

Measuring Income Inequality: The Lorenz Curve and the Gini Index

The *Lorenz Curve* $L(x)$ gives the proportion of the total income earned by the lowest proportion x of the population.

Example: $L(.2) = .15$ says that the poorest 20% of the population earns 15% of the total income.

$L(.9) = .40$ says that the poorest 90% earns 40% of the total income. Equivalently, the richest 10% earn 60% of the total.

Properties:

$$L(0) = 0$$

$$L(1) = 1$$

L is an increasing function

Extreme Cases

(1) Absolute Equality of Income

Everyone earns the exact same income

$$L(x) = x \text{ for all } x$$

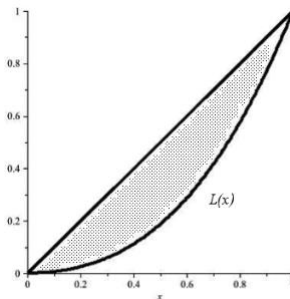
(2) Absolute Inequality of Income

Nobody earns any income except one person

$$L(x) = 0 \text{ for } 0 \leq x < 1; L(1) = 1.$$

The Gini Index

The **Gini Index** is a measure of how much inequality exists in income distribution. An index of 0 corresponds to complete equality and an index of 1 corresponds to complete inequality



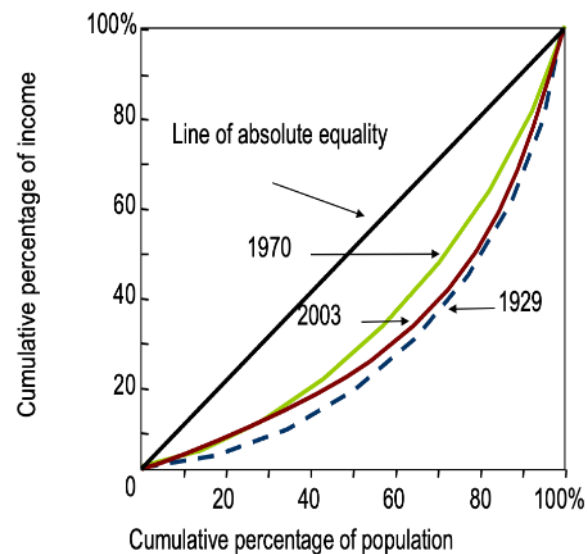
The formal definition of the *Gini Index* G is
$$G = 2 \int_0^1 x - L(x) dx$$

See also <https://www.cia.gov/the-world-factbook/field/gini-index-coefficient-distribution-of-family-income/country-comparison>

Country	Gini Index	Country	Gini Index	Country	Gini Index
Algeria	35.3	Armenia	44.4	Australia	35.2
Austria	31	Azerbaijan	36	Bangladesh	33.6
Belarus	21.7	Belgium	28.7	Bolivia	44.7
Brazil	60.7	Bulgaria	26.4	Burkina Faso	48.2
Burundi	42.5	Cambodia	40	Cameroon	47.7
Canada	31.5	Central African Republic	61.3	Chile	57.1
China	44	Colombia	57.1	Costa Rica	45.9
Cote d'Ivoire	36.7	Croatia	29	Czech Republic	25.4
Denmark	24.7	Dominican Republic	47.4	East Timor	38
Ecuador	42	Egypt	34.4	El Salvador	52.5
Estonia	37	Ethiopia	40	European Union	31.2
Finland	25.6	France	32.7	Georgia	37.1
Germany	30	Ghana	40.7	Greece	35.4
Guatemala	55.8	Guinea	40.3	Honduras	56.3
Hungary	24.4	India	37.8	Indonesia	37
Ireland	35.9	Israel	35.5	Italy	27.3
Jamaica	37.9	Japan	24.9	Jordan	36.4
Kazakhstan	31.5	Kenya	44.9	Korea, South	35.8
Kyrgyzstan	29	Laos	37	Latvia	32
Lesotho	56	Lithuania	34	Madagascar	38.1
Malaysia	49.2	Mali	50.5	Mauritania	37.3
Mauritius	37	Mexico	53.1	Moldova	40.6
Mongolia	44	Morocco	39.5	Mozambique	39.6
Namibia	70	Nepal	36.7	Netherlands	32.6
Nicaragua	55.1	Niger	50.5	Nigeria	50.6
Norway	25.8	Pakistan	41	Panama	48.5
Papua New Guinea	50.9	Paraguay	57.7	Peru	49.8
Philippines	46.6	Poland	31.6	Portugal	35.6
Romania	28.8	Russia	39.9	Rwanda	28.9
Senegal	41.3	Sierra Leone	62.9	Slovakia	26.3
Slovenia	28.4	South Africa	59.3	Spain	32.5
Sri Lanka	34.4	Sweden	25	Switzerland	33.1
Tajikistan	34.7	Tanzania	38.2	Thailand	51.1
Tunisia	41.7	Turkey	42	Turkmenistan	40.8
Uganda	37.4	Ukraine	29	United Kingdom	36.8
United States	45	Uruguay	44.8	Uzbekistan	44.7
Venezuela	49.5	Vietnam	36.1	Yemen	33.4
Zambia	52.6	Zimbabwe	50.1		

