MATH 122A Calculus II
Sample Examination 1

1. (a) Explain the difference between a Lorenz Function and a Gini Index.
The Lorenz Function *L(x)* is the proportion of the total resource owned by bottom *x* proportion of the population.
The Gini Index is twice 2 the area between line of equal distribution *y* = *x* and the actual distribution *L(x)*. It is measure of inequality of the distribution.
(b) Find the Gini Index if

$$L(x) = \begin{cases} \frac{3}{4}x & \text{if } 0 \le x \le \frac{1}{3} \\ \frac{27}{16}\left(x - \frac{1}{3}\right)^2 + \frac{1}{4} & \text{if } \frac{1}{3} \le x \le 1 \end{cases}$$

Gini Index $= 2\int_0^1 x - L(x)\,dx = 2\left[\int_0^{1/3} x - \frac{3}{4}x\,dx + \int_{1/3}^1 x - \left(\frac{27}{16}\left(x-\frac{1}{3}\right)^2 + \frac{1}{4}\right)dx\right] = 2\left[\frac{1}{72} + \frac{1}{9}\right] = \frac{1}{4}$

(c) Suppose $L(x) = x^p$ for some constant p. Determine the value of p if the Gini Index $= \frac{1}{2}$

Gini $= 2\int_0^1 x - x^p\,dx = 2\left[\frac{x^2}{2} - \frac{x^{p+1}}{p+1}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{p+1}\right] = 1 - \frac{2}{p+1}$ **which equals** $\frac{1}{2}$ **when** $p = 3$.

(d) Use your answer for (c), find an expression for the portion of the total owned by the top 5% of the population. $1 - .95^3$

2. Let *f* be the function defined by $f(x) = \frac{3}{8+x^3}$ on the closed interval *I* = [-1,4].
(a) Show that f is a one-to-one function. $f'(x) = \frac{-3\,(3x^2)}{(8+x^3)^2}$ **which is nonpositive for all** x **in** I.
Hence f **is a strictly decreasing function so it is one − to − one.**
An alternative proof: Suppose *f(a)* = *f(b)*. Then $\frac{3}{8+a^3} = \frac{3}{8+b^3}$. **Since numerators are equal, the denominators must also be equal. Thus**

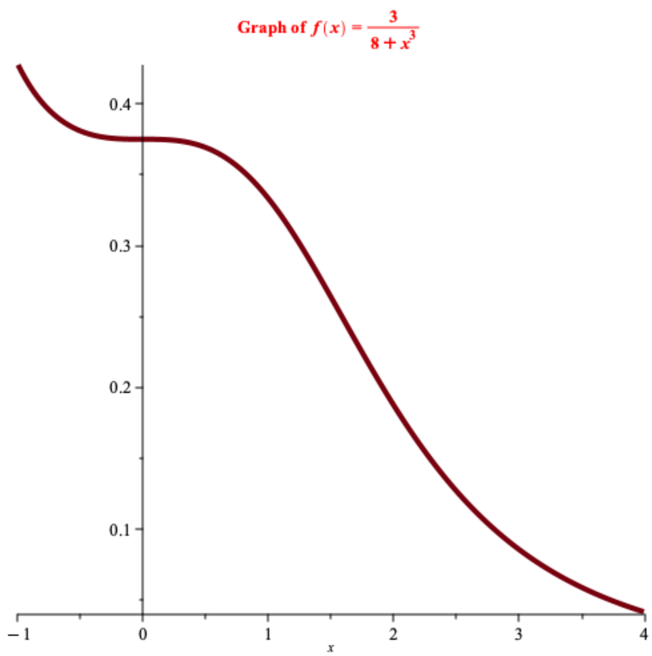
$$8 + a^3 = 8 + b^3 \text{ so } a^3 = b^3 \text{ and hence } a = b.$$

(b) If *g* is the inverse of *f*, determine $g'\left(\frac{1}{3}\right)$ **Since** $f(1) = \frac{3}{8+1} = \frac{3}{9} = \frac{1}{3}$, **we have** $g\left(\frac{1}{3}\right) = 1$. **Thus** $g'\left(\frac{1}{3}\right) = \frac{1}{f'(g(\frac{1}{3}))} = \frac{1}{f'(1)}$
Now $f'(1) = \frac{-3(3)}{(8+1)^2} = \frac{-1}{9}$ **so** $g'\left(\frac{1}{3}\right) = -9$.**Then** $g'\left(\frac{1}{3}\right) = \frac{1}{f'(g(\frac{1}{3}))} = \frac{1}{f'(1)}$
(c) Find f' and determine the maximum and minimum values of *f* on the interval *I*. **See (a) for** f' **. Since** f **is decreasing, the maximum value occurs at *x*= -1 with value** $f(-1) = \frac{3}{7}$ **and minimum value at *x* = 4 with value** $f(4) = \frac{3}{72} = \frac{1}{24}$.

(d) Determine where the graph of *f* is concave up and where it is concave down. Identify all points of inflection.

