b| < \delta_1, \text{ then } |f(g(x)) - f(b)| < \epsilon.$$

Next, turning our attention to the interval $(b - \delta_1, b + \delta_1)$ on l' and using the definition of $\lim_{x \to c} g(x) = b$, we obtain the fact illustrated in Figure 4—that there exists a $\delta > 0$ such that

$$\text{(4)} \quad \text{if } 0 < |x - c| < \delta, \text{ then } |g(x) - b| < \delta_1.$$

Figure 3

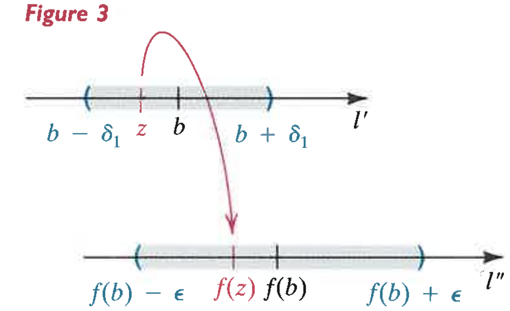
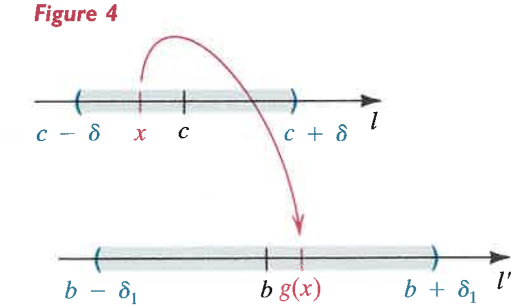


Figure 4



Finally, combining (4) and (3), we see that

$$\text{if } 0 < |x - c| < \delta, \text{ then } |f(g(x)) - f(b)| < \epsilon,$$

which is the desired conclusion (1). ■

Theorem 2.10a

> If n is a positive integer and $f(x) = x^{1/n}$, then
>
> $$f'(x) = \frac{1}{n} x^{(1/n)-1}.$$

PROOF By Definition 2.5,

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^{1/n} - x^{1/n}}{h}.$$

Consider the identity

$$u^n - v^n = (u - v)(u^{n-1} + u^{n-2}v + \cdots + uv^{n-2} + v^{n-1}).$$

If $u \neq v$, then

$$\frac{u-v}{u^n - v^n} = \frac{1}{u^{n-1} + u^{n-2}v + \cdots + uv^{n-2} + v^{n-1}}.$$

Substituting $u = (x+h)^{1/n}$ and $v = x^{1/n}$, we obtain

$$\frac{(x+h)^{1/n} - x^{1/n}}{(x+h) - x} =$$

$$\frac{1}{(x+h)^{(n-1)/n} + (x+h)^{(n-2)/n}x^{1/n} + \cdots + (x+h)^{1/n}x^{(n-2)/n} + x^{(n-1)/n}}.$$

Letting $h \to 0$, we have

$$f'(x) = \frac{1}{x^{(n-1)/n} + x^{(n-1)/n} + \cdots + x^{(n-1)/n} + x^{(n-1)/n}}$$

$$= \frac{1}{nx^{1-(1/n)}} = \frac{1}{n}x^{(1/n)-1}.$$ ■

Chain Rule 2.26

> If $y = f(u)$, $u = g(x)$, and the derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$ both exist, then the composite function defined by $y = f(g(x))$ has a derivative given by
>
> $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)g'(x) = f'(g(x))g'(x).$$

PROOF Since f is differentiable at $g(x)$, we have

$$\lim_{k \to 0} \frac{f(g(x)+k) - f(g(x))}{k} = f'(g(x))$$

so that if $\epsilon > 0$, there is a $\delta_1 > 0$ such that

(1) if $0 < |k| < \delta_1$, then $\left| \frac{f(g(x)+k) - f(g(x))}{k} - f'(g(x)) \right| < \epsilon.$

Since g is differentiable at x, g is also continuous at x, and hence there is a $\delta > 0$ such that

(2) if $|h| < \delta$, then $|g(x+h) - g(x)| < \delta_1$.

We define a function T by the rule

$$T(h) = \begin{cases} \dfrac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} & \text{if } g(x+h) \neq g(x) \\ f'(g(x)) & \text{if } g(x+h) = g(x) \end{cases}$$

Note that

$$\frac{f(g(x+h)) - f(g(x))}{h} = T(h)\frac{g(x+h) - g(x)}{h}$$

for *all* nonzero h (both sides are 0 if $g(x+h) = g(x)$).

Let h be any real number with $|h| < \delta$ and let $k = g(x+h) - g(x)$. Thus, by (2), $|k| < \delta_1$. If $k \neq 0$, then

$$T(h) = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} = \frac{f(g(x)+k) - f(g(x))}{k}.$$

Since $|k| < \delta_1$, by (1) we also have $|T(h) - f'(g(x))| < \epsilon$. If $k = 0$, then $T(h) = f'(g(x))$ so that $|T(h) - f'(g(x))| = 0 < \epsilon$. Thus, given an $\epsilon > 0$, there is a $\delta > 0$ such that if $|h| < \delta$, then $|T(h) - f'(g(x))| < \epsilon$. Hence, T is continuous at 0 with $\lim_{h\to 0} T(h) = f'(g(x))$.