This index measures the degree of inequality in the distribution of family income in a country. The index is calculated from the Lorenz curve, in which cumulative family income is plotted against the number of families arranged from the poorest to the richest. The index is the ratio of (a) the area between a country's Lorenz curve and the 45° helping line to (b) the entire triangular area under the 45° line. The more nearly equal a country's income distribution, the closer its Lorenz curve to the 45° line and the lower its Gini index, e.g., a Scandinavian country with an index of 25. The more unequal a country's income distribution, the farther its Lorenz curve from the 45° line and the higher its Gini index, e.g., a Sub-Saharan country with an index of 50. If income were distributed with perfect equality, the Lorenz curve would coincide with the 45° line and the index would be zero; if income were distributed with perfect inequality, the Lorenz curve would coincide with the horizontal axis and the right vertical axis and the index would be 100.

Lorenz Curve for the U.S. 1929, 1970, and 2003



US income Gini indices over time

Source: http://en.wikipedia.org/wiki/Gini_coefficient

Cited: September 4, 2010

Gini indices for the United States at various times, according to the US Census Bureau:^{[1][2]}

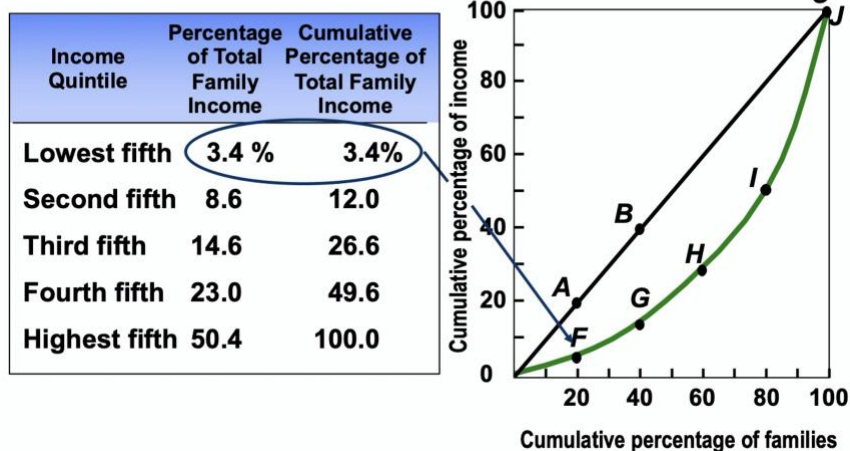
1929: 45.0 (estimated)	2000: 46.2 ^[3]
1947: 37.6 (estimated)	2005: 46.9
1967: 39.7 (first year reported)	2006: 47.0 (highest index reported)
1968: 38.6 (lowest index reported)	2007: 46.3
1970: 39.4	2008: 46.69
1980: 40.3	2009: 46.9
1990: 42.8	2010: 46.9

European Union (EU) Gini index

In 2005 the AVERAGE Gini index for the EU was estimated at 31.^[4]

1. "Gini Ratios for Households, by Race and Hispanic Origin of Householder: 1967 to 2007". *Historical Income Tables - Households*. United States Census Bureau. <http://www.census.gov/hhes/www/income/histinc/h04.html>.
2. "Table 3. Income Distribution Measures Using Money Income and Equivalence-Adjusted Income: 2007 and 2008". *Income, Poverty, and Health Insurance Coverage in the United States: 2008*. United States Census Bureau. p. 17. <http://www.census.gov/prod/2009pubs/p60-236.pdf>.
3. Note that the calculation of the index for the United States was changed in 1992, resulting in an upwards shift of about 2.
4. "Monitoring quality of life in Europe - Gini index". *Eurofound*. 26 August 2009. <http://www.eurofound.europa.eu/areas/qualityoflife/eurlife/index.php?template=3&radioindic=158&idDomain=3>.
5. Noss, Amanda, "Household Income for States: 2009 and 2010," *American Community Survey Briefs*, ACSBR/1002, United States Census Bureau (Issued September, 2011).

