Math 121 - Calculus I
Exam 3: Practice Exam

Name: Solutions
Section:

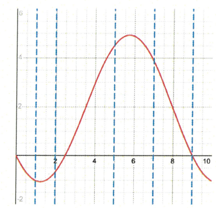
Please be sure to neatly **show and explain all of your work** and clearly label your answers. Except for your index card, this exam is a closed-book, closed-notebook exam. Calculators are not allowed.

Please write and sign the Honor Pledge here when you are done:

Signed:

Problem	Points
1	/10
2	/10
3	/10
4	/8
5	/12
6	/10
Total	/60

1. Consider the following graph of the function $f(x)$.



Consider the values $x = 1, 2, 5, 7, 9$. For each question below, list all of these values that apply. For each, explain your reasoning.

(a) For which of these values is $f(x) > 0$?

x = 5, 7. The graph of f lies above the x-axis at these x-values.

(b) For which of these values is $f'(x) > 0$?

x = 2, 5. The slope of f is positive at these x-values.

(c) For which of these values is $f(x)$ increasing?

x = 2, 5, for the same reason as part (b).

(d) For which of these values is $f'(x)$ increasing?

x = 1, 2, 9. f'(x) is increasing when f''(x) > 0. This coincides with the graph of f being concave up, which is true at x = 1, 2, 9.

(e) For which of these values is the slope of $f(x)$ negative?

x = 1, 7, 9. The graph of f is decreasing at these points.

(f) For which of these values is the slope of $f'(x)$ negative?

x = 5, 7. f''(x) measures the slope of f'(x). f(x) is concave down at x = 5, 7, so these are the values when f''(x) is negative, ie the slope of f'(x) is negative.

2. Suppose that
$$f(x) = (3x+8)^{1/3}.$$

(a) Use a linearization to estimate the value of $f(0.1)$.

We need the tangent line to $f(x)$ at $a=0$.

We will linearize at $a=0$ since this is close to $x=0.1$.

Point: $(0, f(0)) = (0, 8^{1/3}) = (0,2)$

Slope: $f'(x) = \frac{1}{3}(3x+8)^{-2/3}(3) = \frac{1}{(3x+8)^{2/3}}$

$\to f'(0) = \frac{1}{8^{2/3}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{4}$.

Tangent line: $\frac{y-2}{x-0} = \frac{1}{4} \Rightarrow y-2 = \frac{1}{4}(x-0) \Rightarrow \underbrace{y = \frac{1}{4}(x-0)+2}_{L(x)}$

$f(0.1) \approx L(0.1) = \frac{1}{4}(0.1-0)+2 = 0.25(0.1)+2 = \boxed{2.025}$

(b) Is your estimate an overestimate or an underestimate? Please explain.

consider the concavity of $f(x)$ at $x=0$:

$f'(x) = (3x+8)^{-2/3}$

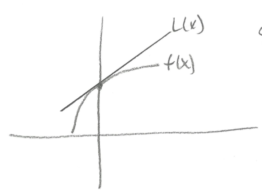
$f''(x) = -\frac{2}{3}(3x+8)^{-5/3}(3)$

chain rule

$= \frac{-2}{(3x+8)^{5/3}}$.

So, $f''(0) = -\frac{2}{8^{5/3}} =$ negative.

Thus, f is concave down at $x=0$



Since $L(x)$ lies above $f(x)$, $L(0.1)$ is an overestimate

3. Suppose that the *derivative* of $f(x)$ is given by
$$f'(x) = \frac{(x-1)^2 x}{(x+1)^3}.$$

Find the local maxima and minima of $f(x)$ assuming that the domain of $f(x)$ is all $x \neq -1$.

We start by finding critical numbers.

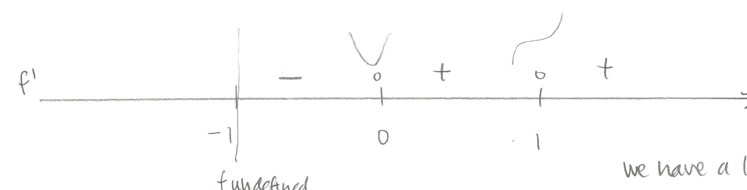
$f'(x)$ is undefined at $x=-1$.

not domain, so can't be a local max or local min.

$f'(x)=0$ when $(x-1)^2 x = 0$,

ie. $x-1=0$ or $x=0$ $\Rightarrow \underbrace{x=0,1}_{\text{CNs}}$

We now use the first derivative test:



$f'(x) = \frac{(x-1)^2 x}{(x+1)^3} \leadsto f'(-\frac{1}{2}) = \frac{(pos)(neg)}{pos} = neg$

$f'(\frac{1}{2}) = \frac{(pos)(pos)}{pos} = pos$

$f'(10) = \frac{(pos)(pos)}{pos} = pos$

We have a local minimum at $x=0$ and no local maxima.