Finally, we note that

$$f'(g(x))g'(x) = \lim_{h\to 0} T(h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} T(h)\frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = (f \circ g)'(x).$$ ■

Theorem 4.22

> If f is integrable on $[a, b]$ and c is any number, then cf is integrable on $[a, b]$ and
>
> $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

PROOF If $c = 0$, the result follows from Theorem (4.21). Assume, therefore, that $c \neq 0$. Since f is integrable, $\int_a^b f(x)\,dx = I$ for some number I. If P is a partition of $[a, b]$, then each Riemann sum R_P for the function cf has the form $\sum_k cf(w_k)\Delta x_k$ such that for every k, w_k is in the kth subinterval $[x_{k-1}, x_k]$ of P. We wish to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $\|P\| < \delta$,

(1) $$\left| \sum_k cf(w_k)\Delta x_k - cI \right| < \epsilon$$

for every w_k in $[x_{k-1}, x_k]$. If we let $\epsilon' = \epsilon/|c|$, then, since f is integrable, there exists a $\delta > 0$ such that whenever $\|P\| < \delta$,

$$\left| \sum_k f(w_k)\Delta x_k - I \right| < \epsilon' = \frac{\epsilon}{|c|}.$$

Multiplying both sides of this inequality by $|c|$ leads to (1). Hence,

$$\lim_{\|P\|\to 0} \sum_k cf(w_k)\Delta x_k = cI$$

$$= c\int_a^b f(x)\,dx.$$ ■

Theorem 4.23

If f and g are integrable on $[a,b]$, then $f+g$ and $f-g$ are integrable on $[a,b]$ and

(i) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(ii) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

PROOF **(i)** By hypothesis, there exist real numbers I_1 and I_2 such that

$$\int_a^b f(x)\,dx = I_1 \quad \text{and} \quad \int_a^b g(x)\,dx = I_2.$$

Let P denote a partition of $[a,b]$ and let R_P denote an arbitrary Riemann sum for $f+g$ associated with P—that is,

(1) $$R_P = \sum_k [f(w_k)+g(w_k)]\,\Delta x_k$$

such that w_k is in $[x_{k-1}, x_k]$ for every k. We wish to show that for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $\|P\| < \delta$, $|R_P - (I_1 + I_2)| < \epsilon$. Using Theorem (4.11)(i), we may write (1) in the form

$$R_P = \sum_k f(w_k)\Delta x_k + \sum_k g(w_k)\Delta x_k.$$

Rearranging terms and using the triangle inequality, we obtain

(2) $$\begin{aligned}|R_P - (I_1+I_2)| &= \left|\left(\sum_k f(w_k)\Delta x_k - I_1\right) + \left(\sum_k g(w_k)\Delta x_k - I_2\right)\right| \\ &\le \left|\sum_k f(w_k)\Delta x_k - I_1\right| + \left|\sum_k g(w_k)\Delta x_k - I_2\right|.\end{aligned}$$

By the integrability of f and g, if $\epsilon' = \epsilon/2$, then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that whenever $\|P\| < \delta_1$ and $\|P\| < \delta_2$,

(3) $$\left|\sum_k f(w_k)\Delta x_k - I_1\right| < \epsilon' = \epsilon/2 \quad \text{and}$$
$$\left|\sum_k g(w_k)\Delta x_k - I_2\right| < \epsilon' = \epsilon/2$$

for every w_k in $[x_{k-1}, x_k]$. If δ denotes the smaller of δ_1 and δ_2, then whenever $\|P\| < \delta$, both inequalities in (3) are true and hence, from (2),

$$|R_P - (I_1+I_2)| < (\epsilon/2) + (\epsilon/2) = \epsilon,$$

which is what we wished to prove.

(ii) By Theorem (4.22) with $c = -1$, we know that $-g$ is integrable on $[a,b]$ and $\int_a^b -g(x)\,dx = -1\int_a^b g(x)\,dx$. Thus, by part (i), $f - g = f + (-g)$ is integrable with

$$\begin{aligned}\int_a^b [f(x)-g(x)]\,dx &= \int_a^b [f(x)+(-g(x))]\,dx \\ &= \int_a^b f(x)\,dx + \int_a^b -g(x)\,dx \\ &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.\end{aligned}$$ ■

Theorem 4.24

If $a < c < b$ and if f is integrable on both $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

PROOF By hypothesis, there exist real numbers I_1 and I_2 such that

(1) $$\int_a^c f(x)\,dx = I_1 \quad \text{and} \quad \int_c^b f(x)\,dx = I_2.$$

Let us denote a partition of $[a,c]$ by P_1, of $[c,b]$ by P_2, and of $[a,b]$ by P. Arbitrary Riemann sums associated with P_1, P_2, and P will be denoted by R_{P_1}, R_{P_2}, and R_P, respectively. We must show that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\|P\| < \delta$, then $|R_P - (I_1 + I_2)| < \epsilon$.

