MATH 122A Some Notes on Sample Final Examination
Spring 2023

1 *Short Answer Questions*

(a) What is the relationship between the natural logarithm and natural exponential functions?
They are inverses of each other.

(b) Our text shows Maclaurin series for several commonly used functions such as the sine and cosine but does not display a Maclaurin series for the cotangent. Why?
Maclaurin series requires that the function and all its derivatives be defined at 0, but cotangent *x* has no value at *x* = 0.

(c) Give an example of a positive series $\sum_{n=1}^{\infty} a_n$ which converges but $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges.
$a_n = \frac{1}{n^2}$

(d) Explain why the integration by parts formula is correct. **Integrate both sides of the product rule for differentiation:**

$$\int (uv)' = \int u'v + \int uv' \text{ so } \int uv' = uv - \int u'v$$

(e) Give an example to show the claim "Every bounded sequence converges" is false.
$\{(-1)^n\}$

2. (a) Let C be the curve described by parametric equations $x = sin^2 t, y = \cos 2t, 0 \le t \le \pi/2$

(i) Sketch a graph of C. **Since $y = cos\ 2t = 1 - 2sin^2 t = 1 - 2x$, the graph is the straight line segment from (0,1) to (1,-1).**

(ii) Find the equation of the normal line to the curve at $t = \pi/4$ **The slope of the tangent line is $\frac{y'(t)}{x'(t)} = \frac{-2\sin 2t}{2\sin t\cos t}$ which has value -2 at $t = \pi/4$. Slope of normal line is $^1/_2$. The point is (1/2, 0) so the equation of the normal line is $y = \frac{1}{2}(x - \frac{1}{2})$**

(ii) Determine the length of the curve. **Note that $y'(t) = -2\sin 2t = -2\ (2\ sin\ t\ cost) = -4\ sin\ t\cos t$.**

Then $\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{20sin^2t\ cos^2t} = 2\sqrt{5}\ sin\ t\ cos\ t = \sqrt{5}\ sin\ 2t$ so

Length = $\int_0^{\pi/2} \sqrt{5}\ sin\ 2t\ dt = \sqrt{5}\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2} = \sqrt{5}$

(iv) Find *y* as an explicit function of *x* **See above: $y = 1 - x$.**

(v) Rewrite the equation you found in (iv) in polar coordinates. **Use $x = r\cos\theta, y = r\sin\theta$. Then y = 1- x becomes $r\sin\theta = 1 - r\cos\theta$ or $r = \frac{1}{\sin\theta + \cos\theta}$**

(b) Find a power series solution for the differential equation $f'(x) = 5x + 2f(x)$ if $f(3) = 7$.

$f''(x) = 5 + 2\ f'(x), f^{(3)}(x) = 2\ f''(x), f^{(4)}(x) = 2\ f^{(3)}(x), \dots, f^{(n+1)}(x) = 2\ f^{(n)}(x)$ for $n > 2$

Thus $f(3) = 7, f'(3) = 5 \times 3 + 2 \times 7 = 29, f''(3) = 5 + 2 \times 29 = 63, f^{(3)}(3) = 2 \times 63,, f^{(4)}(3) = 2^2 \times 63$ and, in general for $n > 2$, $f^{(n)}(3) = 2^{n-2} \times 63$. The series solution is

$$7 + 29(x-3) + \frac{63}{2!}(x-3)^2 + \frac{2 \times 63}{3!}(x-3)^3 + \cdots + \frac{2^{n-3} \times 63}{n!}(x-3)^n + \cdots$$

3. Let $f(x) = \frac{\ln x}{x}$ $for\ x > 0.$

(a) Determine the intervals where the function is increasing and where it is decreasing.

By Quotient Rule, $f'(x) = \frac{1-\ln x}{x^2}$ which is positive for $1 < x < e$ and negative for $x > e$. Thus f is increasing on $(0, e)$ and decreasing on (e, ∞)

(b) Find the maximum value of the function. **Maximum occurs at $x = e$ where value is $1/e$.**

(c) Discuss the concavity of the graph of the function and determine all points of inflection.