4. Find the absolute maximum and absolute minimum values of

$$f(x) = x + \frac{1}{x} \quad = \quad x + x^{-1}$$

on the interval $[0.2, 4]$.

We will use the Extreme Value Theorem:

$$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

$f'(x)$ is undefined at $x = 0$ ← not in the interval.

$f'(x) = 0$ when $x^2 - 1 = 0$, so $\underset{\text{CN to consider.}}{x = 1}$ and $\underset{\text{not in the interval}}{x = -1}$.

By the Extreme Value Theorem, we test endpoints and the CN in the interval:

$f(0.2) = 0.2 + \frac{1}{0.2} = 0.2 + 5 = \boxed{5.2}$ ← abs max value

$f(1) = 1 + \frac{1}{1} = \boxed{2}$ ← abs min value

$f(4) = 4 + \frac{1}{4} = 4.25$

5. Use first and second derivative information to sketch a graph of

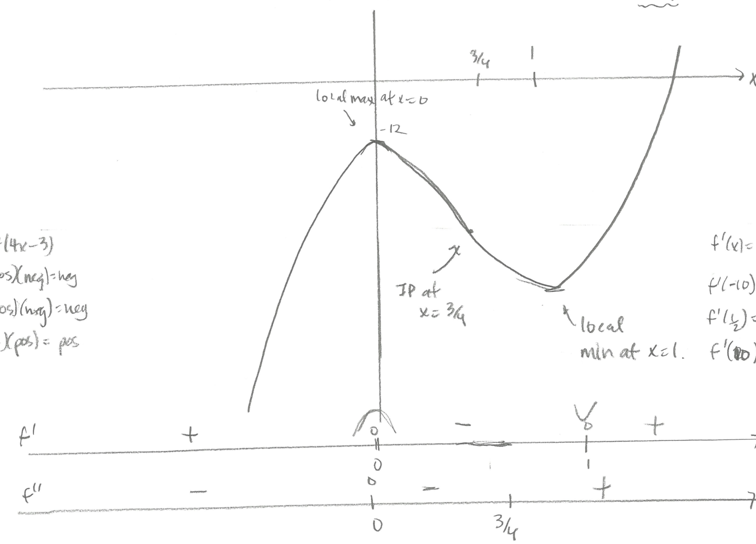
$$f(x) = 4x^5 - 5x^4 - 12.$$

Clearly label the coordinates of the y-intercept and the x-values of any local maxima, local minima, or inflection points.

Domain: $(-\infty, \infty)$. y-intercept: $f(0) = -12$. no HA or VA. ✓

$f'(x) = 20x^4 - 20x^3 = 20x^3(x-1) = 0$ when $\underset{\text{CNs.}}{x = 0, 1}$. ← defined for all x ✓

$f''(x) = 80x^3 - 60x^2 = 20x^2(4x-3) = 0$ when $x = 0, \frac{3}{4}$. ← defined for all x ✓



ncavity

↓

$''(x) = 20x^2(4x-3)$

$''(-10) = (pos)(neg) = neg$

$''(\frac{1}{2}) = (pos)(neg) = neg$

$''(10) = (pos)(pos) = pos$

inc/dec

↓

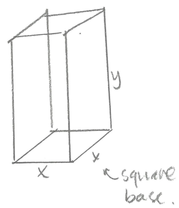
$f'(x) = 20x^3(x-1)$

$f'(-10) = (neg)(neg) = pos$

$f'(\frac{1}{2}) = (pos)(neg) = neg$

$f'(10) = (pos)(pos) = pos$

6. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used. Confirm that your dimensions minimize the amount of material used.



surface area   2 variables!

$SA = x^2 + 4xy$ ← minimize this

base   four sides

$V = x^2 y = 32{,}000$ ← subject to this constraint

From $x^2 y = 32000$, $\boxed{y = \frac{32000}{x^2}}$

Substitute into SA: $SA(x) = x^2 + 4x\left(\frac{32000}{x^2}\right) = x^2 + \frac{128000}{x}$.

Now, find x that minimizes SA(x):

$SA'(x) = 2x - \frac{128000}{x^2} = \frac{2x^3 - 128000}{x^2}$.

$64000 = 8 \cdot 8 \cdot 1000 = 2^3 2^3 10^3$

$\sqrt[3]{64000} = 40$

$SA'(x)$ undefined at $x = 0$ ← this makes volume 0 ... ignore.

$SA'(x) = 0$ when $2x^3 - 128000 = 0 \Rightarrow x^3 = 64{,}000 \Rightarrow x = 40$ CN

There is exactly one CN, so we use the modified 1st derivative test: $\boxed{\begin{matrix} x=40 \\ y=20 \end{matrix}}$

$SA'(10) = \frac{neg}{pos} = neg$

$SA'(100) = \frac{pos}{pos} = pos$

Since $SA'(x) < 0$ when $x < 40$ and $SA'(x) > 0$ when $x > 40$ we have an abs max at $x = 40$. So dimensions are