If we let $\epsilon' = \epsilon/4$, then, by (1), there exist positive numbers δ_1 and δ_2 such that if $\|P_1\| < \delta_1$ and $\|P_2\| < \delta_2$, then

(2) $$|R_{P_1} - I_1| < \epsilon' = \epsilon/4 \quad \text{and} \quad |R_{P_2} - I_2| < \epsilon' = \epsilon/4.$$

If δ denotes the smaller of δ_1 and δ_2, then both inequalities in (2) are true whenever $\|P\| < \delta$. Moreover, since f is integrable on $[a,c]$ and $[c,b]$, it is bounded on both intervals and hence there exists a number M such that $|f(x)| \le M$ for every x in $[a,b]$. We shall now assume that δ has been chosen so that, in addition to the previous requirement, we also have $\delta < \epsilon/(4M)$.

Let P be a partition of $[a,b]$ such that $\|P\| < \delta$. If the numbers that determine P are

$$a = x_0, x_1, x_2, \ldots, x_n = b,$$

then there is a unique half-open interval of the form $(x_{d-1}, x_d]$ that contains c. If $R_P = \sum_{k=1}^n f(w_k)\Delta x_k$, we may write

(3) $$R_P = \sum_{k=1}^{d-1} f(w_k)\Delta x_k + f(w_d)\Delta x_d + \sum_{k=d+1}^{n} f(w_k)\Delta x_k.$$

Let P_1 denote the partition of $[a, c]$ determined by $\{a, x_1, \ldots, x_{d-1}, c\}$, let P_2 denote the partition of $[c, b]$ determined by $\{c, x_d, \ldots, x_{n-1}, b\}$, and consider the Riemann sums

(4)
$$R_{P_1} = \sum_{k=1}^{d-1} f(w_k)\Delta x_k + f(c)(c - x_{d-1}) \quad \text{and}$$
$$R_{P_2} = f(c)(x_d - c) + \sum_{k=d+1}^{n} f(w_k)\Delta x_k.$$

Using the triangle inequality and (2), we obtain

(5)
$$\begin{aligned}\left|(R_{P_1} + R_{P_2}) - (I_1 + I_2)\right| &= \left|(R_{P_1} - I_1) + (R_{P_2} - I_2)\right| \\ &\le \left|R_{P_1} - I_1\right| + \left|R_{P_2} - I_2\right| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.\end{aligned}$$

It follows from (3) and (4) that

$$\left|R_P - (R_{P_1} + R_{P_2})\right| = |f(w_d) - f(c)|\,\Delta x_d.$$

Employing the triangle inequality and the choice of δ gives us

(6)
$$\begin{aligned}\left|R_P - (R_{P_1} + R_{P_2})\right| &\le (|f(w_d)| + |f(c)|)\Delta x_d \\ &\le (M + M)[\epsilon/(4M)] = \epsilon/2,\end{aligned}$$

provided $\|P\| < \delta$. If we write

$$\begin{aligned}|R_P - (I_1 + I_2)| &= \left|R_P - (R_{P_1} + R_{P_2}) + (R_{P_1} + R_{P_2}) - (I_1 + I_2)\right| \\ &\le \left|R_P - (R_{P_1} + R_{P_2})\right| + \left|(R_{P_1} + R_{P_2}) - (I_1 + I_2)\right|,\end{aligned}$$

then it follows from (6) and (5) that whenever $\|P\| < \delta$,

$$|R_P - (I_1 + I_2)| < (\epsilon/2) + (\epsilon/2) = \epsilon$$

for every Riemann sum R_P. This completes the proof. ■

Theorem 4.26

> If f is integrable on $[a, b]$ and $f(x) \ge 0$ for every x in $[a, b]$, then
> $$\int_a^b f(x)\,dx \ge 0.$$

PROOF We shall give an indirect proof. Let $\int_a^b f(x)\,dx = I$, and *suppose* that $I < 0$. Consider any partition P of $[a, b]$, and let $R_P = \sum_k f(w_k)\Delta x_k$ be an arbitrary Riemann sum associated with P. Since $f(w_k) \ge 0$ for every w_k in $[x_{k-1}, x_k]$, it follows that $R_P \ge 0$. If we let $\epsilon = -(I/2)$, then, according to Definition (4.15), whenever $\|P\|$ is sufficiently small,

$$|R_P - I| < \epsilon = -\frac{I}{2}.$$

It follows that $R_P < I - (I/2) = I/2 < 0$, a contradiction. Therefore, the supposition $I < 0$ is false and hence $I \ge 0$. ■

Theorem 6.6

> If f is continuous and increasing on $[a, b]$, then f has an inverse function f^{-1} that is continuous and increasing on $[f(a), f(b)]$.

PROOF If f is increasing, then f is one-to-one and so f^{-1} exists. To prove that f^{-1} is increasing, we must show that if $w_1 < w_2$ in $[f(a), f(b)]$, then $f^{-1}(w_1) < f^{-1}(w_2)$ in $[a, b]$. Let us give an indirect proof of this fact. *Suppose* $f^{-1}(w_2) \le f^{-1}(w_1)$. Since f is increasing, it follows that $f(f^{-1}(w_2)) \le f(f^{-1}(w_1))$ and hence $w_2 \le w_1$, which is a contradiction. Consequently, $f^{-1}(w_1) < f^{-1}(w_2)$.