Use Quotient Rule again to find $f''(x) = \frac{-3+2\ln x}{x^3}$, which changes sign from negative to positive at $x = e^{3/2}$ where there is the only point of inflection.

(d) Find $\lim_{x\to\infty} f(x)$. **This is an $\frac{\infty}{\infty}$ indeterminate form so we can use l'Hopital's Rule: $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$.**

4. (a) Use the Integral Test to determine if $\sum_{n=3}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.

To find an antiderivative, use integration by parts with $u = \ln x$, $dv = \frac{1}{x^2}$ so $du \frac{1}{x}$, $v = -\frac{1}{x}$. Then $\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = \frac{-\ln x - 1}{x}$. The value of the improper integral is $\lim_{b\to\infty} \left(\frac{-\ln b - 1}{b} + \frac{\ln 3 + 1}{3}\right) = \frac{\ln 3 + 1}{3}$. The Improper Integral converges so the series converges

(b) For $n > 1$, determine which is larger: $\frac{n\ln n}{(n+1)^3}$ or $\frac{\ln n}{n^2}$

Since $(n+1)^3 > n^3$. we have $\frac{n\ln n}{(n+1)^3} < \frac{n\ln n}{n^3} = \frac{\ln n}{n^2}$

(c) Use the results of (a) and (b) to determine whether the series with $a_n = \frac{n\ln n}{(n+1)^3}$ converges or diverges.

The a_n series is positive and term by term smaller than a convergent series. By the Comparison Test, it also converges

5. (a) Use your knowledge of Geometric Series to show that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$$

Right hand side is a geometric series with a= 1 and r = -x. Series sums to $\frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}$ if $|x| < 1$

(b) Use the result of (a) to find a Maclaurin series for $ln\ (1 + x)$

Integrate each side of the equation to obtain $\ln 1 + x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots + (-1)^{n-1}\frac{x^n}{n} + ..$

(c) What is the radius of convergence for the series in (b) **Same as geometric series: 1**

(d) If we use the first 5 terms of the power series for $\ln(1+x)$ to approximate $\ln(\frac{1}{2})$, estimate the error.

As we have an alternating series, error has size less than 6th term: $\left(\frac{1}{2}\right)^6/6$

(e) Find a Maclaurin Series for $\frac{\ln(1+x)}{x}$ **Divide each term in (b) by *x*.**

6. A colony of harmful bacteria grows at a constant rate (300 bacteria per hour) and reaches a level of 20,000 when it begins to be felt as a harmful threat to a human it has invaded. At this moment, an antibiotic agent is injected which kills 2% of the remaining bacteria per hour in a continuous manner.

(a) Discuss why the differential equation $B'(t) = 300 - .02B(t)$ is a good model for subsequent growth of the colony.

(b) Without solving the differential equation for *B* as an explicit function of t, discuss what will happen in the long run to the bacteria population. **Population will decline as long as *B'(t)* is negative. Since *B'* = 0 when *B* = $\frac{300}{.02}$= 15,000, population will have limit of 15,000**

(c) Now solve the differential equation and determine the number of bacteria 24 hours after the injection.

Separate variables and integrate:

$\int \frac{B'(t)}{300-.02B(t)}\,dt = \int 1\,dt$ to obtain $-\frac{1}{.02}\,ln(300-.02B(t)) = t + C$ so $ln(300-.02B(t)) = -.02\,t + C$

and then 300 - .02B(t) = C $e^{-.02t}$. Apply initial condition B(0) = 20000 to get 300 – 400 = -100 = C.

Thus B(t) = 15000 +5000 $e^{-.02t}$. Number of bacteria after 24 hours is $B(24) = 15000 + 5000e^{-.48t}$