MATH 122A Sample Final Examination
Spring 2023

1 *Short Answer Questions*

(a) What is the relationship between the natural logarithm and natural exponential functions?
(b) Our text shows Maclaurin series for several commonly used functions such as the sine and cosine but does not display a Maclaurin series for the cotangent. Why?
(c) Give an example of a positive series $\sum_{n=1}^{\infty} a_n$ which converges but $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges.
(d) Explain why the integration by parts formula is correct.
(e) Give an example to show the claim "Every bounded sequence converges" is false.

2. (a) Let C be the curve described by parametric equations $x = sin^2 t, y = \cos 2t, 0 \leq t \leq \pi/2$
(i) Sketch a graph of C.
(ii) Find the equation of the normal line to the curve at $t = \pi/4$
(ii) Determine the length of the curve.
(iv) Find *y* as an explicit function of *x*.
(v) Rewrite the equation you found in (iv) in polar coordinates.

(b) Find a power series solution for the differential equation $f'(x) = 5x + 2f(x)$ if $f(3) = 7$.

3. Let $f(x) = \frac{\ln x}{x}\ for\ x > 0$.
(a) Determine the intervals where the function is increasing and where it is decreasing.
(b) Find the maximum value of the function.
(c) Discuss the concavity of the graph of the function and determine all points of inflection.
(d) Find $\lim_{x\to\infty} f(x)$

4. (a) Use the Integral Test to determine if $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.
(b) For *n* > 1, determine which is larger: $\frac{n \ln n}{(n+1)^3}$ or $\frac{\ln n}{n}$
(c) Use the results of (a) and (b) to determine whether the series with $a_n = \frac{n \ln n}{(n+1)^3}$ converges or diverges.

5. (a) Use your knowledge of Geometric Series to show that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$$

(b) Use the result of (a) to find a Maclaurin series for $ln\ (1 + x)$
(c) What is the radius of convergence for the series in (b)
(d) If we use the first 5 terms of the power series for $\ln(1 + x)$ to approximate $\ln(\frac{1}{2})$, estimate the error.
(e) Find a Maclaurin Series for $\frac{\ln(1+x)}{x}$

6. A colony of harmful bacteria grows at a constant rate (300 bacteria per hour) and reaches a level of 20,000 when it begins to be felt as a harmful threat to a human it has invaded. At this moment, an antibiotic agent is injected which kills 2% of the remaining bacteria per hour in a continuous manner.

(a) Discuss why the differential equation $B'(t) = 300 - .02B(t)$ is a good model for subsequent growth of the colony.

(b) Without solving the differential equation for B as an explicit function of t, discuss what will happen in the long run to the bacteria population.

(c) Now solve the differential equation and determine the number of bacteria 24 hours after the injection.