We next prove that f^{-1} is continuous on $[f(a), f(b)]$. Recall that $y = f(x)$ if and only if $x = f^{-1}(y)$. In particular, if y_0 is in an open interval $(f(a), f(b))$, let x_0 denote the number in the interval (a, b) such that $y_0 = f(x_0)$, or, equivalently, $x_0 = f^{-1}(y_0)$. We wish to show that

(1)
$$\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0.$$

A geometric representation of f and its inverse f^{-1} is shown in Figure 5. The domain $[a, b]$ of f is represented by points on an x-axis and the domain $[f(a), f(b)]$ of f^{-1} by points on a y-axis. Arrows are drawn from one axis to the other to represent function values. To prove (1), consider any interval $(x_0 - \epsilon, x_0 + \epsilon)$ for $\epsilon > 0$. It is sufficient to find an interval $(y_0 - \delta, y_0 + \delta)$, of the type sketched in Figure 6 on the following page, such that whenever y is in $(y_0 - \delta, y_0 + \delta)$, $f^{-1}(y)$ is in $(x_0 - \epsilon, x_0 + \epsilon)$. We may assume that $x_0 - \epsilon$ and $x_0 + \epsilon$ are in $[a, b]$. As in Figure 7 on the following page, let $\delta_1 = y_0 - f(x_0 - \epsilon)$ and $\delta_2 = f(x_0 + \epsilon) - y_0$.

Figure 5

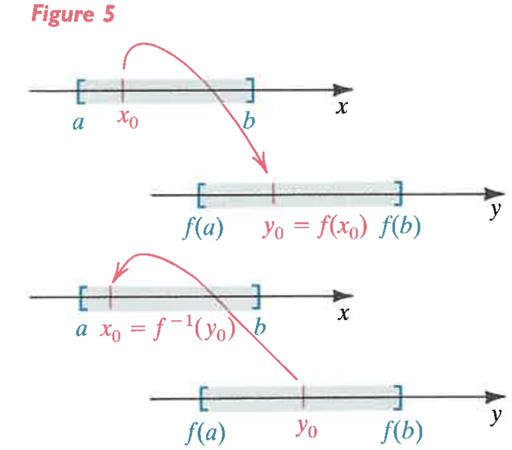


Figure 6

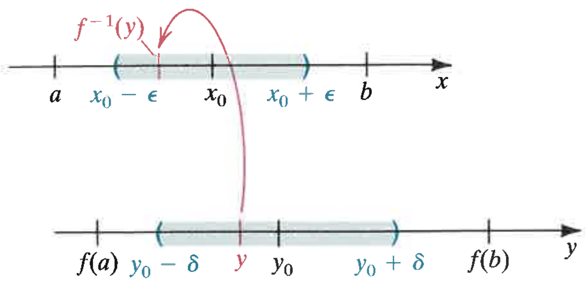
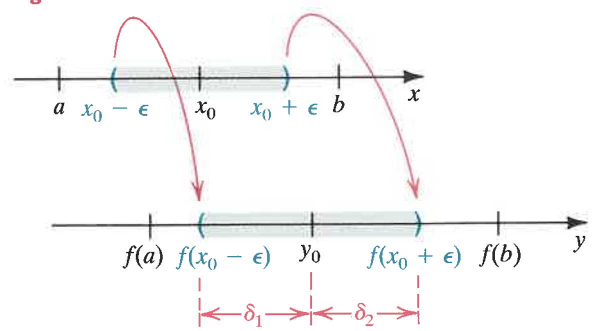


Figure 7



Since f determines a one-to-one correspondence between the numbers in the intervals $(x_0-\epsilon, x_0+\epsilon)$ and $(y_0-\delta_1, y_0+\delta_2)$, the function values of f^{-1} that correspond to numbers in $(y_0-\delta_1, y_0+\delta_2)$ must lie in $(x_0-\epsilon, x_0+\epsilon)$. Let δ denote the smaller of δ_1 and δ_2. It follows that if y is in $(y_0-\delta, y_0+\delta)$, then $f^{-1}(y)$ is in $(x_0-\epsilon, x_0+\epsilon)$, which is what we wished to prove.

The continuity at the endpoints $f(a)$ and $f(b)$ of the domain of f^{-1} may be proved in a similar manner using one-sided limits. ■

Theorem 13.35

If $x = f(u, v)$, $y = g(u, v)$ is a transformation of coordinates, then

$$\iint_R F(x, y)\,dx\,dy = \pm \iint_S F(f(u, v), g(u, v))\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$$

As (u, v) traces the boundary K of S once in the positive direction, the corresponding point (x, y) traces the boundary C of R once in either the positive direction, in which case the plus sign is chosen, or the negative direction, in which case the minus sign is chosen.