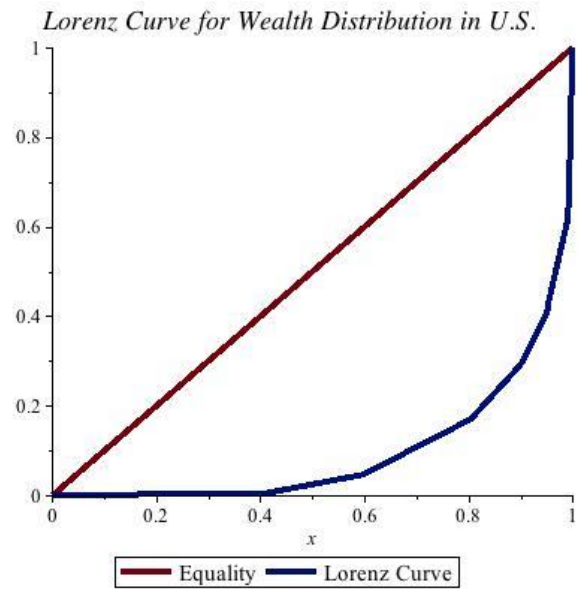
Lorenz Curve of U.S. Income



Wealth Inequality in the United States

<i>Bottom x %</i>	Cumulative % of Total Wealth
0	0
40	0.2
60	4.7
80	16.6
90	29.1
95	40.6
99	61.9
100	100

Interpretation: The poorest 60% of the population controls only 4.7 percent of the nation's wealth. The top 1% own almost 38% of the wealth. Since the lowest 95% have just over 40% of the wealth, the richest 1% have nearly as much wealth as 95% of the entire population.



The Gini Index is about .809 or 80.9 on a 100 point scale

Exercises

1. A country in Northern Europe has a Lorenz curve for family incomes given by the function

$$L(x) = \frac{e^x - 1}{e - 1} \text{ for } 0 \leq x \leq 1.$$

- a. Show that this function is a valid candidate to be a Lorenz curve.
- b. What are the values $L(0)$, $L(0.5)$, and $L(1)$? Interpret each.
- c. What is the Gini coefficient? Give your interpretation and commentary.
- d. Roughly how does the income inequality compare to that in the U.S. at the present time?

2. Find the Gini index corresponding to the Lorenz curve $f(x) = x^3$

3. Find the Gini index corresponding to the Lorenz curve $f(x) = \frac{x}{4} + \frac{3}{4} x^3$

4. The CIA website reports the Gini index for the distribution of family income in the United States to be .45.

(a) Determine the number p so that the Gini index is .45 if the Lorenz curve has the form of a power function $f(x) = x^p$

(b) According to this model, how much of the family income is earned by the top 5% of families?

5. One type of function often used to model Lorenz curves is

$$f(x) = a x + (1 - a) x^p$$

Suppose that $a = 1/4$ and that the Gini index for the distribution of wealth in a country is known to be $9/16$.

(a) Find the value of p that fits this situation.

(b) According to this model, how much of the wealth is owned by the wealthiest 5% of the population?

6. Two-class societies. In theory, it could happen that one portion of the total resources is distributed equally among one class, with the rest being shared equally by another class. Here are functions that represent **two different** two-class societies:

$$f_1(x) = \begin{cases} x/2 & 0 \leq x \leq 1/2 \\ 3x/2 - 1/2 & 1/2 \leq x \leq 1 \end{cases}$$

$$f_2(x) = \begin{cases} x/2 & 0 \leq x \leq 3/4 \\ 5x/2 - 3/2 & 3/4 \leq x \leq 1 \end{cases}$$

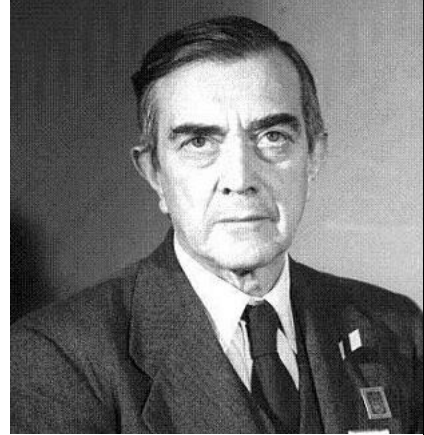
Compute the Gini index for each and decide which is the more equitable society. In each case, how much of the total resources are owned by the richest half of the population?



Max Otto Lorenz

September 19, 1876 – July 1, 1959

<https://www.hetwebsite.net/het/profiles/lorenz.h>



Corrado Gini

May 23, 1884 – March 13, 1965

<https://mathshistory.st-andrews.ac.uk/Biographies/Gini>