MATH 122A Calculus II
Notes on Sample Examination 3

1. Define what it means for a sequence to **bounded**
A sequence $\{a_n\}$ is bounded if there is a positive number B such that $|a_n| \leq B$ for all n.

2. .(a) Determine whether the sequence $\{a_n\}$ where $a_n = 1/5^n$ is increasing, decreasing, or not monotonic. Is the sequence bounded? Does it converge?
Since $5^{n+1} = 5(5^n) > 5^n, \{5^n\}$ is a monotonically increasing sequence and thus the sequence of reciprocals is a monotonically decreasing sequence of positive numbers bounded above by 5. The sequence converges to 0.
(b) Give an example of a sequence which bounded but does not converge.
$\{(-1)^n\}$is a simple example.

3. For each of the two series below, determine if it converges or diverges. If it converges, find the sum:
(a) 1 + 0.4+0.16 + .0064 + … This is a geometric series with a = 1 and r = .4. Since | r | <1, the series converges to $\frac{1}{1-r} = \frac{1}{1-.4} = \frac{1}{.6} = \frac{10}{6} = \frac{5}{3}$.
(b) $\sum_{n-1}^{\infty} \frac{(-6)^{n-1}}{5^n}$. This is a geometric series with $|r| = |-\frac{6}{5}| = \frac{6}{5} > 1$ so it diverges.

4. Determine whether the series is convergent or divergent: $\sum_{n=1}^{\infty} \frac{7n-n^{1/3}}{n^5}$
Here $a_n = \frac{7n-n^{1/3}}{n^5} < \frac{7n}{n^5} = \frac{7}{n^4}$and this series is less than the convergent series $7\sum \frac{1}{n^4}$ (p-series with p = 4 > 1) and so it converges by the Comparison Test.

An alternative approach is to write the series as the difference of two series using $\sum \frac{7n-n^{1/3}}{n^5} = \sum \frac{7n}{n^5} - \sum \frac{n^{\frac{1}{3}}}{n^5} = \sum \frac{7}{n^4} - \sum \frac{1}{n^{14/3}}$, each of which converges by p-test .

5. Use the integral test to determine if the following series converges or diverges

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

The series behaves like the improper integral $\int_2^{\infty} \frac{1}{x^2-1}\, dx$. You can find an antiderivative by a partial fraction decomposition or a trigonometric substitution. We'll illustrate the first approach:
$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$, so $\int \frac{1}{x^2-1} dx = \frac{1}{2}(\ln(x-1) - \ln(x+1)) + C =$ $= \frac{1}{2} ln\left(\frac{x-1}{x+1}\right)$ +C. Then $\int_2^b \frac{1}{x^2-1} dx = \frac{1}{2} ln\left(\frac{b-1}{b+1}\right) - \frac{1}{2} ln\left(\frac{1}{3}\right)$. Since $\lim_{b\to\infty} \left(\frac{b-1}{b+1}\right) = 1$ and $\ln 1 = 0$, we have $\int_2^{\infty} \frac{1}{x^2-1}\, dx. = \lim_{b\to\infty} \int_2^b \frac{1}{x^2-1} dx = -\frac{1}{2} ln\left(\frac{1}{3}\right)$. Since the improper integral converges, so the infinite series.

6. Test the series for convergence

$$\sum_{n=1}^{\infty}(-1)^n \frac{2n}{4n^2+1}$$

The terms of the series clearly alternate in sign with $\lim_{n\to\infty}\frac{2n}{4n^2+1} = \lim_{n\to\infty}\frac{2/n}{4++1/n^2} = 0.$ Finally, we check that the terms monotonically decrease by looking at the derivative of $f(x) = \frac{2x}{4x^2+1}$ which is $f'(x) = \frac{2(1-4x^2)}{(1+4x^2)^2}$ which is negative for all $x > ½$. The series converges by the Alternating Series Test.

7. Test for absolute convergence

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n!}$$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)}\ \frac{n!}{3^n} = \frac{3}{n+1}$ which has limit 0 and $n \to \infty$ so series converges absolutely.

8. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{x^n}{n3^n}$$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)3^{n+1}}\ \frac{n\,3^n}{|x|^n} = |x|\frac{n}{n+1}\frac{1}{3} \to \frac{|x|}{3}$ so power series converges for $|x| < 3$;that is, -3 < x < 3. At x = 3, series is the divergent harmonic series $\sum\frac{1}{n}$ but at x =-3, the series is the convergent alternating harmonic series $\sum\frac{(-1)^n}{n}$. Interval of converges is [-3, 3].

9. (a) Either give an example of an infinite series that sums to 10^{2023} or show that no series can add up to that large a number.
Let $a_1 = 10^{2023}$ and $a_n = 0$ for $n \geq 1$. Then $\sum_{n=1}^{\infty} a_n = 10^{2023}$.

(b) Suppose $\{a_n\}$ is a sequence of positive numbers such that $\sum a_n$ converges. Provide a careful argument that $\sum a_n^2$ must also converge.
Since the series $\sum a_n$ converges, the individual terms a_n have limit 0. Thus after some index M, all the terms a_n are positive numbers less than 1. Then their squares a_n^2 will be less than a_n The Comparison Test implies $\sum a_n^2$ will also converge.

10. Determine which of the following improper integrals converge and which diverge:
(a) $\int_1^{\infty}\frac{1}{\sqrt{x}}\,dx \quad \lim_{b\to\infty}\int_1^b\frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}(2\sqrt{b} - 2\sqrt{1})$ = ∞; diverges

(b) $\int_0^1\frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1\frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}(2\sqrt{1} - 2\sqrt{a})$ = 2; converges.

(c) $\int_0^{\infty}\frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}\,dx = \lim_{b\to\infty}(\arctan b - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$;converges

(d) $\int_2^4\frac{1}{x-4}dx = \lim_{a\to 4^-}\int_2^a\frac{1}{x-4}dx = \lim_{a\to 4^-}(\ln(|a-4|) - \ln(|2-4|))$