PROOF Let us begin by choosing $G(x, y)$ such that $\partial G/\partial x = F$. Applying Green's theorem (14.19) with $G = N$ gives us

(1) $$\iint_R F(x, y)\,dx\,dy = \iint_R \frac{\partial}{\partial x}[G(x, y)]\,dx\,dy = \oint_C G(x, y)\,dy.$$

Suppose the curve K in the uv-plane has a parametrization

$$u = \phi(t),\ v = \psi(t); \quad a \le t \le b.$$

From our assumptions on the transformation, parametric equations for the curve C in the xy-plane are

(2) $$\begin{aligned} x &= f(u, v) = f(\phi(t), \psi(t)) \\ y &= g(u, v) = g(\phi(t), \psi(t)) \end{aligned}$$

for $a \le t \le b$. We may therefore evaluate the line integral $\oint_C G(x, y)\,dy$ in (1) through formal substitutions for x and y. To simplify the notation, let

$$H(t) = G[f(\phi(t), \psi(t)), g(\phi(t), \psi(t))].$$

Applying a chain rule to y in (2) gives us

$$\frac{dy}{dt} = \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt} = \frac{\partial y}{\partial u}\phi'(t) + \frac{\partial y}{\partial v}\psi'(t).$$

Consequently,

$$\begin{aligned} \oint_C G(x, y)\,dy &= \oint_C H(t)\frac{dy}{dt}\,dt \\ &= \int_a^b H(t)\left[\frac{\partial y}{\partial u}\phi'(t) + \frac{\partial y}{\partial v}\psi'(t)\right]dt. \end{aligned}$$

Since $du = \phi'(t)\,dt$ and $dv = \psi'(t)\,dt$, we may regard the last line integral as a line integral around the curve K in the uv-plane. Thus,

(3) $$\oint_C G(x, y)\,dy = \pm\oint_K G\frac{\partial y}{\partial u}\,du + G\frac{\partial y}{\partial v}\,dv.$$

For simplicity, we have used G as an abbreviation for $G(f(u, v), g(u, v))$. The choice of the $+$ sign or the $-$ sign is made by letting t vary from a to b and noting whether (x, y) traces C in the same direction or the opposite direction, respectively, as (u, v) traces K.

The line integral on the right in (3) has the form

$$\oint_K M\,du + N\,dv$$

with $$M = G\frac{\partial y}{\partial u} \quad \text{and} \quad N = G\frac{\partial y}{\partial v}.$$

Applying Green's theorem, we obtain

$$\begin{aligned} \oint_K M\,du + N\,dv &= \iint_S \left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}\right)du\,dv \\ &= \iint_S \left(G\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial v} - G\frac{\partial^2 y}{\partial v\,\partial u} - \frac{\partial G}{\partial v}\frac{\partial y}{\partial u}\right)du\,dv \\ &= \iint_S \left[\left(\frac{\partial G}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial u}\right)\frac{\partial y}{\partial v} - \left(\frac{\partial G}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial v}\right)\frac{\partial y}{\partial u}\right]du\,dv \\ &= \iint_S \frac{\partial G}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial v}{\partial x}\right)du\,dv. \end{aligned}$$

Using the fact that $\partial G/\partial x = F(x, y)$, together with the definition of Jacobian (13.34), gives us

$$\oint_K M\,du + N\,dv = \iint_S F(f(u, v), g(u, v))\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$$

Combining this formula with (1) and (3) leads to the desired result. ■

II TABLE OF INTEGRALS

Basic forms

1 $\int u\,dv = uv - \int v\,du$

2 $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C,\ n \neq -1$

3 $\int \frac{du}{u} = \ln|u| + C$

4 $\int e^u\,du = e^u + C$

5 $\int a^u\,du = \frac{1}{\ln a}a^u + C$

6 $\int \sin u\,du = -\cos u + C$

7 $\int \cos u\,du = \sin u + C$

8 $\int \sec^2 u\,du = \tan u + C$

9 $\int \csc^2 u\,du = -\cot u + C$

10 $\int \sec u \tan u\,du = \sec u + C$

11 $\int \csc u \cot u\,du = -\csc u + C$

12 $\int \tan u\,du = -\ln|\cos u| + C$

13 $\int \cot u\,du = \ln|\sin u| + C$

14 $\int \sec u\,du = \ln|\sec u + \tan u| + C$

15 $\int \csc u\,du = \ln|\csc u - \cot u| + C$

16 $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

17 $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18 $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

19 $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20 $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

Forms involving $\sqrt{a^2 + u^2}$

21 $\int \sqrt{a^2 + u^2}\,du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2 + u^2}\right| + C$

22 $\int u^2\sqrt{a^2 + u^2}\,du = \frac{u}{8}(a^2 + 2u^2)\sqrt{a^2 + u^2} - \frac{a^4}{8}\ln\left|u + \sqrt{a^2 + u^2}\right| + C$

23 $\int \frac{\sqrt{a^2 + u^2}}{u}\,du = \sqrt{a^2 + u^2} - a\ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$

24 $\int \frac{\sqrt{a^2 + u^2}}{u^2}\,du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln\left|u + \sqrt{a^2 + u^2}\right| + C$

25 $\int \frac{du}{\sqrt{a^2 + u^2}} = \ln\left|u + \sqrt{a^2 + u^2}\right| + C$

26 $\int \frac{u^2\,du}{\sqrt{a^2 + u^2}} = \frac{u}{2}\sqrt{a^2 + u^2} - \frac{a^2}{2}\ln\left|u + \sqrt{a^2 + u^2}\right| + C$