Using the Quotient Rule on $f'(x)$ and simplifying, we have $f''(x) = \frac{36x(x^3-4)}{(x^3+8)^3}$. The denominator is positive for all x in the interval so the sign of the second derivative is the sign of $x(x^3-4)$ which is positive for $x < 0$, negative for $0 < x < \sqrt[3]{4}$ and positive for $x > \sqrt[3]{4}$.
Graph of f is concave up for $x < 0$ and $x > \sqrt[3]{4}$ while it is concave down for $0 < x < \sqrt[3]{4}$. There are points of inflection at (0, 3/8) and $(\sqrt[3]{4}, \frac{1}{4})$.

(e) Sketch a graph of f.



3. Differentiate each of the following functions with respect to x: **Note: Chain Rue is essential**

(a) $P(x) = \ln(\sin x)$ has $\boldsymbol{P'(x) = \frac{1}{\sin x}(\sin x)' = \frac{\cos x}{\sin x} = \cot x}$

(b) $Q(x) = e^{x^2+7x}$ **has $Q'(x) = e^{x^2+7x}(x^2+7x)' = e^{x^2+7x}(2x+7)$**

(c) $R(x) = log_8(\cos x)$ has

(d) $S(x) = \ln(e^x)$ **Note $S(x) = x$ so $S'(x) = 1$**

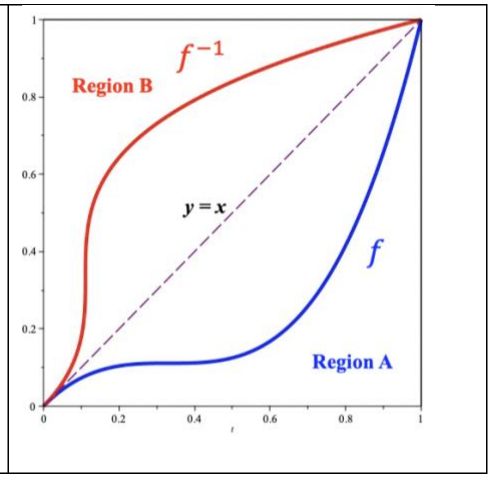
(e) $T(x) = x^x$: **Note $\ln(T(x) = \ln x^x = x \ln x$. Now differentiate: $\frac{T\prime(x)}{T(x)} = x\,\frac{1}{x} + 1 \ln x$ [Product Rule] $= 1 + \ln x$**
so $T'(x) = (1 + \ln x)\, T(x) = (1 + \ln x)\, x^x$

(f) $U(x) = \int_2^x \frac{\sin t}{t}\, dt$ **BY FTC: $U'(x) = \frac{\sin x}{x}$**

4. $V(x) = \int_1^{\sqrt{x}} \frac{3}{8+t^2}$ dt. **Let F be an antiderivative of the integrand; that is, $F'(t) = \frac{3}{8+t^2}$. Then $V(x) = F(\sqrt{x}) - F(1)$. Taking the derivative, $V'(x) = F'(\sqrt{x})(\sqrt{x})' - 0 = \frac{3}{8+(\sqrt{x})^2}\ \frac{1}{2\sqrt{x}} = \frac{3}{2\sqrt{x}\,(8+x)}$**

5. Suppose all we know about a continuous function f is that $f(0) = 0, f(1) = 1, f'(x) > 0$, all x, and $\int_0^1 f(x)\,dx = \frac{1}{4}$.
Find $\int_0^1 f^{-1}(x)\,dx$ where f^{-1} is the inverse of f.

**See Picture on the Right. Region B is the mirror image of Region A across the y=x line so the two regions have the same area.
Now the total area in the square box is 1.
Then $\int_0^1 f^{-1}(x)\,dx$ is the area under the red curve which is 1 – area (Region B) which Equals 1 – area(Region A)
$= 1 - \int_0^1 f(x)\,dx = 1 - \frac{1}{4} = \frac{3}{4}$**



6. (a) Give a careful statement of the Fundamental Theorem of Calculus. **"If G is defined by $G(x) = \int_a^x f(t)\,dt$ for every x in [a,b], then G is an antiderivative of f on [a,b]" OR "If F is any antiderivative of f on [a,b], then $\int_a^b f(x)\,dx = F(b) - F(a)$".**
(b) What is the definition of the natural logarithm function? **$\ln x = \int_1^x \frac{1}{t}\,dt,$ for $x > 0$**
(c) How is the number e defined**?** **e is the number whose natural logarithm is 1**

7. The amount x of light affects the rate y of photosynthesis by the relationship

$$y = f(x) = x^a\, e^{(\frac{a}{b})(1-x^b)} = x^a exp\left(\left(\frac{a}{b}\right)1 - x^b\right)$$

where $x > 0$ and a and b are positive constants . Show that f has a maximum at $x = 1$.
Start by finding the derivative of f using the Product and Chain Rules:

`$f'(x) = a\,x^{a-1}\ e^{(\frac{a}{b})(1-x^b)} + x^a\ e^{(\frac{a}{b})(1-x^b)}\ \frac{a}{b}(0 - b\,x^{b-1})$ which we can simplify to $ax^{a-1}e^{(\frac{a}{b})(1-x^b)}\,[1 - x\,x^{b-1}]$

Thus $f'(x) = [\,ax^{a-1}]\ \left[e^{(\frac{a}{b})(1-x^b)}\right][1 - x^b]$. The first two factors are each positive and third factor changes from positive to negative at $x^b = 1$; that is, $x = 1$.