27 $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

28 $\int \frac{du}{u^2\sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$

29 $\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 + u^2}} + C$

Forms involving $\sqrt{a^2 - u^2}$

30 $\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

31 $\int u^2\sqrt{a^2 - u^2}\,du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

32 $\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

33 $\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

34 $\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

35 $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

36 $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

37 $\int (a^2 - u^2)^{3/2}\,du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

38 $\int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms involving $\sqrt{u^2 - a^2}$

39 $\int \sqrt{u^2 - a^2}\,du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

40 $\int u^2\sqrt{u^2 - a^2}\,du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

41 $\int \frac{\sqrt{u^2 - a^2}}{u}\,du = \sqrt{u^2 - a^2} - a\sec^{-1}\frac{u}{a} + C$

42 $\int \frac{\sqrt{u^2 - a^2}}{u^2}\,du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

43 $\int \frac{u^2\,du}{\sqrt{u^2 - a^2}} = \frac{u}{2}\sqrt{u^2 - a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$

44 $\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

45 $\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

46 $\int \frac{u^2\,du}{(u^2 - a^2)^{3/2}} = \frac{-u}{\sqrt{u^2 - a^2}} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

Forms involving $a + bu$

47 $\int \frac{u\,du}{a+bu} = \frac{1}{b^2}(a + bu - a\ln|a+bu|) + C$

48 $\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|] + C$

49 $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

50 $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

51 $\int \frac{u\,du}{(a+bu)^2} = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

52 $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

53 $\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

54 $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$

55 $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$

56 $\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a+bu} + C$

57 $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, \quad \text{if } a > 0$

$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$

58 $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

59 $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

60 $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

61 $\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

62 $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

Trigonometric forms

63 $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

64 $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

65 $\int \tan^2 u\,du = \tan u - u + C$

66 $\int \cot^2 u\,du = -\cot u - u + C$

67 $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

68 $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

69 $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

70 $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u = -\ln|\sin u| + C$

71 $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

72 $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

73 $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

74 $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

75 $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

76 $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

77 $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

78 $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

79 $\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

80 $\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

81 $\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

82 $\int u\sin u\,du = \sin u - u\cos u + C$

83 $\int u\cos u\,du = \cos u + u\sin u + C$

84 $\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

85 $\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

86 $\int \sin^n u\cos^m u\,du = -\frac{\sin^{n-1} u\cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u\cos^m u\,du$

$= \frac{\sin^{n+1} u\cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u\cos^{m-2} u\,du$

Inverse trigonometric forms

87 $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

88 $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

89 $\int \tan^{-1} u\,du = u\tan^{-1} u - \frac{1}{2}\ln(1+u^2) + C$

90 $\int u\sin^{-1} u\,du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

91 $\int u\cos^{-1} u\,du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

92 $\int u\tan^{-1} u\,du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

93 $\int u^n\sin^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1} u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

94 $\int u^n\cos^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1} u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

95 $\int u^n\tan^{-1} u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1} u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Exponential and logarithmic forms

96 $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

97 $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

98 $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

99 $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

100 $\int \ln u\,du = u\ln u - u + C$

101 $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

102 $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic forms

103 $\int \sinh u\,du = \cosh u + C$

104 $\int \cosh u\,du = \sinh u + C$

105 $\int \tanh u\,du = \ln\cosh u + C$

106 $\int \coth u\,du = \ln|\sinh u| + C$

107 $\int \operatorname{sech} u\,du = \tan^{-1}\sinh u + C$

108 $\int \operatorname{csch} u\,du = \ln\left|\tanh\tfrac{1}{2}u\right| + C$

109 $\int \operatorname{sech}^2 u\,du = \tanh u + C$

110 $\int \operatorname{csch}^2 u\,du = -\coth u + C$

111 $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$

112 $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Forms involving $\sqrt{2au-u^2}$

113 $\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

114 $\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

115 $\int \frac{\sqrt{2au-u^2}}{u}\,du = \sqrt{2au-u^2} + a\cos^{-1}\left(\frac{a-u}{a}\right) + C$

116 $\int \frac{\sqrt{2au-u^2}}{u^2}\,du = -\frac{2\sqrt{2au-u^2}}{u} - \cos^{-1}\left(\frac{a-u}{a}\right) + C$

117 $\int \frac{du}{\sqrt{2au-u^2}} = \cos^{-1}\left(\frac{a-u}{a}\right) + C$

118 $\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2} + a\cos^{-1}\left(\frac{a-u}{a}\right) + C$

119 $\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{u+3a}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

120 $\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

III THE BINOMIAL SERIES

The binomial theorem states that if k is a positive integer, then for all numbers a and b,

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}a^{k-n}b^n + \cdots + b^k.$$

If we let $a = 1$ and $b = x$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n + \cdots + x^k.$$

If k is not a positive integer (or 0), it is useful to study the power series $\sum a_n x^n$ with $a_0 = 1$ and $a_n = k(k-1)\cdots(k-n+1)/n!$ for $n \ge 1$. This infinite series has the form

$$1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n + \cdots$$

and is called the **binomial series**. If k is a nonnegative integer, the series reduces to the finite sum given in the binomial theorem. Otherwise, the series does not terminate. Using the formula for a_n, we can show that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{k-n}{n+1}\right||x| = |x|.$$

Hence, by the ratio test (8.35), the series is absolutely convergent if $|x| < 1$ and is divergent if $|x| > 1$. Thus, the binomial series represents a function f such that

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{k(k-1)\cdots(k-n+1)}{n!}x^n \quad \text{if} \quad |x| < 1.$$

We have already noted that if k is a nonnegative integer, then $f(x) = (1+x)^k$. We shall now prove that the same is true for *every* real number k. Differentiating each term of the binomial series gives us

$$f'(x) = k + k(k-1)x + \cdots + \frac{nk(k-1)\cdots(k-n+1)}{n!}x^{n-1} + \cdots$$

and therefore,

$$xf'(x) = kx + k(k-1)x^2 + \cdots + \frac{nk(k-1)\cdots(k-n+1)}{n!}x^n + \cdots.$$

If we add corresponding terms of the preceding two power series, then the coefficient of x^n is

$$\frac{(n+1)k(k-1)\cdots(k-n)}{(n+1)!}+\frac{nk(k-1)\cdots(k-n+1)}{n!},$$

which simplifies to

$$[(k-n)+n]\frac{k(k-1)\cdots(k-n+1)}{n!}=ka_n.$$

Consequently,

$$f'(x)+xf'(x)=\sum_{n=0}^{\infty}ka_nx^n=kf(x),$$

or, equivalently,

$$f'(x)(1+x)-kf(x)=0.$$

If we define the function g by $g(x)=f(x)/(1+x)^k$, then

$$g'(x)=\frac{(1+x)^kf'(x)-f(x)k(1+x)^{k-1}}{(1+x)^{2k}}$$

$$=\frac{(1+x)f'(x)-kf(x)}{(1+x)^{k+1}}=0.$$

It follows that $g(x)=c$ for some constant c—that is,

$$\frac{f(x)}{(1+x)^k}=c.$$

Since $f(0)=1$, we see that $c=1$ and hence $f(x)=(1+x)^k$, which is what we wished to prove. The next statement summarizes this discussion.

Binomial Series

If $|x|<1$, then for every real number k,

$$(1+x)^k=1+kx+\frac{k(k-1)}{2!}x^2+\cdots$$

$$+\frac{k(k-1)\cdots(k-n+1)}{n!}x^n+\cdots.$$

EXAMPLE ■ 1

(a) Find a power series representation for $f(x)=\sqrt[3]{1+x}$.

(b) Plot the graphs of f and $g(x)=1+\frac{1}{3}x$.

(c) Use the graphs to estimate the largest closed interval of $[-1,1]$ on which $|f(x)-g(x)|<\frac{1}{10}$.

SOLUTION

(a) Using the binomial series with $k=\frac{1}{3}$, we obtain

$$\sqrt[3]{1+x}=1+\frac{1}{3}x+\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}x^2+\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}x^3+\cdots$$

$$+\frac{\frac{1}{3}(\frac{1}{3}-1)\cdots(\frac{1}{3}-n+1)}{n!}x^n+\cdots,$$

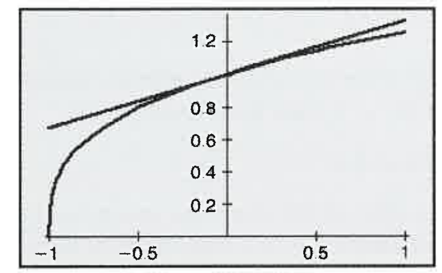
which may be written as

$$\sqrt[3]{1+x}=1+\frac{1}{3}x-\frac{2}{3^2\cdot 2!}x^2+\frac{1\cdot 2\cdot 5}{3^3\cdot 3!}x^3+\cdots$$

$$+(-1)^{n+1}\frac{1\cdot 2\cdots(3n-4)}{3^n\cdot n!}x^n+\cdots$$

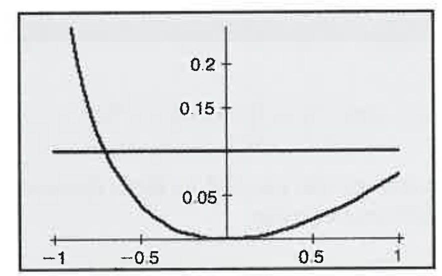
for $|x|<1$. The formula for the nth term of this series is valid provided $n\geq 2$.

Figure 1
$f(x)=\sqrt[3]{1+x}$, $g(x)=1+\frac{1}{3}x$
$-1\leq x\leq 1, 0\leq y\leq 1.4$



(b) The function $g(x)=1+\frac{1}{3}x$ consists of the first two terms of the power series representation of f. By graphing these functions, we can gain a sense of how closely the first-degree polynomial g approximates f. We use a graphing utility to plot f and g for $-1\leq x\leq 1$ and $0\leq y\leq 1.4$ on the same axes, as shown in Figure 1. We see that $g(x)$ appears to be at least as large as $f(x)$ over the entire interval. We also note that the values of f and g are relatively close to each other for $-0.5<x<1$. The closer x is to -1, the farther apart are the values $f(x)$ and $g(x)$.

Figure 2
$h(x)=|f(x)-g(x)|$
$-1\leq x\leq 1, 0\leq y\leq 0.25$



(c) To find the x-values where f and g are within $\frac{1}{10}$ unit of each other, we graph the constant function $\frac{1}{10}$ and the function $h(x)=|f(x)-g(x)|$. Figure 2 shows the graphs. By using the trace operation or Newton's method, we find that the graphs cross at approximately -0.70664905. Thus, we conclude that $|f(x)-g(x)|\leq\frac{1}{10}$ on the interval $[-0.70664905, 1]$. If we use $1+(x/3)$ to approximate $\sqrt[3]{1+x}$ for any x in this interval, the error will be less than $\frac{1}{10}$.

EXAMPLE ■ 2 Find a power series representation for $\sqrt[3]{1+x^4}$.

SOLUTION The power series can be obtained by substituting x^4 for x in the series of Example 1. Hence, if $|x|<1$, then

$$\sqrt[3]{1+x^4}=1+\frac{1}{3}x^4-\frac{2}{3^2\cdot 2!}x^8+\cdots$$

$$+(-1)^{n+1}\frac{1\cdot 2\cdots(3n-4)}{3^n\cdot n!}x^{4n}+\cdots.$$

EXAMPLE 3 Approximate $\int_0^{0.3} \sqrt[3]{1+x^4}\,dx$.

SOLUTION Integrating the terms of the series obtained in Example 2 gives us

$$\int_0^{0.3} \sqrt[3]{1+x^4}\,dx = 0.3 + 0.000162 - 0.000000243 + \cdots.$$

Consequently, the integral may be approximated by 0.300162, which is accurate to six decimal places, since the error is less than 0.000000243. (Why?)

The binomial series can be used to obtain polynomial approximation formulas for $(1+x)^k$. To illustrate, if $|x| < 1$, then from Example 1,

$$\sqrt[3]{1+x} \approx 1 + \tfrac{1}{3}x.$$

Since the series is alternating and satisfies (8.30) from the second term onward, the error involved in this approximation is less than the third term, $\frac{1}{9}x^2$.

EXERCISES

Exer. 1–12: Find a power series representation for the expression, and state the radius of convergence.

1 (a) $\sqrt{1+x}$ (b) $\sqrt{1-x^3}$

2 (a) $\dfrac{1}{\sqrt[3]{1+x}}$ (b) $\dfrac{1}{\sqrt[3]{1-x^2}}$

3 $(1+x)^{-2/3}$

4 $(1+x)^{1/4}$

5 $(1-x)^{3/5}$

6 $(1-x)^{2/3}$

7 $(1+x)^{-2}$

8 $(1+x)^{-4}$

9 $(1+x)^{-3}$

10 $x(1+2x)^{-2}$

11 $\sqrt[3]{8+x}$ (*Hint:* Consider $2\sqrt[3]{1+\frac{1}{8}x}$.)

12 $(4+x)^{3/2}$

Exer. 13–14: (a) Obtain a power series representation for $f(x)$ by using the given relationship. (b) Find the radius of convergence.

13 $f(x) = \sin^{-1} x;\quad \sin^{-1} x = \int_0^x (1/\sqrt{1-t^2})\,dt$

14 $f(x) = \sinh^{-1} x;\quad \sinh^{-1} x = \int_0^x (1/\sqrt{1+t^2})\,dt$

c **Exer. 15–18: Approximate the integral to three decimal places, using the indicated exercise.**

15 $\int_0^{1/2} \sqrt{1+x^3}\,dx$ (Exercise 1)

16 $\int_0^{1/2} \dfrac{1}{\sqrt[3]{1+x^2}}\,dx$ (Exercise 2)

17 $\int_0^{0.3} \dfrac{1}{(1+x^3)^2}\,dx$ (Exercise 7)

18 $\int_0^{0.1} \dfrac{1}{(1+5x^2)^4}\,dx$ (Exercise 8)

c **Exer. 19–20: For the given k, graph $f(x) = (1+x)^k$ and $g(x) = 1 + kx$ on the same xy-plane for $-1 \le x \le 1$. Use the graphs to estimate the largest closed interval of $[-1, 1]$ on which $|f(x) - g(x)| \le \frac{1}{10}$.**

19 $k = \frac{1}{2}$

20 $k = \frac{5}{2}$

21 Refer to Exercise 82 of Section 7.7. The formula for the period T of a pendulum of length L, initially displaced from equilibrium through an angle of θ_0 radians, is given by the improper integral

$$T = 2\sqrt{\frac{2L}{g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,d\theta.$$

By making the substitution $\sin u = (1/k)\sin\frac{1}{2}\theta$, with $k = \sin\frac{1}{2}\theta_0$, it can be shown that

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2 u}}\,du.$$

(a) Use the binomial series for $(1-x)^{-1/2}$ to show that

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{4}k^2\right).$$

(b) Approximate T if $\theta_0 = \pi/6$.